ISSN 2590-9770
The Art of Discrete and Applied Mathematics 6 (2023) #P1.01
https://doi.org/10.26493/2590-9770.1563.5f6
(Also available at http://adam-journal.eu)

On graphs admitting two disjoint maximum independent sets*

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Received 13 September 2021, accepted 11 July 2022, published online 26 September 2022

Abstract

An independent set S is *maximal* if it is not a proper subset of an independent set, while S is *maximum* if it has a maximum size. The problem of whether a graph has a pair of disjoint maximal independent sets was introduced by Berge in the early 1970s. The class of graphs for which every induced subgraph admits two disjoint maximal independent sets was characterized by Schaudt in 2015.

In this paper, we are focused on finding conditions ensuring the existence of two disjoint maximum independent sets.

Keywords: Maximum independent set, shedding vertex, König-Egervàry graph, almost bipartite graph, unicyclic graph.

Math. Subj. Class.: 05C69, 05C75, 13F55

*We express our gratitude to Isabel Beckenbach, who suggested a number of remarks that helped us make proofs of some theorems clearer. It is also our pleasure to thank the anonymous reviewer for drawing our attention to the fact that the original proof of Lemma 2.10 is suitable for a stronger claim concerning the matching number instead of the independence number.

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1 Introduction

Throughout this paper $G = (V, E)$ is a finite, undirected, loopless graph without multiple edges, with vertex set $V = V(G)$ of cardinality $|V(G)| = n(G)$, and edge set $E = E(G)$ of size $|E(G)| = m(G)$. If $X \subset V$, then $G[X]$ is the graph of G induced by X. By $G - U$ we mean the subgraph $G[V - U]$, if $U \subset V(G)$. We also denote by $G - F$ the subgraph of G obtained by deleting the edges of F, for $F \subset E(G)$, and we write shortly $G - e$, whenever $F = \{e\}$.

The *neighborhood* $N(v)$ of $v \in V(G)$ is the set $\{w : w \in V(G) \textit{ and } vw \in E(G)\}$, while the *closed neighborhood* $N[v]$ of v is the set $N(v) \cup \{v\}$. The *neighborhood* $N(A)$ of $A \subseteq V(G)$ is $\{v \in V(G) : N(v) \cap A \neq \emptyset\}$, and $N[A] = N(A) \cup A$. We may also use $N_G(v), N_G[v], N_G(A)$ and $N_G[A]$, when referring to neighborhoods in the graph G.

$C_n, K_n, P_n, K_{p,q}$ denote respectively, the cycle on $n \geq 3$ vertices, the complete graph on $n \geq 1$ vertices, the path on $n \geq 1$ vertices, and the complete bipartite graph on $p + q$ vertices, where $p, q \geq 1$.

A *matching* is a set M of pairwise non-incident edges of G, and by $V(M)$ we mean the vertices covered by M. If $V(M) = V(G)$, then M is a *perfect matching*. The size of a largest matching is denoted by $\mu(G)$. If every vertex of a set A is an endpoint of an edge $e \in M$, while the other endpoint of e belongs to some set B, disjoint from A, we say that M is a *matching from A into B*, or A is *matched into B* by M. In other words, M may be interpreted as an injection from the set A into the set B.

The *disjoint union* $G_1 \cup G_2$ of the graphs G_1 and G_2 with $V(G_1) \cap V(G_2) = \emptyset$ is the graph having $V(G_1) \cup V(G_2)$ and $E(G_1) \cup E(G_2)$ as a vertex set and an edge set, respectively. In particular, qG denotes the disjoint union of $q \geq 2$ copies of the graph G.

Let $\mathcal{H} = \{H_v : v \in V(G)\}$ be a family of graphs indexed by the vertex set of a graph G. The corona $G \circ \mathcal{H}$ of G and $\mathcal{H}$ is the disjoint union of G and $H_v, v \in V(G)$, with additional edges joining each vertex $v \in V(G)$ to all the vertices of H_v. If $H_v = H$ for every $v \in V(G)$, then we denote $G \circ H$ instead of $G \circ \mathcal{H}$ [6].

