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Star-critical Gallai-Ramsey numbers involving the disjoint union of triangles

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Abstract

In the Ramsey theory of graphs, star-critical Gallai-Ramsey numbers serve as a measure of the strength of their corresponding Gallai-Ramsey numbers. In this paper, we evaluate some classes of star-critical Gallai-Ramsey numbers when the arguments are the disjoint unions of K_3-subgraphs. The methods used are constructive and in all cases considered, we find that the removal of a single edge destroys the Ramsey property.

Keywords: Deleted edge number, critical coloring, Gallai partition

Math. Subj. Class.: 05C55, 05D10, 05C35

1 Introduction

First introduced by Hook [9] in 2010, and developed in [10] and [11], the star-critical Ramsey number of a collection of graphs focuses on the number of edges that must be added between a vertex and a critical coloring of a certain complete graph to guarantee the Ramsey property. In a sense, it measures the strength of the corresponding Ramsey number. In their brief existence, star-critical Ramsey numbers have attracted the interest

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of numerous combinatorialists (e.g., see [2, 8, 12, 13, 14, 15, 18] and [20]). In a recent article, Su and Liu [17] considered the analogous concept for Gallai-Ramsey numbers. We seek to evaluate several star-critical Gallai-Ramsey numbers for disjoint unions of K_3-subgraphs, building off of some recent results by Zhang, Zhu, and Chen [19] involving the corresponding Gallai-Ramsey numbers.

In order to describe our results, we must first establish the definitions and notations to be used throughout this paper. A *t-coloring* of a graph $G = (V(G), E(G))$ is a map $\tau\colon E(G) \longrightarrow \{1, 2, \dots, t\}$, where the codomain is thought of as a set of "colors". Such a map is not assumed to be surjective. If $G_1, G_2, \dots, G_t$ are graphs, then the *Ramsey number* $r(G_1, G_2, \dots, G_t)$ is the least natural number p such that every t-coloring of K_p contains a monochromatic subgraph isomorphic to G_i in some color i, where $1 \le i \le t$. When $G_1 = G_2 = \cdots = G_t$, we simplify the notation and just write $r^t(G_1)$. A t-coloring of $K_{r(G_1,G_2,\dots,G_t)-1}$ that lacks a copy of G_i in color i, for all $1 \le i \le t$, is called a *critical coloring for* $r(G_1, G_2, \dots, G_t)$. It should be noted that the removal of a single vertex (and all edges incident with that vertex) from a t-colored $K_{r(G_1,G_2,\dots,G_t)}$ destroys the Ramsey property. The star-critical Ramsey number seeks to measure the precise number of such edges that are needed to retain the Ramsey property.

To be more precise, let K denote a complete graph of order n. For any $1 \le \ell \le n$, define the graph $K \sqcup K_{1,\ell}$ to consist of the union of K and a single vertex, with exactly ℓ edges connecting the two. The *star-critical Ramsey number* $r_*(G_1, G_2, \dots, G_t)$ is then defined to be the least ℓ such that every t-coloring of $K_{r(G_1,G_2,\dots,G_t)-1} \sqcup K_{1,\ell}$ contains a monochromatic subgraph isomorphic to G_i in some color i, where $1 \le i \le t$. As with the Ramsey number, we use the notation $r_*^t(G_1)$ when $G_1 = G_2 = \cdots = G_t$. In the special case where

$$r_*(G_1, G_2, \dots, G_t) = r(G_1, G_2, \dots, G_t) - 1,$$

we say that $(G_1, G_2, \dots, G_t)$ is *Ramsey-full.*. We say that G_1 is *t-Ramsey-full* when $r_*^t(G_1) = r^t(G_1) - 1$. It is well known that any collection of complete graphs is Ramsey-full (e.g., see [4] or Theorem 1 of [5]).

