ISSN 2590-9770
The Art of Discrete and Applied Mathematics 6 (2023) #P1.04
https://doi.org/10.26493/2590-9770.1537.cf9
(Also available at http://adam-journal.eu)

Self-dual polyhedra of given degree sequence

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Received 29 March 2022, accepted 3 September 2022, published online 27 September 2022

Abstract

Given vertex valencies admissible for a self-dual polyhedral graph, we describe an algorithm to explicitly construct such a polyhedron. Inputting in the algorithm permutations of the degree sequence can give rise to non-isomorphic graphs.

As an application, we find as a function of $n \geq 3$ the minimal number of vertices for a self-dual polyhedron with at least one vertex of degree i for each $3 \leq i \leq n$, and construct such polyhedra. Moreover, we find a construction for non-self-dual polyhedral graphs of minimal order with at least one vertex of degree i and at least one i-gonal face for each $3 \leq i \leq n$.

Another application is to rigidity theory, since the constructed families of polyhedra are generic circuits, and globally rigid in the plane.

Keywords: Algorithm, planar graph, degree sequence, polyhedron, self-dual, quadrangulation, radial graph, valency, rigidity.

Math. Subj. Class.: 05C85, 05C07, 05C35, 05C10, 52B05, 52B10, 52C25

1 Introduction

1.1 Results

This paper is about topological properties of polyhedra, namely the number of edges incident to a given vertex (degrees or valencies of vertices), and the number of faces adjacent to a given face ('degrees' or valencies of faces).

The 1-skeleton of a polyhedron is a planar, 3-connected graph – the Rademacher-Steinitz Theorem. These graphs are embeddable in a sphere in a unique way (an observation due to Whitney). We will call them polyhedral graphs, or polyhedra for short. The dual graph of a polyhedron is a polyhedron. Vertex and face valencies swap in the dual.

*The author was supported by Swiss National Science Foundation project 200021_184927. Many thanks to an anonymous referee for useful comments on a previous version.

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We call a polyhedron self-dual if it is isomorphic to its dual. A self-dual polyhedron on p vertices has p faces and $2p-2$ edges (straightforward consequence of Euler's formula).

In [10] we considered the problem of minimising the number of vertices of a polyhedron containing at least one vertex of valency i, for each $3 \leq i \leq n$. We established, among other results, that the minimal order (i.e. number of vertices) for such graphs is

$$\left\lceil \frac{n^2 - 11n + 62}{4} \right\rceil, \qquad n \geq 14.$$

The dual problem, that has therefore the same answer, is about imposing instead that there is at least one i-gonal face, for each $3 \leq i \leq n$. In this paper, we assume both conditions.

Definition 1.1. We say that a polyhedron G has the *property* $\mathcal{S}_n$ if it comprises at least one vertex of degree i for every $3 \leq i \leq n$, *and* at least one i-gonal face for every $3 \leq i \leq n$.

Our first consideration is that if we ask instead for the minimal number of *faces*, and assume we have such a graph G, then its dual G^* also satisfies $\mathcal{S}_n$, and has minimal vertices. Thereby, the answer to both questions must be the same. Moreover, it is natural to also seek self-dual solutions.

Proposition 1.2. *Let $n \geq 3$ and G be a polyhedral graph satisfying $\mathcal{S}_n$. Then the minimal number of vertices of G is*

$$\frac{n^2 - 5n + 14}{2}, \qquad \forall n \geq 3. \tag{1.1}$$

Moreover, Algorithm 2.1 *constructs for each $n \geq 6$ a non-self-dual polyhedron H_n of order* (1.1) *satisfying $\mathcal{S}_n$, whereas* Algorithm 3.2 *constructs for each $n \geq 3$ a self-dual polyhedron G_n of order* (1.1) *satisfying $\mathcal{S}_n$. The speed of the said algorithms is quadratic in n, i.e., linear in the graph order.*

Proposition 1.2 will be proven in section 2. The construction of the self-dual solutions is a special case of the following more general result, to be proven in section 3.

