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Chromatic uniqueness of zero-divisor graphs*

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Abstract

The *zero-divisor graph* $\Pi(R)$ of a commutative ring R is the graph whose vertices are the elements of R such that the vertices u and v are adjacent if and only if $uv = 0$. If the graphs G and H have the same chromatic polynomial, then we say that they are *chromatically equivalent (or $\chi-$equivalent)*, written as $G \sim H$. Suppose a graph is uniquely determined by its chromatic polynomial. Then it is said to be *chromatically unique (or χ-unique)*.

In this paper, we discuss the question: For which numbers n is the graph $\Pi(Z_n)$ χ-unique?

While Z_n is one of the simplest rings, we proved that for any graph A_0, for some n, $\Pi(Z_n)$ contains an induced subgraph isomorphic to A_0. The first result in the subject states that for $n \geq 10$ even, $\Pi(Z_n)$ is not χ-unique (Gehet, Khalaf). By definition, n is *square-free* if it is prime or the product of different prime numbers. Our main result is the following. If $n \geq 10$ is neither square-free nor the square of a prime then it is not χ-unique. Here and in our preceding work, we use a common method.

For odd square-free non-prime n, the problem is open, though on the structure of $\Pi(Z_n)$ we know much in this case.

Keywords: Zero-divisor graph, chromatic equivalence, chromatic uniqueness.

Math. Subj. Class.: 05C15, 05C25, 05C31

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1 Introduction

Definitions and notation used here can be found in the Preliminaries.

Getting much more information on a structure by constructing an object which is seemingly too far from the original one is a rare but very useful method in mathematics. Such are group representations, the application of finite planes in coding theory and cryptography, etc. In the construction of the zero-divisor graph for a ring, we find the same approach.

Beck first created the idea of a zero-divisor graph for a commutative ring in 1988 [5]. Anderson and Livingston [3] introduced the first simplification of Beck's graph in 1999. Further modifications appeared due to Mulay [17]. This history shows that several definitions of the zero-divisor graph can be useful, according to the specific goals. (For example, in our case, it was needed to omit loops).

The notion of chromatic polynomials is much older. In 1932, Whitney [22] defined the chromatic polynomial for a graph made from a map and found many essential results. The motivation was the Four Color Conjecture, among others. In 1968, Read [21] raised novel interest in the study of chromatic polynomials. Among others, he asked two questions.

1. "Is it possible to find a set of necessary and sufficient algebraic conditions for a polynomial to be the chromatic polynomial of some graph?"
2. "What is the necessary and sufficient condition for two graphs to be chromatically equivalent; that is, to have the same chromatic polynomial?"

The last question is already close to our topics, the χ-uniqueness of graphs.

In 2013, Gehet and Khalaf [10] proved some results on the chromatic uniqueness of the graph $\Pi(Z_n)$, for the cases when n is even, $n = p^2$ or $n = p_1 \times p_2$.

Concerning non-commutative rings, Khalaf et al. investigated the right zero- divisor graph of a ring on the 2×2 matrices over the ring Z_3. They also presented an algorithm to find vertices and edges of certain right zero-divisor graphs [12].

Our goal is to analyze the zero-divisor graph $\Pi := \Pi(Z_n)$ of the ring Z_n and to ask whether Π is χ-unique. We developed a method, by constructing the so-called t-clique join graphs which can help us to prove the non-χ-uniqueness for some graphs. We applied the same method already in [2], studying a graph with a chromatic polynomial having only integer roots.

There we have given a complete answer. In the present problem, the odd square-free non-prime natural numbers are the only values for which the question of χ-uniqueness is open.

We deal also with two additional subjects, the contraction of $\Pi(Z_n)$ and its surprising properties for square-free n.

2 Preliminaries

All graphs used in this paper are simple undirected graphs, $G = (V, E)$ with $V = V(G)$ means the vertex set of G and $E = E(G)$ means the edge set of G. The degree of vertex v ($v \in V(G)$) is the number of edges incident to v, it is denoted by $d(v)$. The complete (empty) graph on n vertices is denoted by K_n ($\overline{K_n}$). $G[V']$ is the subgraph of G induced V' throughout. Let C be a vertex set in a graph $G = (V, E)$, $G - C$ means the graph $G[V - C]$. Two graphs G and H are isomorphic if there is a bijective function $f\colon V(G) \to V(H)$ such that the vertices $f(u)$ and $f(v)$ are adjacent in H iff the vertices u and v are adjacent in G. The notation is $G \cong H$. We denote the complement of graph G by $\overline{G}$.

For a graph G, a mapping $f\colon V(G) \to \{1, 2, \ldots, \lambda\}$, $\lambda \in \mathbb{N}$, is called a λ-coloring of G if $f(u) \neq f(v)$, whenever u and v are adjacent. It can be proved that the number of λ-colorings of a graph G is a polynomial of λ. It is called the *chromatic polynomial* of G and denoted by $P(G, \lambda)$. If $P(G, \lambda) = P(H, \lambda)$, then the graphs G and H are *chromatically equivalent* (or $\chi-$equivalent), written as $G \sim H$.
Let $p(\lambda)$ and $q(\lambda)$ be two polynomials. $p(\lambda) \mid q(\lambda)$ (That is, $q(\lambda)$ can be divided by $p(\lambda)$) if there exists a polynomial $r(\lambda)$ with $q(\lambda) = r(\lambda)p(\lambda)$. The following notion is one of the most important concepts in this work.

Definition 2.1. Given a graph G, suppose that for every H, $G \sim H$ implies that H is isomorphic to G. In this case, G is *chromatically unique (or χ-unique).*

A polynomial is called an *integral-root polynomial* if all its roots are integers. A graph G is an *integral-root graph* if $P(G, \lambda)$ is an integral-root polynomial. The chromatic polynomial $\lambda(\lambda - 1) \ldots (\lambda - n + 1)$ of a complete graph on n vertices is denoted by $p_n(\lambda)$. Similarly, it is a well-known fact that $P(\overline{K_n}, \lambda) = \lambda^n$.