All of the concepts defined so far have analogues in the setting of Gallai colorings. See [6, 7], and [16] for an overview of Gallai-Ramsey numbers. A *Gallai t-coloring* of a graph G is a t-coloring τ of G that lacks rainbow triangles. That is, for every three distinct vertices $x, y, z \in V(G)$,

$$|\{\tau(xy), \tau(yz), \tau(xz)\}| \le 2.$$

If $G_1, G_2, \dots, G_t$ are graphs, then the *Gallai-Ramsey number* $gr(G_1, G_2, \dots, G_t)$ is the least natural number p such that every Gallai t-coloring of K_p contains a monochromatic subgraph isomorphic to G_i in color i, for some $1 \le i \le t$. We write $gr^t(G_1)$ when $G_1 = G_2 = \cdots = G_t$. A Gallai t-coloring of $K_{gr(G_1,G_2,\dots,G_t)-1}$ that lacks a copy of G_i in color i, for all $1 \le i \le t$ is called a *critical coloring for* $gr(G_1, G_2, \dots, G_t)$.

The *star-critical Galai-Ramsey number* (introduced in [17]) is the least ℓ such that every Gallai t-coloring of $K_{gr(G_1,G_2,\dots,G_t)-1} \sqcup K_{1,\ell}$ contains a monochromatic subgraph isomorphic to G_i in color i, for some $1 \le i \le t$. In the special case where

$$gr_*(G_1, G_2, \dots, G_t) = gr(G_1, G_2, \dots, G_t) - 1,$$

we say that $(G_1, G_2, \dots, G_t)$ is *Gallai-Ramsey-full*. We say that G_1 is *t-Gallai-Ramsey-full* when $gr_*^t(G_1) = gr^t(G_1) - 1$. It was conjectured in [17] that if G is a graph that does not contain any isolated vertices, then it is t-Ramsey-full if and only if it is t-Gallai-Ramsey-full. This certainly holds for complete graphs and was shown in [17] to hold for C_4, for all $t \geq 2$.

Two steps are required to prove that $gr_*(G_1, G_2, \ldots, G_t) = \ell$.

1. Construct a Gallai t-coloring of $K_{gr(G_1,G_2,\ldots,G_t)-1} \sqcup K_{1,\ell-1}$ that avoids a monochromatic copy of G_i in color i, for all $1 \leq i \leq t$. From this construction, it follows that $gr_*(G_1, G_2, \ldots, G_t) \geq \ell$.
2. Prove that every Gallai t-coloring of $K_{gr(G_1,G_2,\ldots,G_t)-1} \sqcup K_{1,\ell}$ contains a monochromatic copy of G_i in color i, for some $1 \leq i \leq t$. From this step, it follows that $gr_*(G_1, G_2, \ldots, G_t) \leq \ell$.

Proving that $(G_1, G_2, \ldots, G_t)$ is Gallai-Ramsey-full only requires the first step: constructing a Gallai t-coloring of

$$K_{gr(G_1,G_2,\ldots,G_t)-1} \sqcup K_{1,gr(G_1,G_2,\ldots,G_t)-2} = K_{gr(G_1,G_2,\ldots,G_t)} - e$$

that avoids a monochromatic copy of G_i in color i, for all $1 \leq i \leq t$. Here, the notation $K - e$ represents the graph formed by removing a single edge from the complete graph K.

If G is any graph and $k \in \mathbb{N}$, denote by kG the disjoint union of k copies of G. In this paper, we prove that $(2K_3, 2K_3, \ldots, 2K_3, K_3, K_3, \ldots, K_3)$ is Gallai-Ramsey-full, when t is even. We also prove that $(K_3, K_3, \ldots, K_3, kK_3)$ is Gallai-Ramsey-full, for all $k \geq 1$ and t odd. The construction processes we will use often involve replacing a vertex x in a Gallai-colored graph G with a Gallai-colored graph H. The result is a graph with vertex set $V(H) \cup (V(G) - \{x\})$ in which edges contained entirely in $V(H)$ or $V(G) - \{x\}$ receive the same color that they received in their respective graphs. Edges of the form uv, with $u \in V(H)$ and $v \in V(G) - \{x\}$, receive the same color that xv received in the original Gallai coloring of G. We leave it as an exercise for the reader to check that the resulting coloring is also a Gallai coloring. For such a construction, we refer to G as the *base graph* and H as a *block*. We conclude our paper with some conjectures about the Gallai-Ramsey-fullness of collections of disjoint unions of complete graphs of order at least three.