Theorem 1.3. *Let $k \geq 0$ and*

$$t_1, t_2, \ldots, t_k, 3^m \tag{1.2}$$

be given [1]*, where the integers t_i are not necessarily distinct, each $t_i \geq 4$, and*

$$m = 4 + \sum_{i=1}^{k} (t_i - 4). \tag{1.3}$$

Then Algorithm 3.2 *constructs a self-dual polyhedral graph of degree sequence* (1.2)*. Inputting in* Algorithm 3.2 *a permutation of the t_i produces, in general, non-isomorphic graphs. The speed of the algorithm is linear in the graph order.*

Remark 1.4. For fixed $t_1, \ldots, t_k$, we need equality (1.3) to hold in order for (1.2) to be the degree sequence of a self-dual polyhedron. Indeed, we have

$$\sum_{i=1}^{k} t_i + 3m = 2q = 2(2p-2) = 4(k+m) - 4$$

[1] The notation 3^m indicates that the number 3 is repeated m times.

by the handshaking lemma and self-duality. Algorithm 3.2 thereby constructs a self-dual polyhedral graph for any given admissible degree sequence.

Our work also has applications to rigidity theory. A *generic circuit* is a graph on at least four vertices satisfying $q = 2p - 2$ and for every proper subset X of the vertices,

$$|X| \geq 2 \Rightarrow i(X) \leq 2|X| - 3,$$

where $i(X)$ is the number of edges induced by X. A *bar-joint framework* is a pair (G, z) with G a graph and $z = \{z_1, \dots, z_p\}$ a configuration of its p vertices in d-dimensional space (graph embedding), where edges correspond to straight line segments. The framework (G, z) is *rigid* if $\exists \epsilon > 0$ s.t. if z' is another configuration where edges have the same length as in z and satisfying $||z_i - z'_i|| < \epsilon$ for all $1 \leq i \leq p$, then in z, z' each corresponding pair of points is at same distance. Intuitively, if we think of vertices as joints and edges as bars, then the framework is rigid if it has no non-trivial continuous deformations. If z is linearly independent over the rationals, the configuration is called *generic*. If z is generic, rigidity depends only on G [16], and accordingly one calls G rigid in $\mathbb{R}^d$ if (G, z) is rigid for every generic configuration in $\mathbb{R}^d$. A graph is *globally rigid in* $\mathbb{R}^d$ if every generic configuration of the points in d-dimensional space with given edge lengths also uniquely determines distances between each pair of points. For a general account on rigidity theory see e.g. [12].

Among the generic circuits, the 3-connected have a simple constructive characterisation – Connelly's Conjecture, proved by Berg-Jordán [3]. Hendrickson [8] proved that if a graph (on at least four vertices) is globally rigid in the plane, then it is 3-connected. Moreover, if $q = 2p - 2$, then it is a generic circuit (via Laman's Theorem) [3, section 6]. The more difficult direction to prove is, a 3-connected generic circuit is globally rigid in $\mathbb{R}^2$. This is the special case $q = 2p - 2$ of Hendrickson's Conjecture in dimension 2, that was established in [5, 9]. Our work constructs families of generic circuits that are 3-connected (and planar, and satisfy $\mathcal{S}_n$). These are thus globally rigid in the plane.

Corollary 1.5. *The families of polyhedra constructed by* Algorithms 2.1 *and* 3.2 *are generic circuits, and globally rigid in* $\mathbb{R}^2$.

Corollary 1.5 will be proven in section 2.3.

1.2 Discussion and related work

Proposition 1.2 solves a natural modification of the questions investigated in [10], as mentioned in section 1.1. The method is to establish a lower bound on the minimal order of graphs satisfying the property $\mathcal{S}_n$, and then to explicitly construct, for each n, solutions of such order via an algorithm. Here the expression for the minimal order (1.1) is cleaner, and the constructions more straightforward than in [10]. Proposition 1.2 will be proven in section 2. The self-dual construction of Proposition 1.2 is an application of Theorem 1.3.

Our main result Theorem 1.3 is about constructing self-dual polyhedra for any admissible degree sequence. The notions of duality and self-duality have been investigated since antiquity, with the Platonic solids. However, it was only relatively recently that the cornerstone achievement of generating all self-dual polyhedra was carried out [1]. This was done by constructing all their *radial graphs*, to be defined in section 3. Indeed, there is a one-to-one correspondence between self-dual polyhedra and their radial graphs.

