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On the achromatic number of the Cartesian product of two complete graphs

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Abstract

A vertex colouring $f\colon V(G) \to C$ of a graph G is complete if for any $c_1, c_2 \in C$ with $c_1 \neq c_2$ there are in G adjacent vertices v_1, v_2 such that $f(v_1) = c_1$ and $f(v_2) = c_2$. The achromatic number of G is the maximum number $\mathrm{achr}(G)$ of colours in a proper complete vertex colouring of G. Let $G_1 \square G_2$ denote the Cartesian product of graphs G_1 and G_2. In the paper $\mathrm{achr}(K_{r^2+r+1} \square K_q)$ is determined for an infinite number of q's provided that r is a finite projective plane order.

Keywords: Graph, complete vertex colouring, achromatic number, Cartesian product, finite projective plane.

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1 Introduction

Consider a finite simple graph G and a finite set of colours C. A vertex colouring $f\colon V(G) \to C$ is *complete* if for any pair c_1, c_2 of distinct colours in C one can find a pair v_1, v_2 of adjacent vertices in G such that $f(v_i) = c_i$, $i = 1, 2$. Obviously, if f is proper (adjacent vertices receive distinct colours) and $|C|$ is minimum possible (*i.e.*, $|C| = \chi(G)$, the chromatic number of G), then f is necessarily complete.

The *achromatic number* of G, in symbols $\mathrm{achr}(G)$, is the *maximum* number of colours in a proper complete vertex colouring of G. This invariant was introduced by Harary, Hedetniemi, and Prins in [6]. The problem of determining the achromatic number is NP-complete even for trees, see Cairnie and Edwards [2]. So, it is not surprising that exact results concerning the achromatic number are quite rare. A comprehensive bibliography for the achromatic number is maintained by Edwards [5].

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Some attention was paid to the achromatic number of graphs created by graph operations. Hell and Miller in [7] analysed $\operatorname{achr}(G_1 \times G_2)$ where $G_1 \times G_2$ is the categorical product of graphs G_1 and G_2 (in the paper we use the notation taken from the monograph Imrich and Klavžar [14]).

The Cartesian product $G_1 \square G_2$ of graphs G_1 and G_2 is the graph with $V(G_1 \square G_2) = V(G_1) \times V(G_2)$, in which (x_1, y_1) is joined by an edge to (x_2, y_2) if and only if either $x_1 = x_2$ and $\{y_1, y_2\} \in E(G_2)$ or $\{x_1, x_2\} \in E(G_1)$ and $y_1 = y_2$.

A motivation for the study of $\operatorname{achr}(K_p \square K_q)$ comes from the observation by Chiang and Fu [3] stating that the assumption $\operatorname{achr}(G_1) = p$ and $\operatorname{achr}(G_2) = q$ implies $\operatorname{achr}(G_1 \square G_2) \geq \operatorname{achr}(K_p \square K_q)$.

Evidently, since the graphs $K_p \square K_q$ and $K_q \square K_p$ are isomorphic to each other, when looking for $\operatorname{achr}(K_p \square K_q)$ we may suppose without loss of generality that $p \leq q$.

Let p, q be integers. In the paper we work with *integer intervals* that are denoted as follows:

$$[p, q] = \{z \in \mathbb{Z} : p \leq z \leq q\}, \qquad [p, \infty) = \{z \in \mathbb{Z} : p \leq z\}.$$

For a finite set A and $k \in [0, |A|]$ the set $\binom{A}{k}$ consists of all k-element subsets of A.

The element in the ith row and the jth column of a matrix M is presented as $(M)_{i,j}$. The submatrix of M corresponding to the ith row of M is denoted by $R_i(M)$.