We emphasize that here a clique means an arbitrary complete subgraph; it may not be maximal. The number of vertices in a maximum clique of G is said to be the clique number of G and denoted by $\omega(G)$. $C \subseteq V(G)$ is called a cutset in a connected graph G if $G - C$ is disconnected. A clique cutset is a special cutset, it induces a clique in the graph. The clique cutsets are used in this paper frequently, and they play an important role in many graph theory subjects, for example, in [4]. An independent set (coclique) is a set of vertices such that there is no edge connecting any two vertices in it. $K_{p,q}$ is a graph whose vertex set can be divided into two independent sets X and Y, with $|X| = p$ and $|Y| = q$, such that any vertex in X and any vertex in Y are connected. G is *chordal* if no induced cycle of length at least 4 exists in G [8]. We give here two non-standard definitions. The *union* $G \cup H$ of G and H is the graph with $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$. The *join* $G + H$ of G and H is the graph with $V(G + H) = V(G) \cup V(H)$ and $E(G + H) = E(G) \cup E(H) \cup \{xy : x \in V(G), y \in V(H)\}$ (meaning that no loops in the new graph) [8]. Often these are defined for disjoint vertex sets only. The *line graph* $L(G)$ of the graph G is the graph on $E(G)$ in which $e, f \in E(G)$ are adjacent as vertices if and only if they are adjacent as edges in G [13]. $L = L(\mathcal{H})$ is the *intersection graph* ('line graph') of $\mathcal{H}$, that is, $L = (E(\mathcal{H}), F)$ where the subsets $Y, Z \subseteq W$ yield an edge YZ in L if and only if they intersect [18]. For a set W, its *power set* is the set of all subsets of W. For a hypergraph $\mathcal{F}$, the hypergraph $\mathcal{B}$ is its *partial hypergraph* if it consists of some hyperedges of $\mathcal{F}$. Let $S_1, S_2, ...S_l$ be a partition of the vertex set in an arbitrary graph G. The *contraction* of these sets yields a graph with vertices $s_1, s_2, ...s_l$ such that s_i, s_j are adjacent if and only if there exists at least one edge between S_i and S_j.

Definition 2.2. For a natural number n, a *totative* t of n is any natural number coprime to n (the greatest common divisor of t and n is equal to 1).

Definition 2.3 ([5])**.** The *zero-divisor graph* $\Pi(R)$ of a commutative ring R is the graph whose vertices are the elements of R such that the vertices u and v are adjacent if and only if $uv = 0$.

Remark 2.4. Later we shall investigate the chromatic polynomial for the zero-divisor graph of Z_n; thus loops are not allowed (graphs with loops are even non-colorable).

Notation 2.5. For a natural number n, the ring Z_n is the ring constructed from the congruence classes modulo n. It is interpreted in numerous forms, by the set $\{0, 1, \ldots, n-1\}$, for example. Congruence modulo n is equivalent to be equal in Z_n. We shall work with a fixed number $n = p_1^{r_1} \times p_2^{r_2} \times p_3^{r_3} \times \cdots \times p_\alpha^{r_\alpha}$ and the corresponding ring throughout.

Definition 2.6. We say that n is *square-free* if it is prime or the product of different prime numbers, that is, $n = p_1 \times p_2 \times \cdots \times p_\alpha$.

3 Auxiliary results

Fact 3.1 (obvious)**.** *The complete graph K_m and the empty graph $\overline{K_m}$ are χ-unique.*

Theorem 3.2 ([10])**.** *For a graph G, if $u \in V(G)$ such that $d(u) = 1$ and the graph $G - u$ has two vertices a and b such that $d(a) = d(b)$, then G is not χ-unique.*

Corollary 3.3 (folklore)**.** *Let G be a chordal graph of order m, then the chromatic polynomial of G is:*

$$P(G, \lambda) = \lambda^{r_0}(\lambda - 1)^{r_1}(\lambda - 2)^{r_2} \cdots (\lambda - s)^{r_s},$$

where $s, r_j \in \mathbb{N}$, $\forall\ 1 \geq j \geq s$ and $\sum_{j=0}^{s} r_j = m$ such that $s = \chi(G) - 1$.

Theorem 3.4 ([2])**.** *If the chromatic polynomial of an integral-root graph G has some roots of multiplicity at least 3, then G is not χ-unique.*

Corollary 3.5 ([9])**.** *Let G be an arbitrary graph with $K_m \subseteq G$. If a graph F is obtained from G by adding a new vertex w which is adjacent to all vertices of K_m (and no others), then its chromatic polynomial is:*

$$P(F, \lambda) = (\lambda - m)P(G, \lambda).$$

Corollary 3.6 ([2])**.** *Let $G = K_m + \overline{K_s}$. Then the following statements are valid:*

(i) $P(G, \lambda) = p_m(\lambda)(\lambda - m)^s$.

(ii) *For $s \geq 3$, the graph G is not χ-unique.*

Definition 3.7 ([2])**.** Let K_m be a complete graph with $W = V(K_m)$, let I_k be the subset of W, $k \leq t$ and $t \in N$. The *t-clique join graph* is obtained from t arbitrary graphs $R_1, R_2, \ldots, R_t$ on pairwise disjoint vertex sets by joining every vertex in R_k with all vertices of I_k, where $R_k \cap W = \phi$. It is denoted by $J = J(W, I_1, I_2, \ldots, I_t, R_1, R_2, \ldots, R_t)$ (see Figure 1). $\mathfrak{J} = \mathfrak{J}(W, j_1, j_2, \ldots, j_t, R_1, R_2, \ldots, R_t)$ is the set of all such graphs with $|I_1| = j_1$, $|I_2| = j_2$, ..., $|I_t| = j_t$.

Remark 3.8. I_1, I_2, ... are not always disjoint from one another.

Theorem 3.9 ([10])**.** *Let $G := \Pi(Z_n)$ be the zero-divisor graph of the ring Z_n, then the following are satisfied:*

1. *G is χ-unique for all $n \leq 9$.*
2. *If $n \geq 10$ is an even number, then G is not χ-unique.*

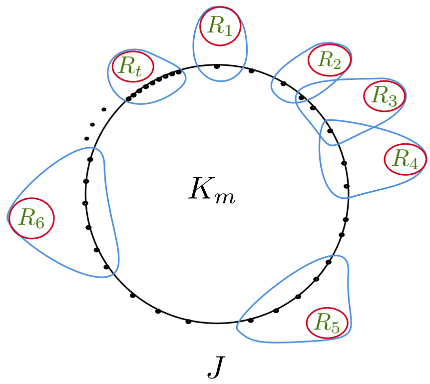

Figure 1: A t-clique join graph.

3. *If $n = p^2$ such that p is prime, then G is χ-unique.*
4. *If $n = pq$ such that $p \neq q$ are primes for all p, q with $q \geq p \geq 3$, then G is χ-unique.*

Definition 3.10 ([11]). The number of totatives of n is *Euler's totient function* $\varphi(n)$.

Lemma 3.11 ([11]). *The following equality is satisfied:*

$$\sum_{d|n} \varphi(d) = n.$$

Remark 3.12. Here d runs over all the divisors of n, namely, 1 and n are involved as well.

Remark 3.13. This statement is well-known but nontrivial and can be used in the proof of Theorem 4.14 below.

4 Preparation for the main result

In this section, we shall discuss the chromatic uniqueness of the zero-divisor graph $\Pi(Z_n)$ of the ring Z_n, using the $t-$clique join graphs as a tool. This tool will help us to answer the question of uniqueness, except for the so-called square-free natural numbers if they are odd and non-prime. Before discussing the uniqueness, we will improve the presentation of a zero-divisor graph.