Their radial graphs are certain 3-connected quadrangulations of the sphere (i.e. polyhedra where all faces are cycles of length 4), namely, those with no *separating* 4-*cycles* (i.e. all 4-cycles are faces). Self-duals and these quadrangulations are thereby intimately related (there is a caveat, a 3-connected quadrangulation of this type is not necessarily the radial of a *self-dual* polyhedron). Now, the generation of all quadrangulations of the sphere is another cornerstone result in graph theory [2, 4]. Equipped with this knowledge, we will prove Theorem 1.3 (section 3.2).

Self-dual polyhedra are a special case of *self-dual maps* on the sphere – see e.g. [13, 14, 15].

Notation. We will usually denote vertex and edge sets of a graph G by $V(G)$ and $E(G)$, and their cardinality by $p = |V(G)|$ (order) and $q = |E(G)|$ (size). We will work with simple graphs (no loops or multiple edges).
For $p \geq 4$, we call W_p the $p-1$-gonal pyramid (or wheel graph), of p *vertices*.
Let $\mathcal{P}$ be an operation on a graph G, that modifies a given subgraph H of G. The notation $\mathcal{P}(G)$ is not well-defined as G may contain no subgraph isomorphic to H, or may be ambiguous when the choice of H is not unique. Given the graphs G, G', we will write $\mathcal{P}[G] \cong G'$ when there exists a subgraph H of G such that the graph obtained from G on applying $\mathcal{P}$ to H is isomorphic to G'.

2 Proof of Proposition 1.2 and Corollary 1.5

2.1 Proof of Proposition 1.2, non-self-dual case

Lower bound. Let the graph G satisfy property $\mathcal{S}_n$. We will now show that the lower bound in Proposition 1.2 actually holds for any planar G of smallest vertex degree $\delta(G) \geq 3$. We have at least one vertex of valency i for every $3 \leq i \leq n$: as shown in [10, proof of Lemma 7], we then have a lower bound on the edges $q = |E(G)|$,

$$2q \geq \frac{(n-2)(n-3)}{2} + 3p.$$

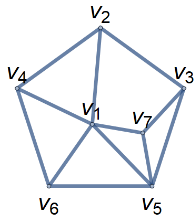
On the other hand, imposing that G has at least one i-gonal face for all $4 \leq i \leq n$ yields

$$2q \leq 6p - 12 - 2\sum_{i=4}^{n}(i-3) = 6p - 12 - (n-3)(n-2).$$

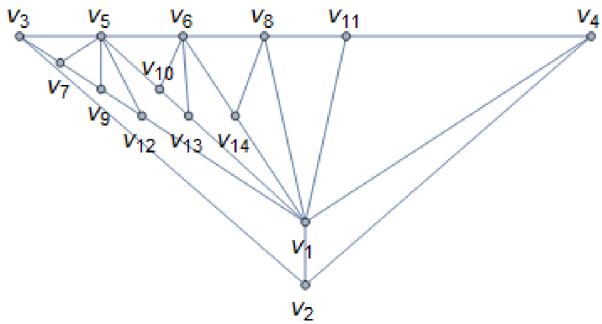
Combining the two inequalities yields the lower bound in Proposition 1.2

$$p \geq \frac{n^2 - 5n + 14}{2}, \qquad \forall n \geq 3.$$

Construction. We now turn to actually constructing 3-polytopes of such order. Consulting [7, Table I], we find that entries 1 (tetrahedron), 2 (square pyramid) and 34 (Figure 1a) are the unique polyhedra of minimal order satisfying $\mathcal{S}_3$, $\mathcal{S}_4$, and $\mathcal{S}_5$ respectively. These are all self-dual. Next, we construct for each $n \geq 6$ a non-self-dual polyhedron of minimal order satisfying $\mathcal{S}_n$.



(a) The only polyhedron $H_5 = G_5$ of minimal order satisfying $\mathcal{S}_5$.



(b) The polyhedron H_7 constructed in Algorithm 2.1.

Figure 1

Algorithm 2.1.

Input. A natural number $N \geq 6$.

Output. For each $6 \leq n \leq N$, a non-self-dual polyhedron H_n of minimal order satisfying $\mathcal{S}_n$.

