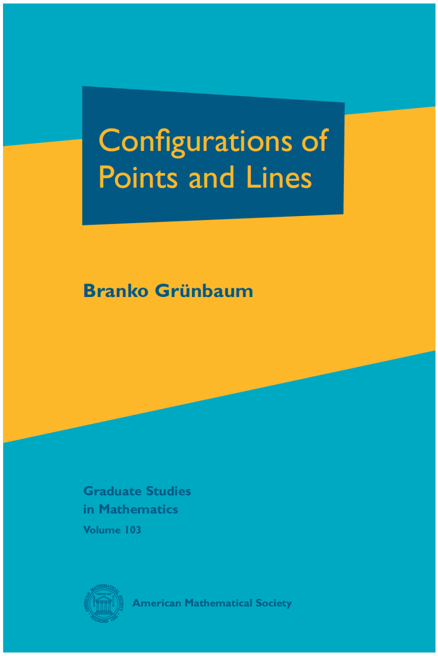
Configurations of Points and Lines
by Branko Grünbaum



B. Grünbaum, *Configurations of Points and Lines*, volume 103 of *Graduate Studies in Mathematics*, American Mathematical Society, Providence, Rhode Island, 2009.

ISBN 978-0-8218-4308-6 (printed edn.), ISBN 978-1-4704-1167-1 (electronic edn.)

What else could be a simpler geometric figure than a point and a straight line? Yet, using some sets of points and lines, a kind of paradise can be created, and we may say, with Hilbert, that no one shall expel us from this paradise. With Branko Grünbaum's monograph, we have even got a guidebook to it.

The rules for organizing a set of points and lines into a structure, called a *configuration*, are very simple: let p, q, n, k be suitable positive integers, and take p points and n lines such that each point is incident with precisely q of the lines, and each line is incident with precisely k of the points. The type of a configuration with these parameters is denoted by (p_q, n_k). For a configuration with an equal number of points and lines, the more concise notation (n_k) is used, and it is called a *balanced configuration*; if the number k is emphasized, we speak of k-configurations.

The archetypal example is the (9_3) *Pappus configuration*. It is associated with the famous incidence theorem due to Pappus of Alexandria (3rd century A.D.); hence, it also exemplifies that certain configurations originate in incidence theorems.

The second half of the 19th century was an era when configurations caught the attention of many outstanding mathematicians including Burnside, Cayley, Cremona, Plücker, Reye, Schönflies, Steiner, Steinitz, Veronese. This led to the discovery of many (actually, infinitely many) examples of configurations. In the book under review, this period, together with the first decade of the 20th century, is called a "classical period".

In the next eight decades there were only few significant publications on this topic. But in 1990, a new era started that may rightly be called a "renaissance" of configurations. The initiator and the leading figure of this renaissance was Branko Grünbaum. Indeed, he published about twenty papers on configurations, starting with a paper in 1990 written jointly with John Rigby [25]; in addition, he gave graduate-level courses, lectures and talks. This work was crowned by the monograph *Configurations of Points and Lines*.

From the "Beginnings", which is also the title of Chapter 1, the monograph grabs the reader's attention; here in particular, by seven introductory sections with carefully chosen topics. One of them (Section 1.2) is entitled "An informal history of configurations"; some details from here we already mentioned above. It not only places the topic in a historical perspective, but also gives criticism of some earlier results and views.

Various notions regarding the symmetry properties of configurations are discussed in Section 1.5. Symmetry (both on a combinatorial and geometric level) appears in a large variety of forms in configurations, hence the notions introduced in this section are often referred to later in the book; among other things, they play important role in efficient construction methods.

The reader can also enjoy a particularly remarkable feature of the book even in the introductory chapter. Indeed, there are great many examples of configurations, throughout the book, which are presented by beautiful drawings (made by various dynamic geometry software). As the author himself remarks in the Preface, *"this is practically inevitable considering the topic"*.

The most well-known configurations are the (n_3) configurations. They are the subject of Chapter 2, the most extensive chapter of the book. The first part of this chapter is devoted to a careful, critical review of early results, going back more than a century. Not only are some deficiences of the original works pointed out and discussed, but relevant recent results are also presented. In particular, a detailed discussion of some parts of the 1894 doctoral thesis of Steinitz is included.