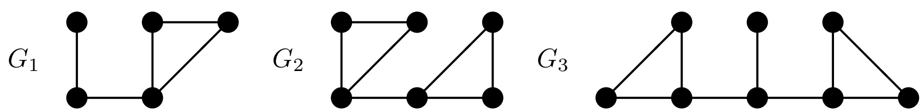

Figure 1: $G_1 = P_2 \circ \{K_1, K_2\}$, $G_2 = P_2 \circ K_2$, $G_3 = P_3 \circ \{K_2, K_1, K_2\}$.

A set $S \subseteq V(G)$ is *independent* if no two vertices from S are adjacent, and by $\mathrm{Ind}(G)$ we mean the family of all the independent sets of G. An independent set A is *maximal* if $A \cup \{v\}$ is not independent, for every $v \in V(G) - A$. An independent set of maximum size is a *maximum independent set* of G. Let $\Omega(G)$ denote the family of all maximum independent sets, and $\alpha(G) = \max\{|S| : S \in \mathrm{Ind}(G)\}$.

Theorem 1.1 ([2]). *In a graph G, an independent set S is maximum if and only if every independent set disjoint from S can be matched into S.*

If $\alpha(G) + \mu(G) = n(G)$, then G is *a König-Egerváry graph* [4, 20]. It is known that every bipartite graph is a König-Egerváry graph as well.

Let $v \in V(G)$. If for every independent set S of $G - N[v]$, there exists some $u \in N(v)$ such that $S \cup \{u\}$ is independent, then v is a *shedding vertex* of G [22]. Clearly, no isolated vertex may be a shedding vertex. On the other hand, every vertex of degree $n(G) - 1$

is a shedding vertex. Let $\mathrm{Shed}(G)$ denote the set of all shedding vertices. For instance, $\mathrm{Shed}(K_1) = \emptyset$, while $\mathrm{Shed}(K_n) = V(K_n)$ for every $n \geq 2$.

The research on the graphs admitting two disjoint maximal independent sets has its roots in [1, 15]. Further, this topic was studied in [3, 5, 8, 16, 18]. A constructive characterization of trees that have two disjoint maximal independent sets of minimum size may be found in [7].

A graph is *well-covered* if all its maximal independent sets are also maximum [17]. By definition, for well-covered graphs, the existence of two disjoint maximal independent sets is the same as the existence of two maximum independent sets.

Theorem 1.2 ([18])**.** *Let G be a well-covered graph without isolated vertices. If G does not contain $C_{2k+1} \circ K_1$ as an induced subgraph for $k \geq 1$, then G has two disjoint maximum independent sets.*

It is known that the corona $H \circ K_1$ has two disjoint maximum independent sets if and only if H is a bipartite graph [13]. The most well-known subclass of graphs with two disjoint maximum independent sets is the family of W_2-graphs. Recall that a graph G belongs to W_2 if every two disjoint independent sets are included in two disjoint maximum independent sets [13, 19].

In this paper, we concentrate on graphs admitting two disjoint maximum independent sets, paying special attention to König-Egerváry graphs and almost bipartite graphs.

2 Results

Theorem 2.1. *A graph G has two disjoint maximum independent sets if and only if there exists a matching M of size $\alpha(G)$ such that $G[V(M)]$ is a bipartite graph.*

Proof. Assume that S_1, S_2 are two disjoint maximum independent sets in G. Then $G[S_1 \cup S_2]$ is a bipartite subgraph in G, having, by Theorem 1.1, a perfect matching M of size $\alpha(G)$. Clearly, $G[V(M)] = G[S_1 \cup S_2]$.

Conversely, suppose that there exists a matching M of size $\alpha(G)$ such that $G[V(M)]$ is a bipartite graph. Consequently, the bipartition $\{A, B\}$ of $G[V(M)]$ provides two disjoint maximum independent sets, namely A and B. □

Theorem 2.1 immediately implies the following.