Description. We start by considering the graph H_5 in Figure 1a with its attached vertex labelling, by setting the integer $n := 6$, and the set of $n - 3$ triples

$$S := \{(v_1, v_4, v_6), (v_5, v_1, v_7), (v_6, v_1, v_5)\}.$$

At each step, given H_{n-1}, we perform the operation depicted in Figure 2, 'edge splitting', to each vertex triple of S in turn, taking for u_1, u_2, u_3 the entries of the triple in order. We label successively $v_8, v_9, \ldots$ the newly inserted vertices via the edge splitting. This yields the graph H_n. The graph H_7 is illustrated in Figure 1b. At the same time, we modify S in the following way. Upon applying edge splitting to (a, b, c), say, we replace it by the new triple (a, b, v), where v is the new vertex introduced by the splitting. Lastly, calling a' the first vertex of the last triple in S, we insert the further triple $(v_{|V(H_{n-1})|+1}, v_1, a')$, and increase n by 1. The algorithm stops as soon as $n = N + 1$.

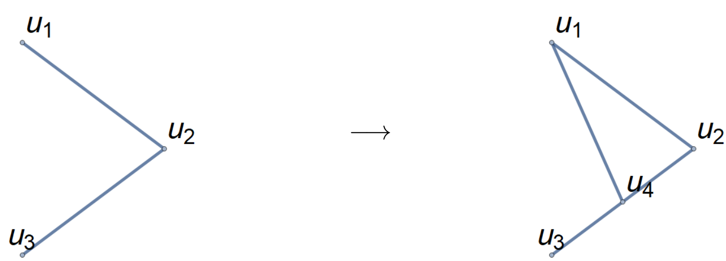


Figure 2: Edge splitting operation on the vertices (u_1, u_2, u_3), consecutive on the boundary of a face.

Remark 2.2. Edge splitting has the effect of raising by one the valencies of the vertex u_1 and of the face containing u_2, u_3 but not u_1. It also introduces the new vertex u_4 of degree 3, and the new triangular face u_1, u_2, u_4.

It is straightforward to check by induction, with base case $n = 6$, that the H_n of Algorithm 2.1 indeed satisfy the sought properties of Proposition 1.2. First, edge splitting is well-defined, as it is always performed on a triple of vertices forming a triangular face. Indeed, once we replace (u_1, u_2, u_3) of Figure 2 with (u_1, u_2, u_4) as in the algorithm, the latter triple forms a face. As for the last triple inserted at each step, note that it is simply

$$(v_{|V(H_{n-1})|+1}, v_1, v_{|V(H_{n-2})|+1}).$$

The vertices $u_1 = v_{|V(H_4)|+1} = v_6$, $u_2 = v_1$, $u_3 = v_{|V(H_3)|+1} = v_5$ form a triangle in H_5. Therefore, after edge splitting, u_1, u_2, and $u_4 = v_{|V(H_5)|+1} = v_8$ are the vertices of a triangle in H_6, and so forth in this fashion.

Second, for the graph order (1.1), each step adds $n-3$ vertices, and we have by induction

$$\frac{(n-1)^2 - 5(n-1) + 14}{2} + n - 3 = \frac{n^2 - 5n + 14}{2}.$$

We record that the inequalities used to derive the lower bound on p now yield $q = 2p - 2$ for all of the H_n.

Third, to obtain H_n from H_{n-1}, we perform $n-3$ edge splittings. These transform a vertex of degree i into one of degree $i+1$, for $3 \leq i \leq n-1$ respectively. Moreover, at the same time an i-gon gets replaced by an $i+1$-gon: indeed, in Figure 2 the face different containing u_2, u_3 but not u_1 loses the edge u_2u_3 and acquires u_2u_4, u_4u_3. We conclude that H_n satisfies $\mathcal{S}_n$.

Fourth, we show that H_n is not self-dual for any $n \geq 6$. On one hand, $\deg_{H_n}(v_1) = n$, $\deg_{H_n}(v_5) = n - 1$, and $v_1v_5 \notin E(H_n)$. On the other hand, in H_n the n-gon and the $n-1$-gon share the edge v_2v_3.

Lastly we note that Algorithm 2.1 may be implemented in linear time in the graph order (quadratic in n).