The value of $\operatorname{achr}(K_p \square K_q)$ is known for each pair (p, q) satisfying $1 \leq p \leq 6$ and $p \leq q$. Besides the trivial case $p = 1$ ($K_1 \square K_q$ is isomorphic to K_q, hence $\operatorname{achr}(K_1 \square K_q) = q$), the case $p \in \{2, 3, 4\}$ was settled by Horňák and Puntigán [13] (for $p \in \{2, 3\}$ the result was rediscovered in [3]), the case $p = 5$ by Horňák and Pčola [11, 12], and the case $p = 6$ by Horňák [9, 8, 10]. The achromatic number of $K_p \square K_q$, where r is an odd order of a finite projective plane, was determined in Chiang and Fu [4] (for $r = 3$ see already Bouchet [1]). Some values of $\operatorname{achr}(K_p \square K_q)$ with $p \leq 6$ will be used in Section 4. They are summarised here:

Theorem 1.1. (1) *If* $q \in [3, \infty)$, *then* $\operatorname{achr}(K_2 \square K_q) = q + 1$.

(2) *If* $q \in [4, \infty)$, *then* $\operatorname{achr}(K_3 \square K_q) = \left\lfloor \frac{3q}{2} \right\rfloor$.

(3) *If* $q \in [25, \infty)$, *then* $\operatorname{achr}(K_4 \square K_q) = \left\lfloor \frac{5q}{3} \right\rfloor$.

(4) *If* $q \in [43, \infty)$, *then* $\operatorname{achr}(K_5 \square K_q) = \left\lfloor \frac{9q}{5} \right\rfloor$.

(5) *If* $q \in [41, \infty)$ *and* $q \equiv 1 \pmod 2$, *then* $\operatorname{achr}(K_6 \square K_q) = 2q + 3$.

(6) *If* $q \in [42, \infty)$ *and* $q \equiv 0 \pmod 2$, *then* $\operatorname{achr}(K_6 \square K_q) = 2q + 4$.

What follows is (up to the notation) a natural and somehow standard (*cf.* [13]) approach to dealing with a proper complete vertex colouring of the Cartesian product of two complete graphs.

Suppose that $p, q \in [1, \infty)$, $V(K_s) = [1, s]$ for $s \in \{p\} \cup \{q\}$, C is a finite set and $f \colon V(K_p \square K_q) \to C$ is a proper complete vertex colouring. Let $M(f)$ be the $p \times q$ matrix with $(M(f))_{i,j} = f(i, j)$. The fact that f is proper means that each row of $M(f)$ consists of q distinct colours of C, and similarly each column of $M(f)$ consists of p distinct colours of C. Because of the completeness of f for any $\{c_1, c_2\} \in \binom{C}{2}$ there is a line (a row or a column) of $M(f)$ that contains both c_1 and c_2. Let $\mathcal{M}(p, q, C)$ be the set of all $p \times q$ matrices M with elements from C such that M has all above properties of the matrix $M(f)$.

Conversely, let $M \in \mathcal{M}(p,q,C)$. It is obvious to see that $f_M\colon V(K_p \Box K_q) \to C$ defined by $f_M(i,j) = (M)_{i,j}$ is a proper complete vertex colouring of the graph $K_p \Box K_q$. Thus, we have just proved

Proposition 1.2. *If $p,q \in [1,\infty)$ and C is a finite set, then the following statements are equivalent:*

(1) *There is a proper complete vertex colouring of the graph $K_p \Box K_q$ using as colours elements of C.*

(2) $\mathcal{M}(p,q,C) \neq \emptyset$.

We shall need a subset $\mathcal{M}^*(p,q,C)$ of $\mathcal{M}(p,q,C)$ consisting of matrices M, which satisfy the additional condition that for any $\{c_1,c_2\} \in \binom{C}{2}$ there is a row (not merely a line) of M with both c_1, c_2.

Let $r \in [2,\infty)$. A *finite projective plane of order* r is a pair $(P,\mathcal{L})$, where P is a finite set of elements called *points*, and $\mathcal{L}$ is a set of subsets of P called *lines*, such that the following axioms are fulfilled:

A_1. If $p_1,p_2 \in P$, $p_1 \neq p_2$, there is exactly one line $L(p_1,p_2) \in \mathcal{L}$ such that $\{p_1,p_2\} \subseteq L(p_1,p_2)$.