4.1 Application of the $t-$clique join graphs

Proposition 4.1. *Let $\omega(G) \geq m$ for a graph G, then $p_m(\lambda)|P(G, \lambda)$.*

Proof. For such a graph, $\chi(G) \geq m$. Consequently, every $k < m$ is a root of $P(G, \lambda)$ and thus $(\lambda - k)|P(G, \lambda)$. Using the properties of polynomials over any field, $p_m(\lambda) = \prod_0^{m-1}(\lambda - k)$ divides $P(G, \lambda)$. □

Corollary 4.2. *For an arbitrary graph G with $K_m \subseteq G$. If a graph F is obtained from G by adding a new graph $\overline{K_s}$ which is adjacent to all vertices of K_m and no others, then the chromatic polynomial of F is:*

$$P(F, \lambda) = (\lambda - m)^s P(G, \lambda),$$

Especially, if $I_j \subset K_m$ and the new graph $\overline{K_s}$ is adjacent to all vertices of I_j and no others $(I_j + \overline{K_s})$, then the chromatic polynomial of F is:

$$P(F, \lambda) = (\lambda - j)^s P(G, \lambda),$$

Proof. The proof works directly by applying Corollaries 3.6, 3.5, and Proposition 4.1. □

Lemma 4.3. *Let J be a 1-clique join graph, $J = J(W, I, R)$, with $W = V(K_m)$, $|R| \geq 2$ and $|I| < m$ i. e, $V(I)$ is a proper subset of W. The empty graph R is adjacent to all vertices of I (and no others), where $R \cap W = \phi$. If for a graph G, $V(G) \cap V(J) = W$, then for $F = G \cup J$, F is not χ-unique.*

Proof. Since $F = G \cup J$, $|R| = s \geq 2$, and $|I| = j \leq m - 1$. We will choose a graph F' that satisfies:

(i) it has the same chromatic polynomial,

(ii) it is not isomorphic to F.

To prove (i), Let us choose $F' = G \cup J'$, such that J' is a 2-clique join graph with $J' = J(W, {I_1}', {I_2}', {R_1}', {R_2}')$, $\left|{I_1}'\right| = j_1$, $\left|{I_2}'\right| = j_2$, with $j_1 = j_2 = j - 1$, but $|V({I_1}') \cap V({I_2}')| = j - 2$, $\left|{R_1}'\right| = s_1$, and $\left|{R_2}'\right| = s_2$, i. e., $s_1 + s_2 = s$. By Corollary 4.2, the two chromatic polynomials are equal to:

$$\begin{aligned} P(F, \lambda) &= (\lambda - (m-1))^s P(G, \lambda), \\ P(F', \lambda) &= (\lambda - (m-1))^{s_1} (\lambda - (m-1))^{s_2} P(G, \lambda) \\ &= (\lambda - (m-1))^s P(G, \lambda), \end{aligned}$$

Clearly we have

Proposition 4.4. *F and F' are not isomorphic.*

Proof. Easy. □

By Proposition 4.4, we achieved the proof of Lemma 4.3. □

Corollary 4.5. *Let J be a t-clique join graph, $J \in \mathfrak{J}(W, j_1, j_2, \ldots, j_t, R_1, R_2, \ldots, R_t)$, with $W = V(K_m)$, and $R_1, R_2, \ldots, R_t$ are empty subgraphs on pairwise disjoint vertex sets such that every vertex in R_k is adjacent to all vertices of I_k (and no others), where $R_k \cap W = \phi$ for any k, and $|R_k| = s_k$. If for a graph G, $V(G) \cap V(J) = W$, then for $F = G \cup J$ and $t \geq 2$:*

$$P(F, \lambda) = (\lambda - j_1)^{s_1} (\lambda - j_2)^{s_2} \cdots (\lambda - j_t)^{s_t} P(G, \lambda),$$

Proof. The proof works directly by repeatedly applying Corollary 4.2. □

Lemma 4.6. *Let J be a t-clique join graph, $J \in \mathfrak{J}(W, j_1, j_2, \ldots, j_t, R_1, R_2, \ldots, R_t)$, with $W = V(K_m)$, and $R_1, R_2, \ldots, R_t$ are empty subgraphs on pairwise disjoint vertex sets such that every vertex in R_k is adjacent to all vertices of I_k (and no others), where $R_k \cap W = \phi$ for any k. Let the graph G have the following property: $V(G) \cap V(J)$ induces a clique K_m.*

For $F = G \cup J$ and $t \geq 2$, F is not χ-unique.

Proof. **A** First, we will give a proof for $t = 2$. Let us consider J above. Our aim is to construct a new 2-clique join graph $J' := J(W, I_1{}', I_2{}', R_1{}', R_2{}')$ in such a way that $F' := G \cup J'$, has the following properties:

(i) It has the same chromatic polynomial.

(ii) It is not isomorphic to F.

Without loss of generality, we may suppose $s_1 \geq s_2$ where $|R_i| = s_i, i = 1, 2$. Take some $x \in I_1$ and $y \in I_2$. Let $I_2{}' := I_2 - y \cup \{x\}$. We let $I_1{}' := I_1$. Thus, for x, the number of its neighbors in $R := R_1 \cup R_2$ (the R-degree) will be $s_1 + s_2$ and for y, this number becomes 0. By Corollary 4.5, the chromatic polynomials of these graphs are equal to:

$$P(F, \lambda) = P(F', \lambda) = (\lambda - j_1)^{s_1}(\lambda - j_2)^{s_2} P(G, \lambda),$$

Using the facts on the R-degrees, it can be easily seen that F and F' are not isomorphic.

B Now we make the construction for general t. We denote $|R_i|$ by s_i $(i = 1, 2, ..., t)$. First we choose two subscripts.

$$s_K := \max\{s_i \mid i = 1, ...t\}$$

and

$$s_L := \max\{s_i \mid i \neq K\}$$

Remark 4.7. In case of a fork, we select subscripts K and L arbitrarily.

Let us take some $x \in I_K$ and some $y \in I_L$. We change I_L only, namely the new subset will be $I_L - x \cup \{y\}$. Thus, for x, its $R - degree$ (the number of its neighbors in $R := \cup_{i=1}^{t} R_i$) will change to $s_K + s_L$, while for y, this number will change to 0.

From Corollary 4.5, the chromatic polynomial of the new graph F' will be tha same as of F.