Two fundamental problems appear here: the enumeration of configurations, and the existence/nonexistence problem. Both occur on three different conceptual levels; namely, a sharp distinction is to be made between *combinatorial*, *topological*, and *geometric* configurations. Every geometric configuration has an underlying abstract incidence structure called a combinatorial (or abstract) configuration. Assume we are given a combinatorial configuration $\mathcal{C}$ and a geometric configuration $\bar{\mathcal{C}}$ such that they are isomorphic (informally speaking, this means that they have the same incidences). In this case we say that $\bar{\mathcal{C}}$ is a *geometric realization* of $\mathcal{C}$. Here we face the problem that combinatorial configurations are more abundant than geometric configurations; in other words, not every combinatorial configuration has a geometric realization. For example, the smallest combinatorial configuration is the (7_3) Fano configuration (also well known e.g. in finite geometry, where the term *Fano plane* is used). The second is the (8_3) Möbius-Kantor configuration. But neither of these can be realized geometrically, with straight lines.

Topological configurations represent an intermediate level: instead of lines, they have *pseudolines*, which are curves mimicking the lines of the real projective plane by their property that any two of them have a unique point of intersection. For a topological configuration which has a geometric realization, Grünbaum uses the term *stretchable*. Here we see again that not every topological configuration is stretchable.

The first enumeration results on 3-configurations occurred near the end of the 19th century, mainly due to Daublebsky von Sterneck, Kantor, Martinetti and Schröter. In more recent research, computer programs are used. For example, Sturmfels and White in 1990 [30], using methods from computer algebra, confirmed the result of Daublebsky who stated that there are 228 nonisomorphic types of combinatorial (12_3) configurations. Nearly a century after Daublebsky's result, Gropp pointed out that there is one additional type; Sturmfels and White also confirmed this, and now the list with 229 types is complete. Grünbaum went on and posed the question whether every geometric configuration has a planar representation for which all point and line coordinates are rational. This is answered in the affirmative for (12_3) configurations, also by Sturmfels and White. We note that this work continues also at present: by a most recent result, all the 2036 combinatorial (13_3) configurations found by Gropp [23] have rational geometric realization (cf. the paper of Kocay in this issue [26]). It seems that at present this is the most that is known about realizability of (n_3) configurations; besides, it also supports Conjecture 2.6.1 by Branko Grünbaum formulated in his book as follows:

Conjecture. *Every 3-connected combinatorial 3-configuration admits geometric realizations by points and straight lines with no incidences except the required ones.*

In addition, a list of the numbers of combinatorial (n_3) configurations is also known up to $n = 19$, and is included in Table 2.2.1, together with data obtained by various computational methods, among others by Betten, Brinkmann and Pisanski [8].

Four sections in the last part of Chapter 3 give an introduction to astral configurations, a remarkable class of highly symmetric objects of plane geometry. Symmetry is meant here as Euclidean symmetry; that is, the symmetry group of a configuration is the group of Euclidean isometries that map the configuration to itself (this does not conflict with the fact that configurations are sometimes considered as embedded into the projective plane, as it is explained in Section 1.5). Since a configuration consists of finitely many points and lines, this group can only be either the dihedral group $\mathbf{D}_m$ or the cyclic group $\mathbf{C}_m$ (here we use the old Schönflies group notation). An (n_k) configuration may have either of these groups as its symmetry group. If, in addition, both its set of points and set of lines decomposes into $h = \lfloor (k+1)/2 \rfloor$ orbits under the action of this group, then it is called a h-astral configuration. This definition has already been introduced also in Section 1.5, together with some refinements in several directions. Here astral 3-configurations with cyclic as well as with dihedral symmetry groups are studied, in two separate sections. Multiastral configurations (i.e. those where the number of orbits of points and of lines is not specified) are also presented (and this is the one of several places of the book where an important and closely related family called *polycyclic configurations* [10] is mentioned as well). Finally, some duality problems of astral 3-configurations are discussed.