2 Main results

One of the first results proved about Gallai-Ramsey numbers was due to Chung and Graham [3], where in 1983, they proved a theorem equivalent to

$$gr^t(K_3) = \begin{cases} 5^{t/2} + 1 & \text{if } t \text{ is even} \\ 2 \cdot 5^{(t-1)/2} + 1 & \text{if } t \text{ is odd,} \end{cases}$$

for all $t \geq 3$. The construction that will serve as the basis for our new constructions is the one that gives the lower bound for $gr^t(K_3)$ when t is even. We follow the notation introduced in [19] and denote by (G, τ) a Gallai t-colored complete graph, with t-coloring τ.

First, define (G_1, τ_1) to be the unique 2-colored K_5 that lacks monochromatic K_3-subgraphs (see the first image in Figure 1). Label the colors in (G_1, τ_1) as colors 1 and 2. For any $i > 1$, define (G_i, τ_i) to be the Gallai $2i$-colored complete graph formed by replacing each vertex in (G_{i-1}, τ_{i-1}) (which uses colors $1, 2, \ldots, 2i-2$) with a 2-colored K_5 (in colors $2i-1$ and $2i$) that lacks monochromatic K_3-subgraphs. For example, (G_2, τ_2) is shown as the second image in Figure 1. This construction produces a Gallai $2i$-colored K_{5^i} that lacks monochromatic K_3-subgraphs, from which it follows that $gr^t(K_3) \geq 5^{t/2} + 1$ when $t > 3$ is even.

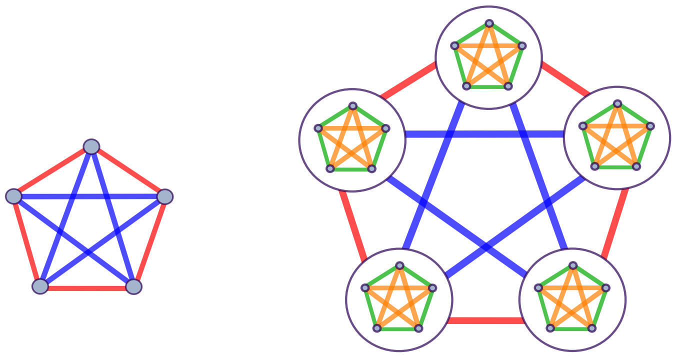

Figure 1: The graphs (G_1, τ_1) and (G_2, τ_2), which imply the lower bounds $r^2(K_3) \geq 6$ and $gr^4(K_3) \geq 26$, respectively.

For $i \geq 1$, define (G'_i, τ'_i) by replacing each vertex in (G_i, τ_i) (which uses colors $1, 2, \ldots, 2i$) with a copy of K_2 in color $2i+1$. The graph (G'_1, τ'_1) is given in Figure 2. This construction produces a Gallai $(2i+1)$-colored $K_{2 \cdot 5^i}$ that lacks monochromatic

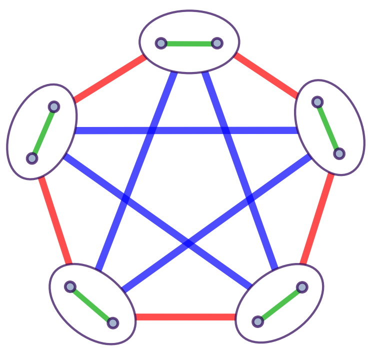

Figure 2: The graph (G'_1, τ'_1), which implies the lower bound $gr^3(K_3) \geq 11$.