Corollary 2.2. *If G has two disjoint maximum independent sets, then $\mu(G) \geq \alpha(G)$.*

Theorem 2.3. *Let $S \in \mathrm{Ind}(G)$ and $S \subseteq \mathrm{Shed}(G)$, then*

(i) *the number of independent sets of size $|S|$ in G is greater or equal to $2^{|S|}$;*

(ii) *there exist some maximal independent set U disjoint from S, and a matching from S into U.*

Proof. Let $A = \{a_1, a_2, ..., a_k\} \subseteq S$. Now, we are constructing an independent set $I_A = (S - A) \cup B$ such that $S \cap B = \emptyset$ and $|A| = |B|$, and a perfect matching M from A into B.

Step 1. Consider $a_1 \in A$. Since $(S - \{a_1\}) \cap N[a_1] = \emptyset$, and $a_1 \in \mathrm{Shed}(G)$, there exists some $b_1 \in N_G(a_1)$, such that

$$S_1 = (S - \{a_1\}) \cup \{b_1\} = (S - A) \cup \{b_1, a_2, a_3, ..., a_k\}$$

is independent.

Step $j = 2, ..., k$. We already have $S_{j-1} = \{b_1, b_2, ..., b_{j-1}, a_j, ..., a_k\}$ independent. Consider $a_j \in A$. Since $(S_{j-1} - \{a_j\}) \cap N[a_j] = \emptyset$, and $a_j \in \mathrm{Shed}(G)$, there exists some $b_j \in N_G(a_j)$, such that

$$S_j = (S_{j-1} - \{a_j\}) \cup \{b_j\} = (S - A) \cup \{b_1, b_2, ..., b_j, a_{j+1}, ..., a_k\}$$

is independent.

In this way, we found $B = \{b_1, b_2 ..., b_k\}$ such that $(S - A) \cup B$ is independent, $S \cap B = \emptyset$ and $|A| = |B|$, and a perfect matching $M = \{a_i b_i : i = 1, 2, ..., k\}$ from A into B.

In other words, every subset of S produces an independent set of the same size, and all these sets are different. Thus the graph G has $2^{|S|}$ independent sets of cardinality $|S|$, at least.

If $A = S$, then I_S is disjoint from S. To complete the proof, one has just to enlarge I_S to a maximal independent set, say U. The sets U and S are disjoint, since there is a matching from S into $I_S \subseteq U$. □

Corollary 2.4. *If G has a maximal independent set S such that $S \subseteq \mathrm{Shed}(G)$, then there exists a maximal independent set U disjoint from S such that $|S| \leq |U|$.*

Corollary 2.5. *If G has a maximum independent set S such that $S \subseteq \mathrm{Shed}(G)$, then $|\Omega(G)| \geq 2^{\alpha(G)}$, while some $I \in \Omega(G)$ is disjoint from S.*

The friendship graph $F_q = K_1 \circ qK_2, q \geq 2$ shows that $2^{\alpha(G)}$ is a tight lower bound for $|\Omega(G)|$ in graphs with a maximum independent set consisting of only shedding vertices.

Notice that each graph from Figure 1 has a maximum independent set containing only shedding vertices, and hence, by Theorem 2.3, each one has two disjoint maximum independent sets.

Combining Proposition 3.5 and Corollary 2.5, we get the following.

Corollary 2.6. *If $p \geq 2$, then $G \circ K_p$ has two disjoint maximum independent sets.*

It is worth mentioning that if G has a pair of disjoint maximum independent sets, it may have $\mathrm{Shed}(G) = \emptyset$; e.g., $G = K_{n,n}$ for $n \geq 2$.

It is known that G is a König-Egervàry graph if and only if every maximum matching matches $V(G) - S$ into S, for each $S \in \Omega(G)$ [12]. Consequently, $\mu(G) \leq \alpha(G)$ holds for every König-Egervàry graph G.