The title of the next chapter is "4-Configurations", and ten sections are devoted to the subject. The gaps in our knowledge compared to 3-configurations is emphasized directly at the beginning. In Theorem 3.2.3 the result of Bokowski and Schewe [14] is cited by which there are no geometric (n_4) configurations for $n \leq 17$. In the same paper the authors also provided the first example of a geometric (18_4) configuration. (We note that here the contrast between combinatorial and geometric configurations is striking: there are precisely 971171 isomorphism classes of combinatorial (18_4) configurations!) However, the question of the next case, (19_4), still remained open. In fact, not long after publication of Branko Grünbaum's book it was proved by Bokowski and Pilaud that geometric (19_4) configuration does not exist [13]. Just in that period, the research for small geometric 4-configurations was particularly active. The current state of knowledge can briefly be summarized as follows: geometric (n_4) configurations exist for all $n \geq 18$ except possibly $n = 23$; the existence of (23_4) is still undecided. A brief summary of how this knowledge was acquired step by step, due to the works of Jürgen Bokowski, Michael Cuntz, Branko Grünbaum, Vincent Pilaud and Lars Schewe, can be found in the introductory part of the paper [6] in this issue; some other details of the story are also referred to (besides the original research articles) in the contribution [11], and in the newly published book by Bokowski [17].

Naturally, only the first part of this period is accounted on in the book under review. Even that account is a captivating reading with many interesting details, in particular with outlines of some proofs taken over from the original contributions.

Here an important characteristic of this research is to be stressed; it is the extensive use of computer-aided methods. On the one hand, this means the application of efficient techniques based on the theory of oriented matroids, mainly in the works of Bokowski and his co-authors. On the other hand, the computer is an indispensable tool in applying these techniques (for example, in the paper [12] the authors even mention that obtaining one of their results needed several months of CPU-time). In the present book Branko Grünbaum does not go into such details (it would fill another book); instead, he refers to relevant sources already in Chapter 1, regarding in particular the theoretical basis of Jürgen Bokowski's methods [16, 15].

In a next section of this chapter a collection of operations is described by which one can build new configurations from old ones. These are ingenious constructions with clever application of various isometric, affine and projective transformations. As in general in the book, they are also illustrated by spectacular examples. The set of these operations has been proved to be a valuable toolkit for constructing many new (n_k) configurations, even beyond the case of $n = 4$, so that later it is presented under the name *Grünbaum Incidence Calculus* in the monograph by Tomaž Pisanski and Brigitte Servatius [29]. It has already been applied in the very recent contribution [6] in proving results for the existence of (n_k) configurations; in particular, now we know that there is a bound such that for any $n \geq 576$ there exists a geometric (n_5) configuration. (Here we note that the smallest known (n_5) example is (48_5) [7]; in the time of writing Branko Grünbaum's book it was so new that it was cited there as a private communication, and only depicted in Figure 4.1.5. The question of existence of a smaller example is still open, and it is considered so important that e.g. Jürgen Bokowski puts it in his book as one of the "beautiful questions" in geometry, and devotes to it a small section [17, Section 2.8]; we also note that finding a smaller example would certainly reduce the bound 576 mentioned above.)

The second part of Chapter 3 deals again with astral configurations, more closely, with a particularly interesting subclass of them. This class is distinguished nowadays by the term *"celestial configurations"*, although the author in the book only mentions this name. In fact, it occurred for the first time in the work by Leah Berman [3] (former PhD student of Branko Grünbaum). Since this class has been studied very extensively, and is the most well-understood class of 4-configurations, it deserves citing its definition here. A h-astral

(n_4) configuration $\mathcal{C}$ is called *celestial* if the following conditions hold: (1) $h \geq 2$ and $n = h \cdot m$ for some $m \geq 7$; (2) the relative position of the h orbits of points of $\mathcal{C}$ is such that all angles subtended by these points from the centre of $\mathcal{C}$ are multiples of π/m (recall that since the symmetry group of $\mathcal{C}$ is finite, it has a unique common fixed point; thus the centre of $\mathcal{C}$ is naturally identified with this point); (3) each line of $\mathcal{C}$ contains two points from each of two point orbits, and likewise, each point is incident with two lines from each of two line orbits.