A_2. If $L_1,L_2 \in \mathcal{L}$, $L_1 \neq L_2$, then $L_1 \cap L_2 \neq \emptyset$.

A_3. There are four distinct points $\tilde{p}_1,\tilde{p}_2,\tilde{p}_3,\tilde{p}_4 \in P$ determining six distinct lines, *i.e.,* $|\bigcup_{\{i,j\}\in\binom{[1,4]}{2}}\{L(\tilde{p}_i,\tilde{p}_j)\}| = 6$.

A_4. There is $\tilde{L} \in \mathcal{L}$ such that $|\tilde{L}| = r+1$.

It is well known that points and lines of a finite projective plane $(P,\mathcal{L})$ of order r have the following basic properties:

B_1. If $L_1,L_2 \in \mathcal{L}$, $L_1 \neq L_2$, then $|L_1 \cap L_2| = 1$.

B_2. If $L \in \mathcal{L}$, then $|L| = r+1$.

B_3. If $p \in P$, then $|\{L \in \mathcal{L} : p \in L\}| = r+1$.

B_4. $|P| = r^2+r+1$.

B_5. $|\mathcal{L}| = r^2+r+1$.

Given $r \in [2,\infty)$, to determine whether there exists a finite projective plane of order r (*i.e.*, whether r is a *finite projective plane order*), is in general a notoriously hard problem of finite combinatorics. All positive results available so far are restricted to $r = q^e$, where q is a prime number and $e \in [1,\infty)$.

2 Some auxiliary results

The following lemma is well known, *cf.* [13]. We include its proof here for a better readability of the paper.

Lemma 2.1. *If $p \in [1, \infty)$, $q \in [p, \infty)$, C is a set of size $a = \operatorname{achr}(K_p \square K_q)$, $M \in \mathcal{M}(p, q, C)$ and l is the smallest of frequencies of elements in M, then the following hold:*

(1) $l \leq p$.

(2) $l \leq \left\lfloor \frac{pq}{a} \right\rfloor$.

(3) $a \leq l(p + q - l - 1) + 1$.

Proof. (1) The vertex colouring f_M is proper, hence each element of C appears in any row of M at most once.

(2) The set of colour classes of f_M is a partition of the set $V(K_p \square K_q)$ of cardinality pq so that $pq \leq al$ and $l \leq \left\lfloor \frac{pq}{a} \right\rfloor$.

(3) Let $\gamma \in C$ be a colour of f_M of frequency l. A matrix created from M by permuting the rows and the columns of M evidently belongs to $\mathcal{M}(p, q, C)$. Therefore, we may suppose without loss of generality that $(M)_{i,i} = \gamma$ for every $i \in [1, l]$. The completeness of f_M means that the neighbourhood N of the colour class $\{(i, i) : i \in [1, l]\}$ corresponding to γ contains a vertex of each colour in $C \setminus \{\gamma\}$. Since $|N| = ql + (p - l)l - l \leq |C| - 1$, we have $a = |C| \leq l(p + q - l - 1) + 1$. □

Lemma 2.2. *If $p \in [3, \infty)$, $q \in [2p - 1, \infty)$, $a = \operatorname{achr}(K_p \square K_q)$, C is an a-element colour set, $\mathcal{M}^*(p, q, C) \neq \emptyset$, and $d \notin C$ for a colour d, then $\mathcal{M}^*(p, q + 1, C \cup \{d\}) \neq \emptyset$ and $\operatorname{achr}(K_p \square K_{q+1}) \geq \operatorname{achr}(K_p \square K_q) + 1$.*

Proof. For $M \in \mathcal{M}(p, q, C)$ we consider the block matrix $M^+ = (M J_p(d))$, in which $J_p(d)$ is the $p \times 1$ matrix with all elements equal to d.