K_3- subgraphs. It follows that when $t \geq 3$ is odd, $gr^t(K_3) \geq 2 \cdot 5^{(t-1)/2} + 1$.

In the following theorem, we consider Gallai-Ramsey numbers involving K_3-subgraphs and $2K_3$-subgraphs, when $t \geq 4$ is assumed to be even. In order to simplify notations, we follow the approach used in [19] and define

$$gr(s;t-s) := gr(\underbrace{2K_3, 2K_3, \ldots, 2K_3}_{s\ terms}, \underbrace{K_3, K_3, \ldots, K_3}_{t-s\ terms})$$

and

$$gr_*(s;t-s) := gr_*(\underbrace{2K_3, 2K_3, \ldots, 2K_3}_{s\ terms}, \underbrace{K_3, K_3, \ldots, K_3}_{t-s\ terms}),$$

for $1 \leq s \leq t$. We now prove that $(2K_3, 2K_3, \ldots, 2K_3, K_3, K_3, \ldots, K_3)$ is Gallai-Ramsey-full when the number of colors used is even and at least 4.

Theorem 2.1. *If $t \geq 4$ is even and $1 \leq s \leq t$, then*

$$gr_*(s;t-s) = gr(s;t) - 1.$$

Proof. It was proved in Theorem 1.6 of [19] that when $t \geq 4$ is even,

$$gr(s;t-s) = 5^{t/2} + 2s + 1.$$

We begin by describing the construction that led to the lower bound that was used to obtain this Gallai-Ramsey number. Let $n = 5^{t/2} + 2s + 1$. Consider $(G_{t/2}, \tau_{t/2})$, which is formed by replacing each of the vertices in the base graph $(G_{t/2-1}, \tau_{t/2-1})$ (which uses colors $1, 2, \ldots, t-2$) with copies of K_5 in colors $t-1$ and t that lack monochromatic K_3-subgraphs. We refer to the copies of K_5 as blocks. Select a block, label it A, and label some vertex in A as a. Among the $5^{t/2} - 5$ vertices not in A, select s vertices and label them $x_1, x_2, \ldots, x_s$. This selection is possible since $s \leq t$ and $t \leq 5^{t/2} - 5$ holds for all $t \geq 4$. For each $1 \leq i \leq s$, replace x_i with a K_3 in color i. In the resulting construction, $2s$ vertices are added to $(G_{t/2}, \tau_{t/2})$ in which there exists a K_3 in colors $1, 2, \ldots, s$, but no $2K_3$ exists in any of these colors. Also, no K_3 exists in colors $s+1, s+2, \ldots, t$. This construction gives the lower bound $gr(s;t-s) \geq n$. In order to form a $K_n - e$ that still avoids the same monochromatic subgraphs, add in a vertex b to the existing $K_{5^{t/2}+2s}$ that we just constructed. For every vertex $x \neq a$, color edge bx the same as edge ax. Let ab be the "missing edge." See Figure 3 for the construction that corresponds with $gr_*(3;1)$. The resulting $K_n - e$ lacks a monochromatic copy of K_3 in colors $s+1, s+2, \ldots, t$ since any newly created K_3 in one of these colors would have to contain b, but no such K_3 included a in the original $K_{5^{t/2}+2s}$. While there exist K_3-subgraphs in each of the colors $1, 2, \ldots, s$, any K_3-subgraph in color i must contain at least two vertices from the K_3 that replaced vertex x_i. As there are only three vertices in this block, no copy of $2K_3$ exists in any of the colors $1, 2, \ldots, s$. Since we have provided a coloring of $K_n - e$ that avoids a $2K_3$ in colors $1, 2, \ldots, s$ and a K_3 in colors $s+1, s+2, \ldots, t$, it follows that $gr_*(s;t-s) = 5^{t/2} + 2s$, completing the proof of the theorem. □