Theorem 2.7. *G is a König-Egervàry graph with two disjoint maximum independent sets if and only if G is a bipartite graph having a perfect matching.*

Proof. Let $S_1, S_2 \in \Omega(G)$ and $S_1 \cap S_2 = \emptyset$. Since G is a König-Egerváry graph and $S_1 \subseteq V(G) - S_2$, we get

$$\alpha(G) = |S_1| \leq |V(G) - S_2| = n(G) - \alpha(G) = \mu(G) \leq \alpha(G).$$

It follows that $S_1 = V(G) - S_2$ and $\mu(G) = \alpha(G)$. Hence, $G = (S_1, S_2, E(G))$ is a bipartite graph with a perfect matching.

The converse is evident. □

There exist regular graphs without disjoint maximal (maximum) independent sets [16]. On the other hand, if $r > n + 2 - 2\sqrt{2n}$, then every r-regular graph has two disjoint maximal independent sets [5].

Proposition 2.8. *If G is a regular König-Egervàry graph, then G is bipartite with a perfect matching.*

Proof. Assume that $deg(v) = r$ for every $v \in V(G)$. Let $S \in \Omega(G)$. Then

$$\begin{aligned}|E(G)| - \alpha(G)\cdot r &= \frac{|V(G)|}{2}\cdot r - \alpha(G)\cdot r\\ &= \frac{r}{2}\cdot(\alpha(G)+\mu(G)-2\alpha(G))\\ &= \frac{r}{2}\cdot(\mu(G)-\alpha(G)) \geq 0,\end{aligned}$$

which implies $\mu(G) = \alpha(G)$, because $\mu(G) \leq \alpha(G)$ holds for every König-Egervàry graph. Consequently, $|E(G)| - \alpha(G)r = 0$, i.e., there are no edges between the vertices in $V(G) - S$. Hence, G is bipartite with a perfect matching. Thus the bipartition of G comprises of two maximum independent sets. □

Theorem 2.7 and Proposition 2.8 imply the following.

Corollary 2.9. *If G is a regular König-Egervàry graph, then it has two disjoint maximum independent sets.*

Lemma 2.10. *If G is a König-Egervàry graph, then $|\Omega(G)| \leq 2^{\mu(G)}$. Moreover, the equality $|\Omega(G)| = 2^{\mu(G)}$ holds if and only if $G = \mu(G)K_2 \cup (\alpha(G) - \mu(G))\, K_1$.*

Proof. Let $S \in \Omega(G)$ and M_G be a maximum matching of G. Then $|V(G) - S| = |M_G| = \mu(G)$ and each maximum matching of G matches $V(G) - S$ into S, since G is a König-Egervàry graph. Thus every maximum independent set different from S must contain vertices belonging to $V(G) - S$. Let us define a graph H as follows: $V(H) = S \cup (V(G) - S)$ and $E(H) = M_G$. In other words, $H = \mu(G)K_2 \cup (\alpha(G) - \mu(G))\, K_1$. Clearly, every vertex of H is contained in some maximum independent set, and adding an edge to $E(H)$ reduces the number of maximum independent sets. Thus $\Omega(G) \subseteq \Omega(H)$, since $E(H) \subseteq E(G)$. Hence, we infer that $|\Omega(G)| \leq |\Omega(H)| = 2^{\mu(G)}$, as required.

Suppose $|\Omega(G)| = 2^{\mu(G)}$. Then $\Omega(G) = \Omega(H)$, since $\Omega(G) \subseteq \Omega(H)$.

First, $V(G) = V(H)$. Second, $E(G) = E(H)$, since otherwise, if there is an edge $xy \in E(G) - E(H)$, then each maximum independent set of H containing $\{x, y\}$ does not appear in $\Omega(G)$, in contradiction with $\Omega(G) = \Omega(H)$. Consequently, we obtain

$$G = H = \mu(G)\, K_2 \cup (\alpha(G) - \mu(G))\, K_1.$$

Conversely, if $G = \mu(G)K_2 \cup (\alpha(G) - \mu(G))\, K_1$, then clearly, $|\Omega(G)| = 2^{\mu(G)}$. □

Corollary 2.11. *If G is a König-Egervàry graph, then $|\Omega(G)| \leq 2^{\alpha(G)}$. Moreover, the equality $|\Omega(G)| = 2^{\alpha(G)}$ holds if and only if $G = \alpha(G)K_2$.*

Proof. It is known that $\mu(G) \leq \alpha(G)$ for König-Egervàry graphs. Thus Lemma 2.10 implies that $|\Omega(G)| \leq 2^{\mu(G)} \leq 2^{\alpha(G)}$.