Remark 2.3. There are several other constructions, similar to Algorithm 2.1, yielding polyhedra of minimal order satisfying $\mathcal{S}_n$, e.g. the duals H_n^*. The idea is to apply $n-3$ edge splittings at each step, where each simultaneously increases by 1 the valency of a vertex and of a face.

2.2 Self-dual case of Proposition 1.2, assuming Theorem 1.3

For the last part of Proposition 1.2 we require the further condition of self-duality. However, constructions with edge splitting in general do not preserve the self-duality of H_5 in the new graphs obtained from it. In the next section we will present Algorithm 3.2, that produces a self-dual polyhedron for any given admissible degree sequence, as stated in Theorem 1.3. The self-dual polyhedra of Proposition 1.2 may be constructed independently of the arguments of section 3, although possibly in a less intuitive fashion. Here we complete the proof of Proposition 1.2 assuming Theorem 1.3. To obtain G_n we simply input the tuple $(4, 5, \dots, n)$, i.e. the sequence

$$n, n-1, \dots, 4, 3^{(n^2-7n+20)/2},$$

into Algorithm 3.2.

2.3 Proof of Corollary 1.5

The operation of edge splitting in Figure 2 is a special version of **1-extension** in rigidity theory (sometimes also referred to as edge splitting). This operation deletes an edge u_2u_3, and adds a vertex u_4 together with edges u_4u_1, u_4u_2, and u_4u_3, where $u_1 \neq u_2, u_3$ (Figure 3). It is easy to check that the 1-extension operation preserves 3-connectivity, and moreover a 1-extension of a 3-connected generic circuit is a 3-connected generic circuit. In fact, Connelly conjectured, and Berg-Jordán proved [3], that every 3-connected generic circuit may be obtained from K_4 (i.e. the tetrahedron) by a sequence of 1-extensions.

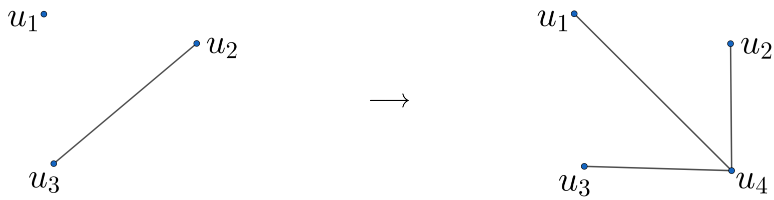


Figure 3: The 1-extension operation.

One quickly checks that H_5 in Figure 1a is obtained from K_4 in this way, and therefore all of the H_n for $n \geq 5$, as well as any other family of graphs derived from H_5 via edge splitting (Remark 2.3). This proves Corollary 1.5 for the non-self-dual construction. The self-duals are also constructed via successive edge-splitting of the tetrahedron – Remark 3.6.

Our particular version of edge splitting assumes also that u_1, u_2, u_3 are consecutive vertices on a face. As established above, this also preserves the property $\mathcal{S}_n$, as well as planarity.

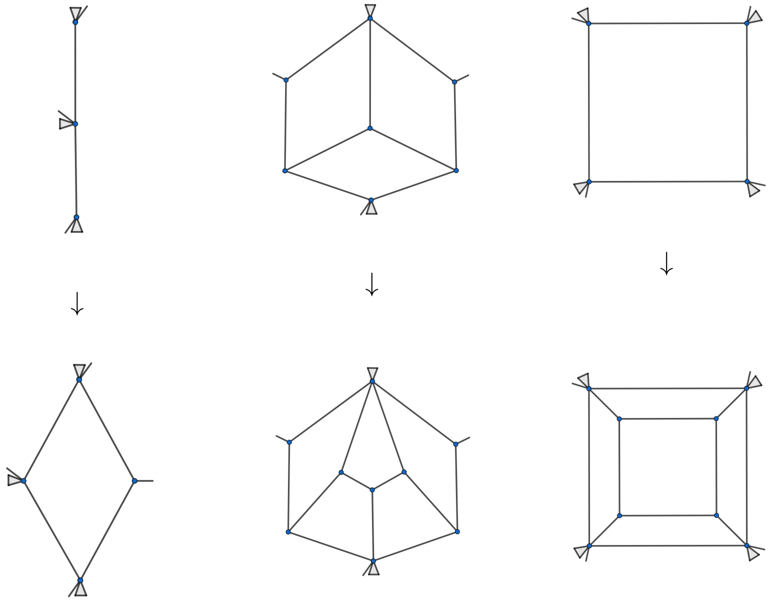
3 Generating self-dual polyhedra

3.1 Radial graphs and quadrangulations

The *radial*, or *vertex-face* graph R_G of a plane graph G is obtained by taking $V(R_G)$ to be the set of vertices and regions of G. We have an edge between two vertices u, v of R_G whenever u is a vertex of G, and v a region of G, such that u lies on the boundary of v in G [11, section 2.8].