For each $i \in [1, p]$ we define recurrently an $i \times (q + 1)$ matrix M_i^+. First, we put $M_1^+ = M^+$. For $i \in [2, p]$, if the matrix M_{i-1}^+ is already defined, we construct a matrix M_i^+ from the block matrix $M_i = \begin{pmatrix} M_{i-1}^+ \\ R_i(M^+) \end{pmatrix}$ by interchanging elements $(M_i)_{i,q+1} = d$ and $(M_i)_{i,j}$ for a suitable $j \in [1, q]$ in such a way that lines of M_i^+ contain pairwise distinct elements (so that $f_{M_i^+}$ is a proper vertex colouring of $K_i \square K_{q+1}$).

To see that this is doable realise that there are two reasons why an integer from $[1, q]$ cannot be chosen as j. The first one is that the jth column of M_{i-1}^+ contains d, and the second one is that $(M_i)_{i,j}$ is an element of the $(q + 1)$th column of M_{i-1}^+. Therefore, the total number of integers from $[1, q]$, which are not a valid choice for j, is $2(i - 1)$, and M_i^+ can be created in a required way, since $q - 2(i - 1) \geq q - 2(p - 1) = q + 2 - 2p \geq 1$.

Thus $f_{M_p^+}$ is a proper vertex colouring of $K_p \square K_{q+1}$. As $M \in \mathcal{M}^*(p, q, C)$, having in mind the construction of M_p^+ and the fact that the colour d is present in all p rows of the matrix M_p^+, it is clear that the colouring $f_{M_p^+}$ is complete, too. Therefore, $M_p^+ \in \mathcal{M}^*(p, q+1, C \cup \{d\}) \neq \emptyset$ and $\operatorname{achr}(K_p \square K_{q+1}) \geq |C \cup \{d\}| = a + 1 = \operatorname{achr}(K_p \square K_q) + 1$. □

Lemma 2.3. *If r is a finite projective plane order and $s \in [r+1, \infty)$, then*

(1) *there exists a colour set C of size $(r^2+r+1)s$ such that $\mathcal{M}^*(r^2+r+1, (r+1)s, C) \neq \emptyset$;*

(2) $\mathrm{achr}(K_{r^2+r+1} \Box K_{(r+1)s}) \geq (r^2+r+1)s$.

Proof. (1) Let $(P, \mathcal{L})$ be a finite projective plane of order r with $P = \{p_k : k \in [1, r^2+r+1]\}$ and $\mathcal{L} = \{L_k : k \in [1, r^2+r+1]\}$ (see the properties B_4, B_5). Consider an $(r^2+r+1) \times (r+1)$ matrix M with elements from P such that, for each $i \in [1, r^2+r+1]$, L_i is equal to the set $\{(M)_{i,j} : j \in [1, r+1]\}$ of elements in the ith row of M.

Given $k \in [1, r^2+r+1]$ and $l \in [1, r+1]$, replace the lth copy of p_k in M with p_k^l (by B_3, p_k appears in $r+1$ distinct lines of $\mathcal{L}$, and so in $r+1$ distinct rows of M); we suppose that the ordering of copies of p_k in M is "inherited" from the lexicographical ordering of pairs $(i,j) \in [1, r^2+r+1] \times [1, r+1]$ with $(M)_{i,j} = p_k$. Denote by M' the $(r^2+r+1) \times (r+1)$ matrix obtained from M if each point of P in M is replaced in the above way with p_k^l, where $(k,l) \in [1, r^2+r+1] \times [1, r+1]$.