By the choice of the new I's, the R-degree of x will be strictly larger than the other R-degrees. From this, it can be easily seen that F and F' are not isomorphic. □

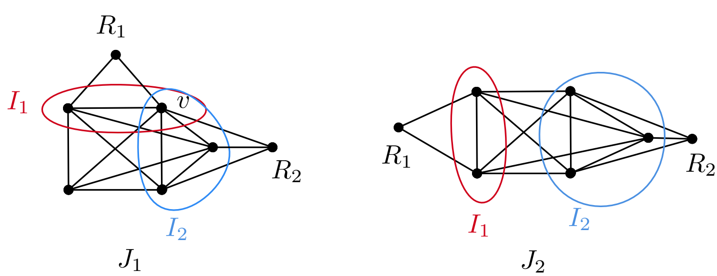

Figure 2: J_1 and J_2 are χ-equivalent.

4.2 The principal equivalence relation $\simeq$

We shall introduce an equivalence relation $\simeq$ on the base set V of any commutative ring R for obtaining a clear picture of its zero-divisor graph. First, we give a general definition of this relation in any simple graph.

Definition 4.8. In a simple graph G, for $x, y \in V(G)$, $x \simeq y$ if for any *third* vertex $z \in R - \{x, y\}$ in the graph, $xz \in E(G) \Leftrightarrow yz \in E(G)$ (in the literature such vertices are called *twins*).

Remark 4.9. It is easy to prove the transitivity of $\simeq$.

Now, if we consider the (most interesting) special case when $G = \Pi(R)$, we obtain

Definition 4.10. For $x, y \in R$, $x \simeq y$ if for any *third* element $z \in R - \{x, y\}$ in the ring, $xz = 0 \Leftrightarrow yz = 0$.

Remark 4.11. As we mentioned in the Preliminaries, in our case, loops are not allowed in the zero-divisor graph. It is the reason for excluding $z = x$ and $z = y$.

Definition 4.12. For $x \in R$, the equivalence class of x in $\simeq$ will be called $Sub(x)$. The name of a subset of the form $Sub(x)$ will be a *class* or a *partial subset*.

From now on, we will use the special ring, Z_n ($n \geq 2$).

Notation 4.13. Let k be an arbitrary proper divisor of n. We will denote by $U(k)$ for the auxiliary subset of Z_n, $U(k) := \{\lambda k | 1 \leq \lambda \leq n/k, (\lambda, n/k) = 1\}$.

The following statement is folklore and thus we do not present its proof.

Theorem 4.14. *The equivalence classes for the relation $\simeq$ are the sets $U(k)$ where k runs over all proper divisors of n (in other words, $Sub(k) = U(k)$), furthermore, the one-element class $\{0\}$.*

Remark 4.15. From a practical point of view, it is useful to keep the zero element of the ring in the zero-divisor graph and make it adjacent to everything, in spite of the fact that zero-divisors are defined to be non-zero.

For $k = 1$, we obtain the set of all totatives of n. It is exactly the set of non-zero-divisors of Z_n, different from zero.

In fact, the following two simple statements are very useful concerning the structure of our zero-divisor graph. Note that they are valid for any graph.

Proposition 4.16. *For a graph $G = (V, E)$, let S and T be two different classes with respect to $\simeq$, suppose we find an edge $e = st$ such that $s \in S$, and $t \in T$. Then any vertex in S and any vertex in T are adjacent. Clearly, in contrast, if we find a pair of vertices such that $s \in S$, $t \in T$ and st is not an edge, then any vertex in S and any vertex in T are non-adjacent.*

For any simple graph and every class S, S is a clique or an independent set.

Proof. (A) For all $y \in T$, $sy \in E$ since $y \simeq t$, For all $x \in S$, $xt \in E$ since $x \simeq s$, For all $x \in T$, $y \in T$, $xy \in E$ since $x \simeq t$.

(B) Suppose there are $s \in S$, $t \in T$, such that $st \notin E$. We will prove by way of contradiction that for any x, y, such that $x \in S$, and $y \in T$. If $xy \in E$, then we could apply (A) and $st \in E$ would be proved, a contradiction.

(C) If there is an edge in T, then T is a clique. The reason is: suppose for $t_1, t_2 \in T$, $t_1 t_2 \in E$. For every $x \in T - \{t_1, t_2\}$, $xt_1 \in E$ since $t_2 \simeq x$. Similarly, $xt_1 \in E$.
Take an arbitrary $y \in T - \{t_1, t_2\}$. $xy \in E$ since $t_1 \simeq y$.

(D) Suppose $\exists\, t_1, t_2 \in T$, $t_1 t_2 \notin E$. Then we may use (C), similarly as above. □

Remark 4.17. We will apply Proposition 4.16 for the zero-divisor graph of Z_n.

Definition 4.18. Let S_1 and S_2 be two different partial subsets, such that any vertex in S_1, and any vertex in S_2 are adjacent; we say that S_1 is adjacent to S_2.

Now we consider the extension of Theorem 4.14 for arbitrary elements x in Z_n.

Theorem 4.19. *Let x be a general element of Z_n and $k := (x, n)$. Then $Sub(x) = Sub(k)$.*

Proof. Let $\lambda := x/k$. We state that $(\lambda, n/k) = 1$. By way of contradiction, suppose there exists $d > 1$ such that $d|\lambda$ and $d|n/k$. Then dk is a common divisor of x and n, thus $(x, n) > k$, a contradiction. Consequently, $x \in U(k) = Sub(k)$ □

Remark 4.20. [7] and in [20] introduce an equivalence relation such that x and y are equivalent if "they are associates" (see in the papers). By Lauve [14] this relation can be identified with $\simeq$ in our definition.

Representation of the zero-divisor graph of the ring Z_n We will denote by $\Pi(Z_n)$ the zero-divisor graph of the ring Z_n. We represent it most frequently in the following:
For every k, the class $Sub(k)$ will be drawn like a circle of $\varphi(n/k)$ vertices, as shown in Figure 3. This circle is not a subgraph but a figure only. It gives a clearer picture of the graph, making the understanding easier for the reader.

4.3 Using the classes

In this section, we will use the partial subsets frequently.

Claim 4.21. $k^2 \equiv 0 \mod n$, *iff* $k = p_1^{g_1} \times p_2^{g_2} \times p_3^{g_3} \times \cdots \times p_\alpha^{g_\alpha}$, $g_j \geq \lceil \frac{r_j}{2} \rceil$, $1 \leq j \leq \alpha$.

Proof. Easy, applying the statements in Subsection 4.2. □

Lemma 4.22. (i) $k^2 = 0$ *in the ring* $\Leftrightarrow Sub(k)$ *is a clique.*

(ii) $k^2 \neq 0 \Leftrightarrow Sub(k)$ *is an empty induced subgraph.*

Proof. (i) First we will prove ($\Rightarrow$).
It is very easy. $(\lambda_1 k)(\lambda_2 k) = \lambda_1 \lambda_2 k^2$.