If the plane graph G is 2-connected, the newly constructed R_G is a *quadrangulation* of the sphere, i.e. each region is delimited by a 4-cycle [11, section 2.8]. If G is a polyhedron then so is R_G [1, Lemma 2.1]. Moreover, G is a polyhedron if and only if R_G has no separating 4-cycles (i.e. 4-cycles that are not faces, so that removing the cycle disconnects the graph) [11, Lemma 2.8.2].

The radial graph of the tetrahedron is the cube, and more generally the radial graph of the pyramid (or wheel) W_p, $p \geq 4$ is the so-called 'pseudo double wheel' PDW_{2p} (of $2p$ vertices), i.e. the dual graph of the $p-1$-gonal antiprism. As established in [4, Theorem 3], and initially stated in [2], all polyhedral quadrangulations of the sphere are obtained from the cube by applying three transformations $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$, sketched in Figure 4 (cf. [4, Figure 3] and [2, Figure 3]).



(a) $\mathcal{P}_1$ (b) $\mathcal{P}_2$ (c) $\mathcal{P}_3$

Figure 4: The transformations $\mathcal{P}_1$, $\mathcal{P}_2$, and $\mathcal{P}_3$. A half-edge indicates that an edge *must* occur, while a triangle indicates that one or more edges *may* occur.

We introduce the notation $\mathbf{C}(\mathfrak{G}, \mathfrak{P})$ for the set of all graphs that may be obtained from an initial set of graphs $\mathfrak{G}$ by applying the set of transformations $\mathfrak{P}$. Under this notation, the previous statement may be rephrased as,

$\mathbf{C}(\{PDW_8\}, \{\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3\})$ is the set of 3-connected quadrangulations of the sphere.

Moreover,

$$\mathbf{C}(\{PDW_{2p} : p \geq 4\}, \{\mathcal{P}_1\})$$

is the set of all polyhedral quadrangulations without separating 4-cycles [4, Theorem 4]. It follows that

$$\mathbf{C}(\{PDW_{2p} : p \geq 4\}, \{\mathcal{P}_1\}) \quad \textit{is the set of radial graphs of polyhedra.}$$

We note that the transformation $\mathcal{P}_2$ replaces a subgraph of G that is isomorphic to $PDW_8 - v$ with a copy of $PDW_{10} - v$. In particular, $\mathcal{P}_2[PDW_{2p}] \cong PDW_{2p+2}$. Therefore, we have

$$\mathbf{C}(\{PDW_{2p} : p \geq 4\}, \{\mathcal{P}_1\}) \subseteq \mathbf{C}(\{PDW_8\}, \{\mathcal{P}_1, \mathcal{P}_2\}).$$

For G a *self-dual* polyhedron, we have in particular $|V(R_G)| = 2|V(G)|$ and $|E(R_G)| = 2|E(G)| = 2|V(R_G^*)|$. Furthermore, we can recover G from R_G by noting that the latter is always bipartite, and taking for G all of the vertices in either part of R_G, together with edges for G between pairs of vertices belonging to the same face in R_G. The above considerations have the following consequence.

Proposition 3.1. *The radial graph of any polyhedron G may be obtained from the cube via the transformations $\mathcal{P}_1, \mathcal{P}_2$ of* Figure 4. *Moreover, the number of applications of $\mathcal{P}_1$ to generate self-duals is even.*

Proof. By the arguments of the present section, it suffices to prove that when G is self-dual, the number of applications of $\mathcal{P}_1$ on the cube to obtain R_G is indeed even. From Figure 4, we observe that applying $\mathcal{P}_1$ to R_G has the effect of adding an edge to G, and a vertex and an edge to G^*. As opposed to this, applying $\mathcal{P}_2$ to R_G adds one vertex and one edge to both G, G^*. We have thus obtained our parity argument. □

In the next section we prove Theorem 1.3, putting it in the context of the above literature.