For $z \in \mathbb{Z}$ let$(z)_s$ be the unique $t \in [1, s]$ satisfying $t \equiv z \pmod{s}$. Further, let M_k^s be the $(r+1) \times s$ matrix with elements from $\{p_k\} \times [1, s]$ defined by $(M_k^s)_{i,j} = (p_k, (i+j-1)_s)$, *i.e.*,

$$M_k^s = \begin{pmatrix} (p_k,1) & (p_k,2) & \dots & (p_k,s-1) & (p_k,s) \\ (p_k,2) & (p_k,3) & \dots & (p_k,s) & (p_k,1) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (p_k,r) & (p_k,r+1) & \dots & (p_k,r-2) & (p_k,r-1) \\ (p_k,r+1) & (p_k,r+2) & \dots & (p_k,r-1) & (p_k,r) \end{pmatrix}$$

Finally, let M_s be the $(r^2+r+1) \times (r+1)s$ matrix obtained from M' if each p_k^l with $(k,l) \in [1, r^2+r+1] \times [1, r+1]$ is replaced with the $1 \times s$ block matrix equal to the lth row submatrix of M_k^s.

Let us show that $M_s \in \mathcal{M}^*(r^2+r+1, (r+1)s, C)$ where the set of colours $C = \{(p_k, t) : k \in [1, r^2+r+1], t \in [1, s]\}$ is of size $(r^2+r+1)s$. First, the ith row of M_s, $i \in [1, r^2+r+1]$, consists of $(r+1)s$ distinct elements of C (corresponding to $r+1$ points of L_i, see B_2). Next, the assumption $s \geq r$ guarantees that each column of M_s consists of r^2+r+1 distinct elements of C (even if there is a column of M' containing, for some $k \in [1, r^2+r+1]$, all p_k^l with $l \in [1, r+1]$). So, M_s represents a proper vertex colouring.

If $k, l \in [1, r^2+r+1]$, $k \neq l$, by the axiom A_1 there is a unique $i \in [1, r^2+r+1]$ such that $\{p_k, p_l\} \subseteq L_i$. Therefore, for any $t, u \in [1, s]$, both (p_k, t) and (p_l, u) belong to the ith row of M_s. If $k \in [1, r^2+r+1]$ and $t, u \in [1, s]$, $t \neq u$, both (p_k, t) and (p_k, u) appear in $r+1$ rows of M_s (corresponding to $r+1$ rows of M containing p_k). Thus M_s represents a complete vertex colouring, too; moreover, $M_s \in \mathcal{M}^*(r^2+r+1, (r+1)s, C) \neq \emptyset$.

(2) Since $\mathcal{M}(r^2+r+1, (r+1)s, C) \supseteq \mathcal{M}^*(r^2+r+1, (r+1)s, C) \neq \emptyset$ (see Lemma 2.3(1)), using Proposition 1.2 we obtain $\mathrm{achr}(K_{r^2+r+1} \Box K_{(r+1)s}) \geq |C| = (r^2+r+1)s$. □

Note that the structure of the matrix M from the proof of Lemma 2.3 (that depends on the projective plane order r) is "largely" various, but it determines the structure of matrices

M' and M_s in a unique way. For example, in the case $r = 2$ and $s = 3$ we have

$$M = \begin{pmatrix} p_1 & p_2 & p_3 \\ p_3 & p_4 & p_5 \\ p_5 & p_6 & p_1 \\ p_4 & p_1 & p_7 \\ p_7 & p_3 & p_6 \\ p_5 & p_7 & p_2 \\ p_2 & p_6 & p_4 \end{pmatrix}, \quad M' = \begin{pmatrix} p_1^1 & p_2^1 & p_3^1 \\ p_3^2 & p_4^1 & p_5^1 \\ p_5^2 & p_6^1 & p_1^2 \\ p_4^2 & p_1^3 & p_7^1 \\ p_7^2 & p_3^3 & p_6^2 \\ p_5^3 & p_7^3 & p_2^2 \\ p_2^3 & p_6^3 & p_4^3 \end{pmatrix},$$