(ii) ($\Leftarrow$).
Suppose by way of contradiction that $n \not| k^2$ and S is a clique. Then, taking any $v \in S - k$, $vk = 0$. Moreover, $\exists\, \lambda$ such that $v = \lambda k$ and $(\lambda_1, \frac{n}{k}) = 1$. We state that in this case:
$- p_j|\lambda$ and
$- p_j|\frac{n}{k}$, a contradiction.

Because of $n \not| k^2$, $\exists j$ such that, ($g_j < \lceil \frac{r_j}{2} \rceil$ $(*)$), and $n|\lambda k^2$, thus the exponent of λk^2 in p_j has to be larger, consequently, $p_j|\lambda$ indeed.
From $(*)$, $2g_j < r_j \Rightarrow g_j < r_j \Rightarrow p_j|\frac{n}{k}$. □

Remark 4.23. It is not specified in our figures whether a given class induces a clique or an independent set in $\Pi(Z_n)$.

Remark 4.24. Let us suppose n_1 and n_2 have the same numbers of primes and the same values of powers. Then the two corresponding graphs on them have the same number of classes, only the sizes of these classes are differ.
In addition, the way of drawing (picture) of $\Pi(Z_{n_1})$ and $\Pi(Z_{n_2})$ is the same.

Figure 3 represents the way of drawing of $\Pi(Z_n)$, where n has any value of the form $n = p_1 \times p_2 \times p_3$.

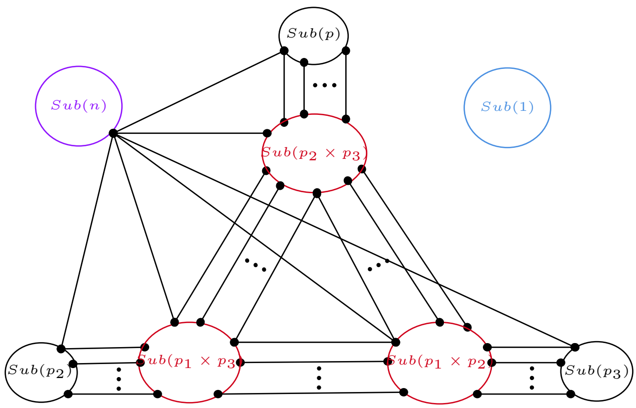

Figure 3: Illustration of Remark 4.24.
$\Pi(Z_n), n = p_1 \times p_2 \times p_3$.

Remark 4.25. From now on, in our results and figures, we will not indicate the two classes $Sub(n)$ and $Sub(1)$, where $Sub(n) = \{0\}$ is adjacent to all vertices of $\Pi(Z_n)$, and $Sub(1)$ is the set of all totatives of n (isolated vertices).

Before presenting the next statement, we need an auxiliary object. The following terminology is so important that we mention it again.

Remark 4.26. For a square-free natural number n, the zero-divisor graph of Z_n is in thorough connection with the so-called line graphs of hypergraphs. (See Section 7). For information on hypergraphs, we propose [6].

Observation 4.27. (obvious) Let $x = p_1^{t_1} \times p_2^{t_2} \times p_3^{t_3} \times \cdots \times p_\alpha^{t_\alpha}$. Then $x = 0$ in Z_n if and only if $t_i \geq r_i$, for every i.

Corollary 4.28. *If $n = p^r$, and the classes are $S_1 := Sub(p)$, $S_2 := Sub(p^2), \cdots$, $S_{r-1} := Sub(p^{r-1})$, then the consequences below are satisfied:*

(i) *If r is even, then the clique number is:*

$$\omega(\Pi) = \sum |S_i|, \quad \left\lceil \frac{r}{2} \right\rceil \leq i \leq r-1$$

(ii) *If r is odd, then the clique number is:*

$$\omega(\Pi) = 1 + \sum |S_j|, \quad \left\lceil \frac{r}{2} \right\rceil \leq j \leq r-1.$$

Proof. Since $S_1, S_2, \cdots, S_{x-1}$ are independent classes, and $S_x, S_{x+1}, \cdots, S_{r-1}$ are clique classes, where $x = \lceil \frac{r}{2} \rceil$.
First, we prove the lower bounds.
In both cases, all clique classes are adjacent (from Observation 4.27).
For case (i), this is enough for the proof of the lower bound.
In case (ii), S_{x-1} is an independent class, and it is adjacent to all classes S_j, $x \leq j \leq r-1$, thus one vertex can be taken from this class.

The upper bounds are valid as well since the union of independent classes is independent, and thus, any clique may contain at most one vertex from there. □

Notation 4.29. The union of all the clique classes will be denoted by cl_Π.

Proposition 4.30. *The clique classes are pairwise adjacent, and cl_Π is a clique.*

Proof. Let S_1 and S_2 be two clique classes have the key vertices k_1 and k_2, respectively. By Lemma 4.22, $k_i = p_1^{g_1} \times p_2^{g_2} \times p_3^{g_3} \times \cdots \times p_\alpha^{g_\alpha}$, $i = 1, 2$; $g_j \geq \lceil \frac{r_j}{2} \rceil$, $1 \leq g_j \leq r_j$, $1 \leq j \leq \alpha$. Since $k_1 \times k_2$ can be divided by n, then there are edges between the vertices of S_1 and the vertices of S_2. □

Lemma 4.31. *If n is a prime power p^r, then the graph $\Pi(Z_n)$ is chordal*

Proof. We can prove it by way of contradiction, assume that the graph $\Pi(Z_n)$ is non-chordal, which means there is at least one induced cycle of order q, $q \geq 4$.
Let $S_i = Sub(p^i)$, $1 \leq i \leq r-1$, be the classes of $\Pi(Z_{p^r})$. So, by Lemma 4.22, S_i is the independent class, where $1 \leq i \leq \lfloor \frac{r-1}{2} \rfloor$. Also, S_i is a clique class, where $\lfloor \frac{r-1}{2} \rfloor + 1 \leq i \leq r-1$. In this graph $\Pi(Z_{p^r})$, if S_i is adjacent to S_j ($p^i \cdot p^j \equiv 0 \mod n$), $1 \leq i \leq \lfloor \frac{r-1}{2} \rfloor$, $\lfloor \frac{r-1}{2} \rfloor + 1 \leq j \leq r-1$, then S_i is adjacent to all classes S_t such that $j < t \leq r-1$, which means it is impossible to get an induced cycle of order $q \geq 4$. □

Definition 4.32. Let L be a class and suppose that, omitting the zero class $Sub(n)$, L is adjacent to exactly one class, say B. Then L is called a *leaf class*. The class B is said to be a *branch class*.

Lemma 4.33. *Let Z_n the leaf classes are $L_i := Sub(p_i)$ for $i = 1, 2, ..., \alpha$. The branch classes are $T_i := Sub(n/p_i)$, for $1 \leq i \leq \alpha$.*

Moreover, for one branch class there exist no further leaf classes adjacent to it. Particularly, if $n = p^r$, $r \geq 3$. Then there is one leaf class which is $Sub(p)$, and one branch class, $Sub(p^{r-1})$. Clearly, if $r = 2$, then there is no leaf class.