3.2 The proof of Theorem 1.3

As it turns out, for any $n \geq 3$, generating a self-dual polyhedron G of minimal order satisfying $\mathcal{S}_n$ may be done by applying only a transformation $\mathcal{P}$ (to be defined below, and similar to $\mathcal{P}_2$ of Figure 4b) to the cube in order to construct R_G, and then passing to G. This generalises readily to Theorem 1.3, as we will now prove.

We begin by defining a function f, that maps a tuple $T = (t_1, t_2, \dots, t_k)$, $k \geq 0$, of integers ≥ 4 to the degree sequence (1.2)

$$f(T) = t_1, t_2, \dots, t_k, 3^m,$$

where m is given by (1.3).

Algorithm 3.2.

Input. A k-tuple of integers $T = (t_1, t_2, \dots, t_k)$, with $t_i \geq 4$ for each i.

Output. A self-dual polyhedron $G(T)$ of degree sequence $f(T)$.

Description. We will construct the radial graph $R_{G(T)}$, and then pass to $G(T)$ as explained in section 3.1. We begin by setting $R_{G(T)}$ to be the cube PDW_8, radial graph of the tetrahedron. We also consider a subgraph H of $R_{G(T)}$ with the vertex labelling of Figure 5a. We define the transformation $\mathcal{P}$ that modifies a subgraph H of a graph G as shown in Figure 5.
We stop when T is empty. Each step entails $t_i - 3$ successive applications of $\mathcal{P}$ to $R_{G(T)}$. Before each subsequent application, we apply to H a graph isomorphism φ such that

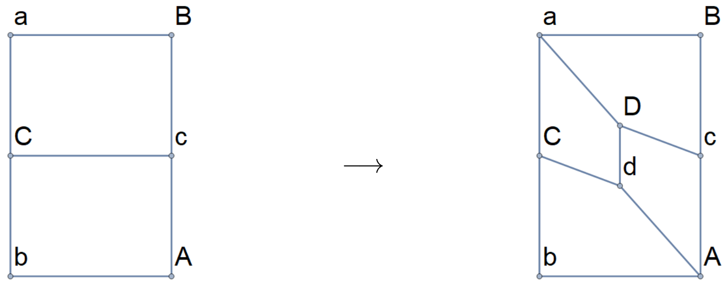
$$\begin{aligned} \varphi(a) = a, \quad \varphi(b) = c, \quad \varphi(c) = d, \\ \varphi(A) = A, \quad \varphi(B) = C, \quad \varphi(C) = D. \end{aligned} \tag{3.1}$$

as labelled in Figure 5. Following all the $t_i - 3$ operations, we instead apply to H the graph isomorphism ψ satisfying

$$\begin{aligned} \psi(a) = c, \quad \psi(b) = a, \quad \psi(c) = d, \\ \psi(A) = C, \quad \psi(B) = A, \quad \psi(C) = D. \end{aligned} \tag{3.2}$$

then we delete t_i from T, and proceed to the next step.

Remark 3.3. There are in general several polyhedra for a given degree sequence (1.2). Algorithm 3.2 does not construct them all. On the other hand, in many cases permutations of the t_i's give rise to non-isomorphic solutions, as may be observed via direct computation.



(a) Subgraph H of R_G. (b) Resulting subgraph.

Figure 5: The transformation $\mathcal{P}$. Vertices of G have lower-case labels, those of G^* upper-case.

Remark 3.4. It follows from Theorem 1.3 that the set

$$\mathbf{C}(\{PDW_8\}, \{\mathcal{P}\})$$

contains the radial graph of at least one self-dual polyhedron for any given degree sequence. As for how many radial graphs of self-dual polyhedra of given size belong to $\mathbf{C}(\{PDW_8\}, \{\mathcal{P}\})$, we have computed the values of Table 1 for small sizes (data available on request).