and then the matrix M_3 is

$$\begin{pmatrix} (p_1,1) & (p_1,2) & (p_1,3) & (p_2,1) & (p_2,2) & (p_2,3) & (p_3,1) & (p_3,2) & (p_3,3) \\ (p_3,2) & (p_3,3) & (p_3,1) & (p_4,1) & (p_4,2) & (p_4,3) & (p_5,1) & (p_5,2) & (p_5,3) \\ (p_5,2) & (p_5,3) & (p_5,1) & (p_6,1) & (p_6,2) & (p_6,3) & (p_1,2) & (p_1,3) & (p_1,1) \\ (p_4,2) & (p_4,3) & (p_4,1) & (p_1,3) & (p_1,1) & (p_1,2) & (p_7,1) & (p_7,2) & (p_7,3) \\ (p_7,2) & (p_7,3) & (p_7,1) & (p_3,3) & (p_3,1) & (p_3,2) & (p_6,2) & (p_6,3) & (p_6,1) \\ (p_5,3) & (p_5,1) & (p_5,2) & (p_7,3) & (p_7,1) & (p_7,2) & (p_2,2) & (p_2,3) & (p_2,1) \\ (p_2,3) & (p_2,1) & (p_2,2) & (p_6,3) & (p_6,1) & (p_6,2) & (p_4,3) & (p_4,1) & (p_4,2) \end{pmatrix}.$$

3 Main theorem

Theorem 3.1. *If r is a finite projective plane order, $s \in [r^3+1, \infty)$ and $t \in [0, r]$, then*

$$(r^2+r+1)s + t \le \operatorname{achr}(K_{r^2+r+1} \square K_{(r+1)s+t}) \le (r^2+r+1)s + rt.$$

Proof. Denote $a_{s,t} = \operatorname{achr}(K_{r^2+r+1} \square K_{(r+1)s+t})$ for $t \in [0, r]$. Since $s \ge r^3 + 1 \ge r + 1$, from Lemma 2.3 we know there is a colour set $C^*_{s,0}$ of size $(r^2+r+1)s$ such that $\mathcal{M}^*(r^2+r+1, (r+1)s, C^*_{s,0}) \neq \emptyset$ and

$$a_{s,0} \ge (r^2+r+1)s. \tag{3.1}$$

If $t \in [0, r-1]$, it is an easy exercise to prove the inequality $(r+1)s+t \ge 2(r^2+r+1)-1$. Therefore, using (3.1) and Lemma 2.2, by induction on t we see that for any $t \in [0, r]$ there exists a colour set $C^*_{s,t}$ of size $(r^2+r+1)s+t$ with $\mathcal{M}^*(r^2+r+1, (r+1)s+t, C^*_{s,t}) \neq \emptyset$, which implies

$$a_{s,t} \ge (r^2+r+1)s + t. \tag{3.2}$$

Now let $C_{s,t}$ be an $a_{s,t}$-element set. By Proposition 1.2 there exists a matrix $M_{s,t} \in \mathcal{M}(r^2+r+1, (r+1)s+t, C_{s,t})$. If $l_{s,t}$ is the minimum frequency of an element of $C_{s,t}$ in $M_{s,t}$, from Lemma 2.1(2) and (3.2) it follows that

$$\begin{aligned} l_{s,t} &\le \left\lfloor \frac{(r^2+r+1)[(r+1)s+t]}{a_{s,t}} \right\rfloor \le \left\lfloor \frac{(r^2+r+1)[(r+1)s+t]}{(r^2+r+1)s} \right\rfloor \\ &= r+1. \end{aligned} \tag{3.3}$$

In the case $l_{s,t} = r+1$ each colour class of the colouring $f_{M_{s,t}}$ is of cardinality at least $r+1$, hence, by (3.2),

$$|V(K_{r^2+r+1} \square K_{(r+1)s+t})| = (r^2+r+1)[(r+1)s+t] \ge (r+1)a_{s,t},$$

so that

$$a_{s,t} \le (r^2+r+1)s + \left\lfloor \frac{(r^2+r+1)t}{r+1} \right\rfloor = (r^2+r+1)s + rt. \tag{3.4}$$

On the other hand, if $l_{s,t} \le r$ (see (3.3)), Lemma 2.1.3 yields

$$\begin{aligned} a_{s,t} &\le l_{s,t}[r^2+r+1+(r+1)s+t-l_{s,t}-1]+1 \\ &\le r[r^2+r+1+(r+1)s+t-r-1]+1 \\ &= r[r^2+(r+1)s+t]+1 \end{aligned} \tag{3.5}$$