Proof. We state that there is exactly one class, namely T_i, which is adjacent to $Sub(p_i)$. We prove this statement for $i = 1$ only.
The adjacency of the two classes is obvious. Suppose there is another class, say $Sub(k)$, adjacent to $Sub(p_1)$. Let $k = p_1^{g_1} \times p_2^{g_2} \cdots \times p_\alpha^{g_\alpha}$. $g_j \leq r_j$ for every j. $p_1 k$ is divisible by n, thus for each $j \geq 2$, $g_j = r_j$. $k < n$. Consequently, $g_1 < r_1$. We use again that $p_1 k$ is divisible by n. It implies $g_1 = r_1 - 1$ and $Sub(k) = T_1$. We have shown that the $Sub(p_i)$'s are leaf classes, and the T_i's are branch classes.
Assume that there is another leaf class, say S^*, that differs from $Sub(p_1)$, $Sub(p_2)$, $\cdots$ and $Sub(p_\alpha)$. Let k^* be the key vertex of S^*. There are two cases.

First, k^* can be divided by the product of at least two primes, say $p_i \times p_j$, so the class S^* is adjacent to both T_i and T_j as we have mentioned above. Thus it is not a leaf, a contradiction.

Second, $k^* = {p_i}^s$ with $s \geq 2$. Then S^* is adjacent both to T_i and to $Sub(n/{p_i}^2)$.

We have proved that no more leaf class. Moreover, it is an easy consequence of the facts above that no more branch class, the T_i's only. □

Corollary 4.34. *Let us consider* $\Pi(Z_n)$, $n \neq p, p^2$. *Then every leaf class is an independent class, and every branch class of the form* $T_i = Sub(\frac{n}{p_i}) = Sub\{p_1^{r_1} \times p_2^{r_2} \times \cdots \times p_i^{r_i - 1} \times \cdots \times p_\alpha^{r_\alpha}\}$, $p_i^{r_i - 1} > 1$, *is a clique class.*

Proof. The proof works directly by applying Lemma 4.22. □

The validity of the following statement can be easily seen.

Observation 4.35. If n is square-free, then every class of $\Pi(Z_n)$ is an independent class.

Lemma 4.36. *Let us consider* $n = p_1^r \times p_2 \times \cdots \times p_\alpha$, $2 \leq r \leq 3$. *Then there is a clique* cl_Π *such that:*

$$|V(cl_\Pi)| = |U_r| + 1,$$

where U_r *is the only clique class in* $\Pi(Z_n)$.

Proof. If $r = 2$, then $U_2 := Sub(p_1 \times p_2 \times p_3 \times \cdots \times p_{\alpha-1} \times p_\alpha)$ is the only clique class in $\Pi(Z_n)$. Since the independent class $Sub(p_1^2)$ is adjacent to U_2. So, we select one vertex v from $Sub(p_1^2)$ to get a clique cl_Π which is larger than U_2 with vertex set $V(cl_\Pi) = V(U_2) \cup \{v\}$, and $|V(cl_\Pi)| = |U_2| + 1$.
The proof for $r = 3$ is the same as for $r = 2$, using $U_3 := Sub(p_1^2 \times p_2 \times p_3 \times \cdots \times p_{\alpha-1} \times p_\alpha)$, and $Sub(p_1^3)$. □

5 The main result

In this section, we will discuss the χ-uniqueness of the zero-divisor graph Z_n.

Theorem 5.1. *Let us consider* $\Pi = \Pi(Z_n)$, $n \geq 10$. *If* n *is none of the following types: odd square-free, or of the form* p^2, *then* Π *is not* χ*-unique.*

Proof. (i) If $n = p^r$, $r \geq 3$ (Π is a complete graph where $r = 2$, then it is χ-unique), then Π is a chordal graph as shown in Corollary 4.28 and Lemma 4.31. So, the chromatic polynomial of Π is an integral-root chromatic polynomial, as shown in Corollary 3.3. In a graph Π, a leaf class $S_1 := Sub(p)$ is always only adjacent to a branch class $S_{r-1} :=$

$Sub(p^{r-1})$, where S_{r-1} is a clique class for all $r \geq 3$.
The number of vertices in S_1 is equal to $\varphi(\frac{n}{p}) = \varphi(p^{r-1}) = p^{r-2}(p-1) = s$, and the number of vertices in S_{r-1} is equal to $\varphi(\frac{n}{p^{r-1}}) = \varphi(p) = p-1$. Thus, there are $(\lambda-(p-1))$ colors to colour the vertices of S_1 (s vertices). Let c be the number of choices for coloring the vertices of S_1. For any coloring of S_1, it can be extended in the same number of ways, thus $c|P$, which means $(\lambda-(p-1))^s|P(\Pi,\lambda)$. So, $p-1$ is a root of $P(\Pi,\lambda)$ with multiplicity s. Since the multiplicity of a root $p-1$ is not less than 3, which satisfies Theorem 3.4, then Π is not χ-unique.

(ii) Let $n = p_1^r \times p_2 \times \cdots \times p_\alpha$, $2 \leq r \leq 3$ and $\alpha \geq 2$. We will prove the case $r = 2$ only, i.e., $n = p_1^2 \times p_2 \times \cdots \times p_\alpha$, $\alpha \geq 2$. In this case, $U_2 := Sub(p_1 \times p_2 \times p_3 \times \cdots \times p_{\alpha-1} \times p_\alpha)$ is the only clique class (Lemma 4.33). So, there is an induced clique cl_Π (as shown in Lemma 4.36) which is:

$$|V(cl_\Pi)| = |U_2| + 1,$$

that is, $|U_2| < |(cl_\Pi)|$. Since $Sub(p_1^2)$ is adjacent to U_2, we can choose v from $Sub(p_1^2)$ such that $cl_\Pi := U_2 + K_1$, where $V(K_1) = \{v\}$.

Since the leaf class $Sub(p_1)$ is only adjacent to U_2 (Lemma 4.33), the induced subgraph $cl_\Pi \cup Sub(p_1)$ is a 1-clique join graph. Namely, Π is isomorphic to the graph F in Lemma 4.3, such that the induced subgraph $J = J(W, I, R) \cong cl_\Pi \cup Sub(p_1)$. Moreover, $I \cong U_2$, $R \cong Sub(p_1)$, and $K_m \cong cl_\Pi$, where $W = V(K_m)$. Finally, if G' is the graph obtained by the union of cl_Π and all classes of Π except the leaf class $Sub(p_1)$). G' does not contain any induced subgraph isomorphic to J. The reason— there is only one clique class U_2 in Π (using Lemma 4.36), and the leaf class $Sub(p_1)$ adjacent to U_2, that is, any other leaf class in G' will be adjacent to some independent class. So, we may apply Lemma 4.3 for $G \cong G'$. Consequently, Π is not χ-unique, see Example 5.3 and Figure 4.