If $|\Omega(G)| = 2^{\alpha(G)}$, then $\alpha(G) = \mu(G)$ and

$$G = \mu(G)K_2 \cup (\alpha(G) - \mu(G))\, K_1 = \alpha(G)K_2.$$

The converse is clear. □

Combining Corollary 2.5 and Corollary 2.11, we obtain the following.

Corollary 2.12. *For a König-Egervàry graph G, the following assertions are equivalent:*

(i) *there is a maximum independent set included in* $\mathrm{Shed}(G)$*;*

(ii) $|\Omega(G)| = 2^{\alpha(G)}$*;*

(iii) $\mathrm{Shed}(G) = V(G)$.

Theorem 2.13. *A graph G has two disjoint maximum independent sets if and only if there exists a set $A \subset V(G)$ such that $G - A$ is a bipartite graph having a perfect matching of size $\alpha(G)$.*

Proof. Assume that G has two disjoint maximum independent sets. By Corollary 2.2, we know that $\mu(G) \geq \alpha(G)$.

If G is bipartite, then $\alpha(G) \geq \frac{n(G)}{2}$. Thus

$$\frac{n(G)}{2} \geq \mu(G) \geq \alpha(G) \geq \frac{n(G)}{2},$$

which means that G has a perfect matching.

If G is not bipartite, then there is an odd cycle C_1. Since C_1 cannot be two-colored, at least one vertex $v_1 \in V(C_1)$ does not lie in either of two disjoint maximum independent sets. Thus, $G - v_1$ has two disjoint maximum independent sets as well. One may keep on with this shelling procedure till $G - \{v_1, ..., v_k\}$ turns out to be bipartite with two disjoint maximum independent sets, where k is the frustration number of G. Clearly, $G - \{v_1, ..., v_k\}$ has a perfect matching of size $\alpha(G)$.

Conversely, $G - A$ is a bipartite graph with a perfect matching of size $\alpha(G)$. Hence, its bipartition is comprised of two disjoint maximum independent sets of G. □

A *cycle* is a trail, where the only repeated vertices are the first and last ones. The graph G is *unicyclic* if it has a unique cycle. A graph G is *almost bipartite* if it has a unique odd cycle [14]. Since this cycle is unique, it is chordless, and there is no other cycle of G sharing edges with it. Clearly, every unicyclic graph with an odd cycle is almost bipartite.

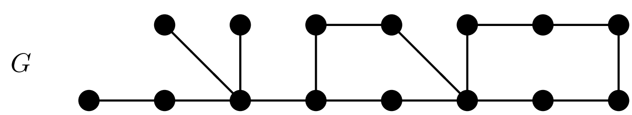

Figure 2: An almost bipartite non-König-Egerváry graph with $\alpha(G) = 8$ and $\mu(G) = 7$.

Lemma 2.14 ([14])**.** *If G is an almost bipartite graph, then*

$$n(G) - 1 \leq \alpha(G) + \mu(G) \leq n(G).$$

Theorem 2.15 ([14])**.** *If G is an almost bipartite non-König-Egerváry graph, then*

$$\alpha(G) - \mu(G) = |\mathrm{core}(G)| - |N(\mathrm{core}(G))|.$$

Proposition 2.16. *Let G be an almost bipartite graph. Then G has two disjoint maximum independent sets if and only if there is a vertex v belonging to its unique odd cycle, such that $G - v$ has a perfect matching of size $\alpha(G)$.*

Proof. Let $C = (V(C), E(C))$ be the unique odd cycle of G, and assume that G has $S_1, S_2 \in \Omega(G)$ such that $S_1 \cap S_2 = \emptyset$. By Theorem 2.7, G is not a König-Egervàry graph. Theorem 2.15 implies $\mu(G) = \alpha(G)$, since $\mathrm{core}(G) = \emptyset$. Hence $n(G) = 2\mu(G) + 1 = 2\alpha(G) + 1$, in accordance with Lemma 2.14. Clearly, $V(C) \nsubseteq S_1 \cup S_2$, because $|V(C)|$ is odd. Hence, there is some $v \in V(C) - S_1 \cup S_2$. Thus, there exists $v \in V(C)$ such that $G - v$ is a bipartite graph with a perfect matching of size $\alpha(G)$.