Theorem 2.2. *If $t \geq 3$ is odd and $k \in \mathbb{N}$, then the t-color star-critical Gallai-Ramsey number satisfies*

$$gr_*(K_3, K_3, \ldots, K_3, kK_3) = 2 \cdot 5^{(t-1)/2} + 3k - 3.$$

Proof. It was proved in Corollary 1.5 of [19] that when $t \geq 3$ is odd, the t-color Gallai-Ramsey number satisfies

$$gr(K_3, K_3, \ldots, K_3, kK_3) = 2 \cdot 5^{(t-1)/2} + 3k - 2.$$

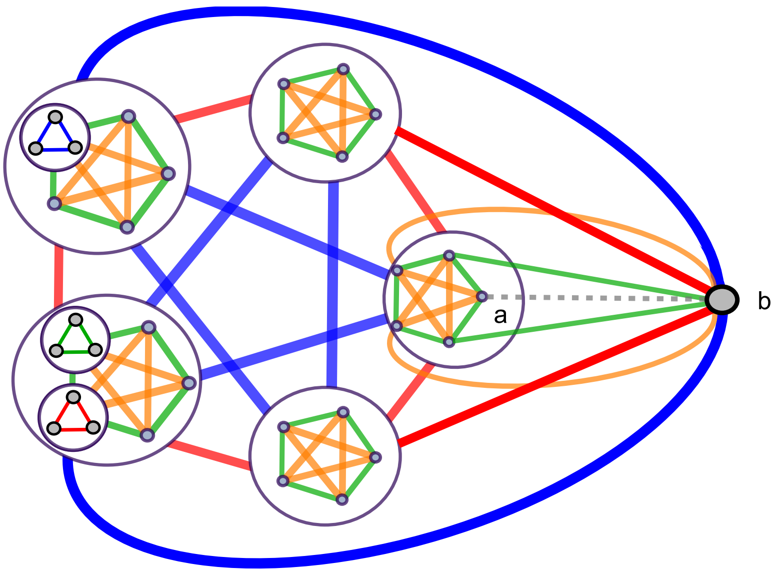

Figure 3: The construction that leads to the evaluation of $gr_*(3;1) = 31$.

We begin by describing the construction that led to the lower bound that was used to obtain this Gallai-Ramsey number. Consider the graph $(G_{(t-1)/2}, \tau_{(t-1)/2})$ which is a $K_{5^{(t-1)/2}}$ with edges in colors $1, 2, \ldots, t-1$. Replace one vertex in $(G_{(t-1)/2}, \tau_{(t-1)/2})$ with a K_{3k-1} in color t and the other vertices with K_2-subgraphs in color t. The resulting $K_{2 \cdot 5^{(t-1)/2}+3k-3}$ lacks a K_3 in colors $1, 2, \ldots, t-1$ and a kK_3 in color t. Select a vertex in one of the K_2-subgraphs in color t and label it a. Add in vertex b, color edge bx the same as ax, and consider ab as the missing edge (e.g., Figure 4 shows the case $t = 3$). No K_3 includes vertex b in colors $1, 2, \ldots, t-1$ since a was not contained in any such K_3. Also, b connects to vertices in the K_{3k-1} via colors other than color t. So, no kK_3 was created in color t. We have produced a Gallai t-colored $K_{2 \cdot 5^{(t-1)/2}+3k-2} - e$ that lacks a monochromatic K_3 in colors $1, 2, \ldots, t-1$ and a kK_3 in color t. It follows that

$$gr_*(K_3, K_3, \ldots, K_3, kK_3) = 2 \cdot 5^{(t-1)/2} + 3k - 3,$$

completing the proof of the theorem. □

3 Conclusion

In recent work (see [1] and [2]), a parameter similar to the star-critical Ramsey number was defined, called the deleted edge number. In order to define a Gallai-Ramsey analogue of it, we first define the *k-deleted Gallai-Ramsey number* $gd_k(G_1, G_2, \ldots, G_t)$ to be the minimum natural number p such that every Gallai t-coloring of $K_p - E(K_{1,k})$ contains a copy of G_i in color i for some $1 \le i \le t$. It is easily checked that

$$gr(G_1, G_2, \ldots, G_t) \le gd_k(G_1, G_2, \ldots, G_t) \le gr(G_1, G_2, \ldots, G_t) + 1,$$