The proof for the other case (when $r = 3$, i. e. $n = p_1^3 \times p_2 \times \cdots \times p_\alpha$) is similar, using $U_3 := Sub(p_1^2 \times p_2 \times p_3 \times \cdots \times p_{\alpha-1} \times p_\alpha)$, $Sub(p_1^3)$, and $Sub(p_1)$.

(iii) We are not in any of the two cases above.

Here we intensively use Defintion 4.32 and Lemma 4.33. We will choose those branch classes that are cliques and we define the set of subscripts

$$Q := \{q \mid T_q \text{ is both a branch class and a clique}\}$$

Such a T_q we call a *clique branch class*. Clearly,

Claim 5.2. *There exists at least one clique branch class in* $\Pi(Z_n) \Leftrightarrow n$ *is not square-free.*

$$L_q := \text{the leaf class adjacent to the clique branch class } T_q$$

Such an L_q we call a *special leaf class*.

Let ζ be the number of clique branch classes.

The branch classes of the form $T_q(q \in Q)$ are pairwise adjacent by Proposition 4.30. Let

$$\mathcal{S} := \cup_{q\in Q} L_q)$$

So, the induced subgraph $J' = cl_\Pi \cup \mathcal{S}$ is a t-clique join graph, where $t = \zeta$. That is, Π is isomorphic to the graph F in Lemma 4.6, such that $J \cong J'$, $J \in \mathfrak{J}(W, j_1, j_2, \ldots, j_t,$

$R_1, R_2, \ldots, R_t$), $t \geq 2$, and $K_m \cong cl_\Pi$, where $W = V(K_m)$. On the other hand, the I_j's are isomorphic to the clique branch classes $T_q, Q \in Q$. Moreover, the R_j's are the classes $L_q, Q \in Q$. Finally, we call G' the graph induced by $\Pi - \mathcal{S}$. Consequently, we may apply Lemma 4.6 and Π is not χ-unique . (See Example 5.4 and Figure 5.) □

Example 5.3. We have the graph $\Pi(Z_{45})$, where $45 = 3^2 \times 5$. The classes of this graph are $= Sub(3)$, $Sub(3^2)$, $Sub(5)$ and $U_2 =: Sub(3 \times 5)$. $\Pi(Z_{45})$ is not isomorphic to F' in Figure 4, but the chromatic polynomials of them are equal using Maple program,

$$\begin{aligned} P(\Pi(Z_{45}), \lambda) = P(F', \lambda) := \lambda(\lambda-1)(\lambda-2)^9(&\lambda^9 - 30\lambda^8 + 432\lambda^7 - 3814\lambda^6 \\ &+ 22380\lambda^5 - 89459\lambda^4 + 241466\lambda^3 - 421599\lambda^2 \\ &+ 430032\lambda - 194615). \end{aligned}$$

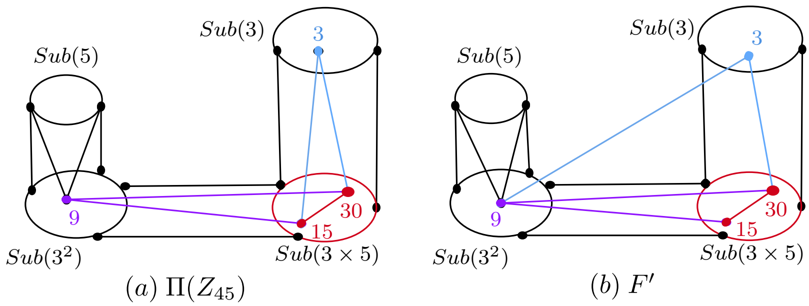

Figure 4: $\Pi(Z_{45})$ and F' are $\chi-$equivalent but non-isomorphic graphs.

Example 5.4. We have the graph $\Pi(Z_{72})$, where $72 = 2^3 \times 3^2$. The classes of this graph are $Sub(2)'$, $Sub(3)'$, $Sub(2^2)$, $Sub(2^3)$, $Sub(3^2)$, $Sub(2 \times 3)$, $Sub(2^2 \times 3)$, $Sub(2 \times 3^2)$ ${T_1}' := Sub(2^3 \times 3)$, and ${T_2}' := Sub(2^2 \times 3^2)$. $\Pi(Z_{72})$ is not isomorphic to F' in Figure 5, but the chromatic polynomials of them are equal using Maple program.

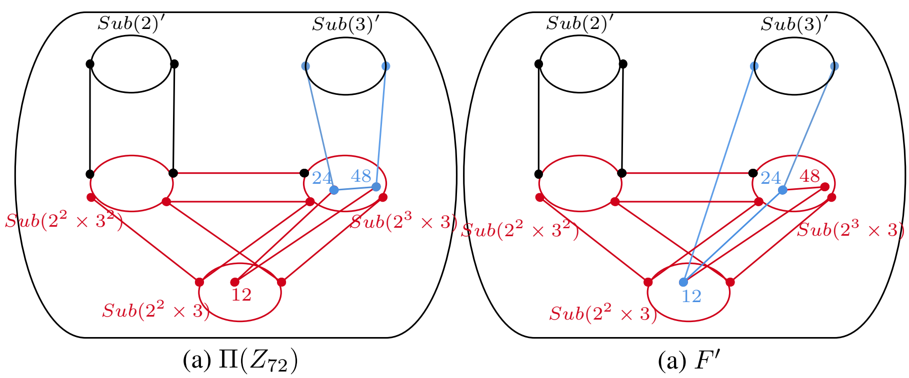

Figure 5: $\Pi(Z_{72})$ and F' are $\chi-$equivalent but non-isomorphic graphs.

6 On the contraction graph

In this section we will introduce a new definition based on the zero-divisor graph.

Definition 6.1. Let $\Pi'(Z_n)$ be obtained from a zero-divisor graph $\Pi(Z_n)$ omitting $Sub(n)$ and $Sub(1)$ with the classes $Sub(k_i)$, where k_i runs over all the divisors of n. The *contraction graph*, denoted by $\mathfrak{C}_n$, can be defined in two ways. First, using the general notion of contraction (see Preliminaries), namely as the contraction of the equivalence classes in $\Pi'(Z_n)$. Second, through the divisors k_i, such that they are the vertices and their adjacency relation is the same as in $\Pi'(Z_n)$.

Remark 6.2. Some authors allow the contraction of two adjacent vertices only. Here no such restriction is made.