The particular action of the symmetry group determined by the third condition makes this class a really remarkable subject of study. Indeed, as it is emphasized in Section 3.5, we have an "easily implementable decision algorithm for checking the membership of either a given configuration to the class, or of a symbol for correspondence to a geometric configuration". The symbol mentioned here is called a "configuration symbol", and it is subject to certain axioms. All these properties are widely utilized in the subsequent sections, separately on 2-astral, 3-astral and k-astral $(k \geq 4)$ configurations. Moreover, the research in this direction continues later on, see e.g. [3, 1].

The title of Chapter 4 is "Other Configurations". Here the author overviews what is known on k-configurations for $k \geq 5$ and on unbalanced configurations. The simple reason that all these configurations can be reviewed in a single chapter (in contrast to the case of 3- and 4-configurations) is the "paucity of knowledge" in this case, as the author admits at the beginning of the chapter.

Some combinatorial aspects are also mentioned here (related to 5-configurations), but to a much lesser extent than in the previous chapters. For example, an interesting combinatorial property is being cyclic. A (p_q, n_k) configuration is *cyclic* if its points can be identified with the elements of the (abstract) cyclic group $\mathbb{Z}_p$ and its set of lines $\mathcal{L}$ is invariant under the action $x \mapsto (x+1) \bmod p$ (in case of a combinatorial configuration $\mathcal{L}$ is the set of abstract lines, or *blocks*). As it is noted in Section 2.1, study of configurations with this property goes back to Levi [27]. Here the contrast between the combinatorial and geometric side of configurations is even more striking: while cyclic combinatorial configurations are a subject of intensive research (see e.g. the recent paper [18] and the references therein), only a few papers deal with cyclic geometric configurations (as a more recent example, see [4]).

Similar differences can be seen in case of unbalanced configurations. For example, there are enumeration results from which we know that the number of nonisomorphic types of combinatorial $(15_6, 30_3)$ configurations is 10177328 (see e.g. Table 7.18 in [24] with data also for other types therein). Such results for geometric configurations appear rather rarely; here a paper by Leah Berman is cited, which reports, among other results, the determination of many $[6, 4]$ configurations (these are configurations in which each point is on six lines and each line passes through four points). Some infinite sequences of unbalanced configurations are also known, obtained mainly by combining the construction methods described earlier.

The situation is even worse regarding (n_k) configurations with $k \geq 6$; more precisely, it was so in the time of writing the book. In Section 4.2, dealing with these configurations, the author explicitly complains about the lack of relevant contributions. First he mentions an interesting observation (which occurs at several places in the book, first in Section 1.1) that geometric $((k^k)_k)$ configurations exist for all k (some graph-theoretical aspects of these configurations are discussed by Tomaž Pisanski [28], where they are called the "generalized

Gray configurations"). As a consequence, we know that for an arbitrary integer k there exists a k-configuration.

But it turns out that the same is known due to a much older observation by Cayley. Indeed, Cayley pointed out in 1846 that for all binomial coefficients of the form $B^{(k)} = \binom{2k-1}{k-1}$ with $k \geq 3$ there exists a $((B^{(k)})_k)$ configuration (we note that for $k = 3$ this is precisely the well-known Desargues configuration). (Much later, Cayley's idea was rediscovered, independently, by Ludwig Danzer, and has been elaborated in more detail by Boben, Gévay and Pisanski [9, 20].)

In the same section on (n_k) configurations with $k \geq 6$ results only from one additional contribution are mentioned [2]. One of them is the theorem stating that no 3-astral 6-configuration exists.

Some examples are also mentioned demonstrating how the "$(5m)$ construction" introduced in Section 3.3 can be generalized so that starting from any (m_k) configuration, it yields a $(((k+2)m)_{k+1})$ configuration. An interesting example is (880_7) obtained from a (110_6) configuration, presented here as the smallest known 7-configuration. We note that not much later this record was improved by reporting the existence of a (288_7) configuration [5]. Together with an example of type (96_6) reported in the same paper, it is still the known record-holder. Thus again, here one may put the following "beautiful question" (in the spirit of the book [17]).