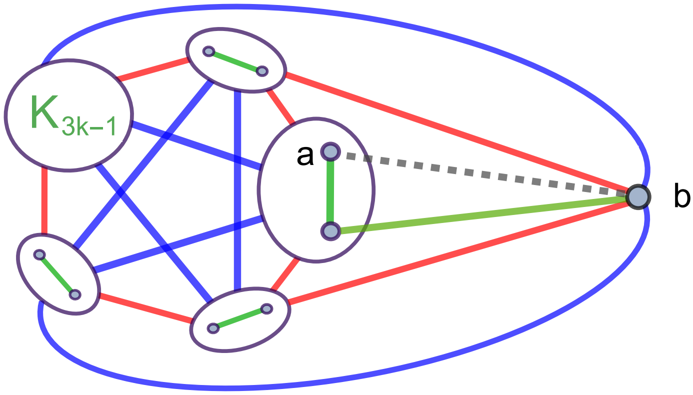

Figure 4: A Gallai 3-coloring of $K_{3k+8} - e$ that avoids a monochromatic K_3 in colors 1 (red) and 2 (blue) and a monochromatic kK_3 in color 3 (green).

for all $1 \leq k \leq gr(G_1, G_2, \ldots, G_t) - 1$. The *Gallai deleted edge number*

$$gde(G_1, G_2, \ldots, G_t)$$

is defined to be the least k such that

$$gd_k(G_1, G_2, \ldots, G_t) = gr(G_1, G_2, \ldots, G_t) + 1.$$

It follows that

$$gde(G_1, G_2, \ldots, G_t) + gr_*(G_1, G_2, \ldots, G_t) = gr(G_1, G_2, \ldots, G_t),$$

so that the computation of a given star-critical Gallai-Ramsey number is equivalent to the computation of a Gallai deleted edge number.

Essentially, the two numbers measure the same phenomenon, one from the perspective of adding edges and the other from the perspective of deleting edges. Many results involving star-critical Gallai-Ramsey numbers are greatly simplified when stated in terms of Gallai deleted edge numbers. In particular, observe that $(G_1, G_2, \ldots, G_t)$ is Gallai-Ramsey-full whenever $gde(G_1, G_2, \ldots, G_t) = 1$. For historic reasons, we chose to approach this paper from the perspective of adding edges.

We conclude with two conjectures, the second of which is a stronger result than the first.

Conjecture 3.1. *Let $m_i \in \mathbb{N}$ for all $1 \leq i \leq t$. Then*

$$gr_*(m_1K_3, m_2K_3, \ldots, m_tK_3) = gr(m_1K_3, m_2K_3, \ldots, m_tK_3) - 1.$$

Conjecture 3.2. *Let $m_i \in \mathbb{N}$ and $n_i \geq 3$, for all $1 \leq i \leq t$. Then*

$$gr_*(m_1K_{n_1}, m_2K_{n_2}, \ldots, m_tK_{n_t}) = gr(m_1K_{n_1}, m_2K_{n_2}, \ldots, m_tK_{n_t}) - 1.$$

If either of these conjectures are true, and if a generalization of the conjecture relating Ramsey-fulless to Gallai-Ramsey-fullness stated by Su and Liu [17] is true, then we will immediately obtain extra information about star-critical Ramsey numbers involving the disjoint unions of complete graphs.

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