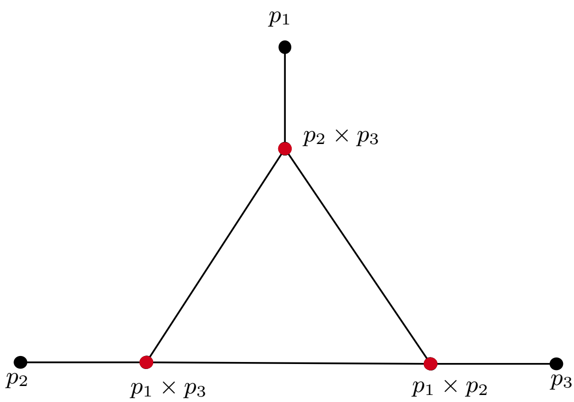

Figure 6: A contraction graph.

Definition 6.3. The key vertices $p_1, p_2, \cdots, p_\alpha$ are the leaf vertices in $\mathfrak{C}_n$. The branch vertices n/p_i (the keys of the branch classes) are denoted by t_i.

Lemma 6.4. *For the contraction graph, the subgraph induced by the branch vertices* $\{t_1, t_2, \cdots, t_\alpha\}$ *is a clique* $K(\Pi')$ *in* $\mathfrak{C}_n$.

Proof. Similar as for Π earlier. □

Corollary 6.5. *If* $n \geq 10$ *is neither of the form* p^2*, nor* $n = p_1 \times p_2$*, then the contraction graph* $\mathfrak{C}_n$ *is not* χ*-unique.*

Proof. We will prove that $\mathfrak{C}_n$ is isomorphic to graph F in Lemma 4.6.
Let $T_1 = \{t_1\}, T_2 = \{t_2\}, \cdots, T_\alpha = \{t_\alpha\}$ be subgraphs with one vertex in $\mathfrak{C}_n$. Let $P_1 = \{p_1\}, P_2 = \{p_2\}, \cdots, P_\alpha = \{p_\alpha\}$ be empty subgraphs which have pairwise disjoint vertex sets such that the leaf vertex in P_i is adjacent to the vertex t_i from $K(\Pi')$ (and no others).

We may apply Lemma 4.6. Namely, for J in the Lemma, we may take J', where $J' = K(\Pi') \cup P_1 \cup P_2 \cup \cdots \cup P_\alpha$. J' is a t-clique join graph with $t = \alpha$, $J' \in \mathfrak{J}(W, j_1, j_2, \ldots, j_t, R_1, R_2, \ldots, R_t)$, where $W \cong K(\Pi')$. Moreover, $I_k \cong T_k$, $j_k = |T_k|$, $1 \leq k \leq t$,, and $R_k \cong P_k$. In addition, the vertices of $\mathfrak{C}_n$ except the leaf vertices form the induced subgraph G'.

So, by Lemma 4.6 $\mathfrak{C}_n$ is not χ-unique. □

7 Connections of a square-free modulus with hypergraphs

In the Abstract we promised the proof of the statement which is a good argument for the complexity of $\Pi(Z_n)$. The reader can find it here.

Let us consider a square-free non-prime n with $n := p_1 p_2 ... p_\alpha$ and the zero-divisor graph $\Pi(Z_n)$.

Theorem 7.1. *For a given n with the properties above, let us consider an arbitrary graph A of order q where*

$(*)\ \ q \leq \frac{1+\sqrt{1+8\alpha}}{2} \Rightarrow \binom{q}{2} \leq \alpha$

Then $\Pi(Z_n)$ contains A as an induced subgraph. (More exactly, it contains an induced subgraph isomorphic to A.)

Proof. Let $W := \{1, 2, ..., \alpha\}$ and let $\mathcal{F}$ be the power set of W. First we give a sketch of the proof.

We begin by representing the graph $B := \overline{A}$ as the intersection graph of a hypergraph $\mathcal{H}$ (Proposition 7.3).

We continue so that we consider the subgraph F induced by the set of all divisors of n and we observe that F is isomorphic to the complement of $L(\mathcal{F})$.

We accomplish the proof, by using these results.

Remark 7.2. In fact, we use the contraction graph also here.

Proposition 7.3. *For any graph B, we have some hypergraph $\mathcal{H}$ such that $B \cong L(\mathcal{H})$*

Proof. First, we consider isolate-free graphs. Let $\mathcal{H} = \mathcal{H}_B$ be the hypergraph with $E(B)$ as a vertex set and let $H_v := \{e = vw \in E(B)\}$ be a hyperedge for every $v \in V(B)$. Clearly, $B \cong L(\mathcal{H})$. If B has isolates, we may add disjoint hyperedges to the structure. □

As in the Preliminaries, the intersection graph ('line graph') of a hypergraph $\mathcal{F}$ is denoted by $L(\mathcal{F})$.

We state that

Theorem 7.4. *The subgraph F of $\Pi(Z_n)$ induced by all the divisors of n can be expressed as the complement of $L(\mathcal{F})$.*

Proof. For $I, J \subseteq \{1, 2, ..., \alpha\}$, let $d = \prod_{i \in I} p_i$ and $\delta = \prod_{j \in J} p_j$, that is, two divisors of n. Obviously, $d\delta = 0$ in $Z_n \Leftrightarrow I \cup J = X$. Let us take the complements, we obtain $d\delta = 0$ in $Z_n \Leftrightarrow \overline{I} \cap \overline{J} = \emptyset$. □

The following statement is clearly valid.

Claim 7.5. *For any hypergraph $\mathcal{H}$ on at most α vertices, $\mathcal{F}$ contains a partial hypergraph $\mathcal{H}'$ isomorphic to $\mathcal{H}$.*

Now we are in the position to achieve the proof of Theorem 7.1. Take an arbitrary graph A with $q \leq \frac{1+\sqrt{1+8\alpha}}{2}$ vertices. Applying Proposition 7.3 for $B := \overline{A}$, we obtain a hypergraph $\mathcal{H}$ on $\binom{q}{2}$ vertices with $L(\mathcal{H}) \cong B$, that is, $\overline{L(\mathcal{H})} \cong A$.

From the condition $(*)$ of Theorem 7.1, $\binom{q}{2} \leq \alpha$ can be deduced. Thus $\mathcal{H}$ has at most α vertices and, by Claim 7.5, $\mathcal{F}$ contains a partial hypergraph $\mathcal{H}'$ isomorphic to $\mathcal{H}$. But this implies that F contains an induced subgraph isomorphic to A, moreover $\Pi(Z_n)$ contains F as an induced subgraph. □

8 Concluding remarks

We did not solve everything in Theorem 5.1:

Open problem: Let n be an odd square-free number. Is $\Pi := \Pi(Z_n)$ $\chi-$unique or not?

Furthermore, the research could be extended.

Question: What is the situation with rings similar to Z_n? Clearly, we mean here finite commutative rings with some zero-divisors.

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