(since $r \ge 2$, the polynomial $x[r^2+r+1+(r+1)s+t-x-1]+1$ in variable x is increasing for $x \le r$), and from (3.5) we obtain

$$a_{s,t} \le r^3 + r(r+1)s + rt + 1 \le s + r(r+1)s + rt = (r^2+r+1)s + rt. \tag{3.6}$$

From (3.4) and (3.6) we see that

$$a_{s,t} \le (r^2+r+1)s + rt \tag{3.7}$$

independently from the value of $l_{s,t}$, and so, by (3.2) and (3.7), for any $t \in [0,r]$ (including $t = k$) we have $(r^2+r+1)s + t \le a_{s,t} \le (r^2+r+1)s + rt$. □

Corollary 3.2. *If r is a finite projective plane order and $s \in [r^3+1, \infty)$, then*

$$\operatorname{achr}(K_{r^2+r+1} \square K_{(r+1)s}) = (r^2+r+1)s.$$

Proof. Take $k = 0$ in Theorem 3.1. □

4 Asymptotic analysis

Theorem 4.1. *If r is a finite projective plane order, then*

$$\lim_{q\to\infty} \frac{\operatorname{achr}(K_{r^2+r+1} \square K_q)}{q} = \frac{r^2+r+1}{r+1}.$$

Proof. Denote $a(p,q) = \operatorname{achr}(K_p \square K_q)$. We have $a(p,q) = a\left(p, (r+1)\left\lfloor \frac{q}{r+1} \right\rfloor) + k\right)$, where $k = q - (r+1)\left\lfloor \frac{q}{r+1} \right\rfloor \in [0,r]$. If $s = \left\lfloor \frac{q}{r+1} \right\rfloor \ge r^3+1$, then, by Theorem 3.1,

$$\begin{aligned} \frac{(r^2+r+1)\left\lfloor \frac{q}{r+1} \right\rfloor}{q} &\le \frac{(r^2+r+1)\left\lfloor \frac{q}{r+1} \right\rfloor + k}{q} \\ &\le \frac{a\left(r^2+r+1, (r+1)\left\lfloor \frac{q}{r+1} \right\rfloor + k\right)}{q} \\ &= \frac{a(r^2+r+1,q)}{q} \le \frac{(r^2+r+1)\left\lfloor \frac{q}{r+1} \right\rfloor + rk}{q} \\ &\le \frac{(r^2+r+1)\left\lfloor \frac{q}{r+1} \right\rfloor + r^2}{q}. \end{aligned} \tag{4.1}$$

Now, having in mind that $\lim_{q\to\infty} \frac{\left\lfloor \frac{q}{r+1} \right\rfloor}{q} = \frac{1}{r+1}$ and $\lim_{q\to\infty} \frac{r^2}{q} = 0$, from (4.1) we obtain $\lim_{q\to\infty} \frac{a(r^2+r+1,q)}{q} = \frac{r^2+r+1}{r+1}$. □

From the known results for $\mathrm{achr}(K_p \square K_q)$ with $p \leq 6$, see [13] ($p \leq 4$), [11] ($p = 5$) and [8, 10] ($p = 6$), we can easily deduce the existence and the value of the limit $l_p = \lim_{q\to\infty} \frac{\mathrm{achr}(K_p \square K_q)}{q}$. Namely, we have $l_1 = 1 = l_2$, $l_3 = \frac{3}{2}$, $l_4 = \frac{5}{3}$, $l_5 = \frac{9}{5}$ and $l_6 = 2$. These facts together with Theorem 4.1 motivate us to formulate

Conjecture 4.2. *If* $p \in [1, \infty)$, *then* $\lim_{q\to\infty} \frac{\mathrm{achr}(K_p \square K_q)}{q}$ *does exist and is a rational number.*

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