The converse is clear. □

Corollary 2.17. *A unicyclic graph G has two disjoint maximum independent sets if and only if, either G is a bipartite graph with a perfect matching, or there is a vertex v belonging to its unique odd cycle, such that $G - v$ has a perfect matching of size $\alpha(G)$.*

Proposition 2.18. *If $vw \in E(G)$, then $\alpha(G) = \max\{\alpha(G - v), \alpha(G - w)\}$.*

Proof. Clearly, $\alpha(G) - 1 \leq \alpha(G - u) \leq \alpha(G)$ for every $u \in V(G)$. Moreover, $\alpha(G) = \alpha(G - u)$ if and only if there is a maximum independent set S such that $u \notin S$. Since v and w are adjacent, then at most one of them belongs to all maximum independent sets of G. □

Since the matching number of a bipartite graph G can be computed in $O\left(n(G)^{\frac{5}{2}}\right)$ time [9], Propositions 2.16, 2.18, and Corollary 2.17 imply the following.

Corollary 2.19. *One can decide in polynomial time whether an almost bipartite (unicyclic) graph has two disjoint maximum independent sets.*

3 Conclusions

In the context of line graphs, to reveal two disjoint maximum independent sets means to find two disjoint maximum matchings in the original graphs.

Theorem 3.1 ([11]). *A bipartite graph G has two disjoint perfect matchings if and only if it has a partition of its vertex set comprising of a family of simple cycles, i.e., G has a 2-factor.*

A polynomial algorithm recognizing 2-factors exists due to the Tutte reduction [21] to the matching problem. Thus to decide whether a line graph of a bipartite graph with a 1-factor has two disjoint maximum independent sets is polynomially tractable. When a bipartite graph has no perfect matchings, the problem is still open.

Conjecture 3.2. *Given a bipartite graph, there is a polynomial algorithm deciding whether its line graph has two disjoint maximum independent sets.*

The same question (concerning the existence of a pair of disjoint maximum independent sets) may be asked about other classes of graphs. Recall that G is an *edge α-critical graph* if $\alpha(G - e) > \alpha(G)$, for every $e \in E(G)$. For instance, every odd cycle C_{2k+1} and its complement are edge α-critical graphs. Moreover, both C_{2k+1} and $\overline{C_{2k+1}}$ have two disjoint maximum independent sets.

Conjecture 3.3 ([10]). *If G is an edge α-critical graph without isolated vertices, then it has two disjoint maximum independent sets.*

The friendship graph F_q is a non-König-Egerváry graph with exactly $2^{\alpha(G)}$ maximum independent sets. Thus Theorem 2.12 motivates the following.

Problem 3.4. Characterize non-König-Egerváry graphs with $|\Omega(G)| = 2^{\alpha(G)}$.

A vertex v of a graph G is *simplicial* if the induced subgraph of G on the set $N[v]$ is a complete graph and this complete graph is called a simplex of G.

Proposition 3.5 ([22]). *If v is a simplicial vertex of G, then $N(v) \subseteq \mathrm{Shed}(G)$.*

A graph G is said to be *simplicial* if every vertex of G belongs to a simplex of G. By Proposition 3.5, if every simplex of a simplicial graph G contains two simplicial vertices at least, then $\mathrm{Shed}(G) = V(G)$. The converse is not necessarily true. For instance, C_5 has no simplicial vertex, while $\mathrm{Shed}(C_5) = V(C_5)$.

Problem 3.6. Characterize graphs with $\mathrm{Shed}(G) = V(G)$.

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