Size q	6	8	10	12	14	16	18	20	22
Radials of self-duals in $\mathbf{C}(\{PDW_8\}, \{\mathcal{P}\})$	1	1	2	5	15	40	140	417	1496
Total self-duals	1	1	2	6	16	50	165	554	1908

Table 1: For given graph size q, the number of radial graphs R_G of self-dual polyhedra G with size q that belong to $\mathbf{C}(\{PDW_8\}, \{\mathcal{P}\})$, compared to the total. For values in the last row, refer e.g. to [6].

Remark 3.5. The transformation $\mathcal{P}$ is similar to $\mathcal{P}_2$ of Figure 4b. More precisely, $\mathcal{P}_2$ is applicable if and only if, $\mathcal{P}$ may be applied and moreover either a, b or A, B belong to the same face in R_G (referring to the labelling of Figure 5).

Remark 3.6. Applying $\mathcal{P}$ to R_G has the same effect on G and G^* as applying the edge splitting of Figure 2 to them, where $u_1 = a$, $u_2 = b$, $u_3 = c$, and $u_4 = d$ (and analogously for vertices A, B, C, D of G^*).

Let us now complete the proof of Theorem 1.3. We start by justifying applicability of the transformation $\mathcal{P}$. The initial cube clearly has a subgraph isomorphic to H in Figure 5a. Furthermore, the graph in Figure 5b also has a subgraph isomorphic to H, where the isomorphism is φ (3.1). The same statement remains true for ψ (3.2).

Starting with the cube $R_{G((3,3,3,3))}$, each operation $\mathcal{P}$ clearly yields another 3-connected quadrangulation of the sphere. We now check that self-duality of G is preserved by the algorithm. Each operation $\mathcal{P}$ on $R_{G(T)}$ transforms $G(T)$ and $G^*(T)$ in the same way (Remark 3.6). As the initial $G((3,3,3,3))$ (tetrahedron) is self-dual, then so will all the successive $G(T)$'s be. Further, the following considerations for lower-case labels a, b, c, d apply verbatim to the upper-case ones by duality.

We now analyse how each step affects the degrees of the vertices in G. First, the degree of a vertex in G is the number of faces that the corresponding vertex lies on in R_G, i.e., $\deg_G(v) = \deg_{R(G)}(v)$ for each v by the definition of radial graph. Now, each application of $\mathcal{P}$ adds 1 to the degree of a (and A of G^*), introduces the new vertex d (and D of G^*), of degree 3, and leaves other valencies unchanged. When we update H via φ (3.1), a is mapped to itself. Therefore, step $i = 1, \dots, k$ has the effect of increasing by $t_i - 3$ the degree of a (and A).

Second, we claim that the algorithm step i increases by $t_i - 4$ the number of vertices of valency 3 in G. By the considerations above, the first application of $\mathcal{P}$ increases one valency of G from 3 to 4, and adds a new vertex of degree 3. Hence the first application of each step leaves the number of vertices of valency 3 in $G(T)$ unchanged. Each subsequent application of $\mathcal{P}$ increases their total by 1. Now step i entails $t_i - 3$ operations of type $\mathcal{P}$, hence the number of vertices of degree 3 increases by $(t_i - 3) - 1$ as claimed.

Third, we claim that, at the beginning of each algorithm step, in $G(T)$ with its attached labelling one has

$$\deg(a) = \deg(A) = 3.$$

We show this claim by induction. In the initial cube all vertices are of valency 3. When we apply ψ (3.2) to H, a is mapped to c, and $\deg(c) = 3$ since $\mathcal{P}$ does not modify its degree.

Putting everything together, after k algorithm steps the degree sequence of $G(T)$ will be

$$t_1, t_2, \dots, t_k, 3^{4+\sum_{i=1}^{k}(t_k-4)}$$

i.e., at the end of the algorithm the resulting sequence will be (1.2) as desired.

As for algorithm speed, the total number of operations to obtain $G(T)$ is proportional to the sum of the t_i's, i.e. to the graph size q, that is to say, to its order p since $q = 2p - 2$. The proof of Theorem 1.3 is complete.

Future work. Our investigation generates a portion of the self-dual polyhedra (recall Table 1), starting from the tetrahedron, by applying $\mathcal{P}$ to its radial graph (Figure 5). This portion includes at least one such graph for every admissible degree sequence. It would be of interest to further analyse the set $\mathbf{C}(\{PDW_8\}, \{\mathcal{P}\})$ and its properties.

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