Question. Does there exist an (n_6) configuration for $n < 96$, or an (n_7) configuration for $n < 288$?

As we noted previously, the $(5m)$ construction (and its generalization, utilized also in [6]) forms part of the "Grünbaum calculus". At the time of writing the book under review, this construction was the only known operation that can be applied to any configuration (n_k) for increasing the incidence number k. But not much later, a new binary operation was introduced, namely, the Cartesian product. This can be applied to any two (balanced or unbalanced) configurations, with the only restriction that the point/line incidence number must be same in both configurations; hence, to be precise, this is a *partial operation* (this term borrowed from universal algebra). It was defined independently for combinatorial configurations by Pisanski and Servatius [29], and for geometric configurations by Gévay [19].

By this operation, it is easy to obtain highly incident configurations, in fact those in which the incidence numbers exceed any bound; but in searching for minimal examples (such as in the question above) it can hardly be used, since the number of points grows very fast in comparison to the incidence number k. (A similar drawback occurs, to a lesser extent, for the generalization of the $(5m)$ operation as well.) Thus, novel clever and sophisticated methods are still badly needed, and hopefully, the wealth of ideas in the book by Branko Grünbaum will give inspiration in this direction as well.

The next two sections are concerned with the results of (at the time) quite recent research on "floral" configurations and topological configurations; both form promising subjects of further study.

The last section is on "unconventional configurations", necessarily a narrow selection from a large topic (that is essentially beyond the scope of this book). The general formula (p_q, n_k) giving the numbers of the elements and the mutual incidence numbers still applies here, precisely in an analogous way as in the case of configurations of points and lines; but instead of lines some other geometric constituents (e.g. circles, or planes, etc.) occur in the given structure. On the other hand, if lines are retained as second constituents, then one may consider configurations consisting of infinitely many points and lines.

In the first case, the examples are restricted to configurations of points and circles, and the beginnings are emphasized. Indeed, this topic goes back to the nice incidence theorem due to August Miquel (1844), which gives rise to a configuration of type $(8_3, 6_4)$. As a famous example, the infinite sequence found by Clifford in the second half of the 18th century is also mentioned; this consists of point-circle configurations, all related to incidence theorems. We note that more recent results on configurations of points and circles, together with further details on the Clifford configurations, appear in the paper by Gévay and Pisanski [22]. Moreover, in a quite recent paper other "unconventional configurations", namely, configurations of points and conics are also studied [21].)

The last chapter is entitled "Properties of Configurations", and covers eight distinct topics. Each one is so interesting that it would deserve a separate paper, but here we restrict ourselves to mention some details on one of them, the *dimension* of a configuration.

All configurations throughout the book are "planar" in the sense that they considered as embedded in either the Euclidean or in the projective plane. This may give the false illusion that they are confined in fact to one of these planes. However, closer scrutiny shows that some of these configurations, by an isomorphic but different arrangement of their points and lines, are able to span a space of dimension higher than two. The precise definition is the following. We say that a configuration $\mathcal{C}$ has dimension d if this is the largest integer for which $\mathcal{C}$ admits a geometric representation (by points and straight lines) in some Euclidean space, such that the affine hull of the embedding has dimension d.

Recall the well-known fact that the (10_3) Desargues configuration can be constructed in such a way that it spans a 3-dimensional space. In this section the author gives a short proof that the dimension of this configuration is 3. A theorem is also proved which states that there exist 3-configurations with arbitrary large dimensions. We note that the construction used in this proof is essentially a spatial version of one of those in the Grünbaum calculus and which is called "parallel switch" in [29]. This raises the question of the effect of various operations mentioned earlier on the dimension of the configurations involved. (The same question can be put related to the numerous spatial constructions given in [19], including the Cartesian product.)

The book is completed by a particularly extensive list of references, with back-references to pages of occurrence. We reproduced here some of them intentionally, and added some more recent ones. With the latter, and also with our remarks regarding the later developments, our aim was to indicate that the research in many directions presented in the book continues unbroken in the time elapsed since its publication. We are certain that it will be even more so in the future, and many research mathematicians will draw inspiration from it. Besides, we warmly recommend it to anybody who is delighted by the beauty of Geometry.

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