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A new perspective on taxicab conic sections

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Abstract

We explore taxicab conic sections from the perspective of slicing taxicab cones by planes, as opposed to the more well-studied approach from the perspective of distance formulations. After establishing some structural framework, a complete characterization of the resulting taxicab conic sections is established, and a number of special cases are explored.

Keywords: Taxicab geometry, Conic Sections

Math. Subj. Class.: 51M05, 51M15

1 Introduction

Taxicab conic sections have been well studied from the perspective of distance formulations involving two-foci and focus-directrix definitions. This is a fruitful exercise and a number of interesting shapes and special cases emerge. See [1, 2, 4, 5, 6] for examples of this type of analysis, and see Figure 1 for the variety of shapes that can arise from these definitions.

While fairly natural, this process turns out to be somewhat separate from the eponymous definition, so in this paper, the notion of a taxicab conic section is explored from the perspective of slicing cones. We define a cone to be the set of points whose distance to a given line ℓ is a multiple of its distance to a given plane P:

$$C(\ell, P, \kappa) = \{x \in \mathbb{R}^3 : d(x, \ell) = \kappa\, d(x, P)\}.$$

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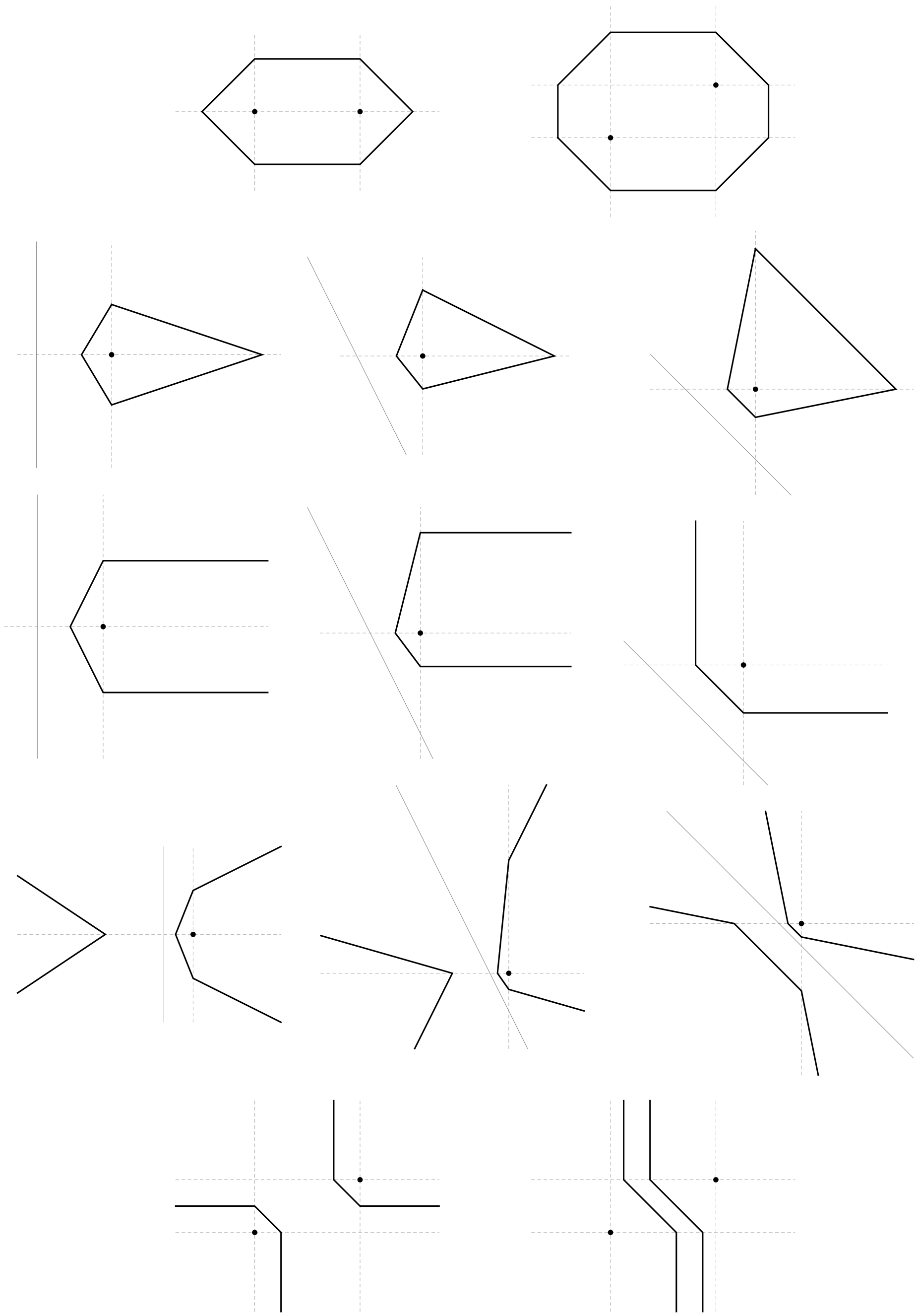


Figure 1: Some taxicab conic sections using the distance definitions. First and last rows: ellipses and hyperbolas using the two-foci definition; second, third, and fourth rows: ellipses, parabolas, and hyperbolas using the focus-directrix definition. The foci and directrices, are also shown.

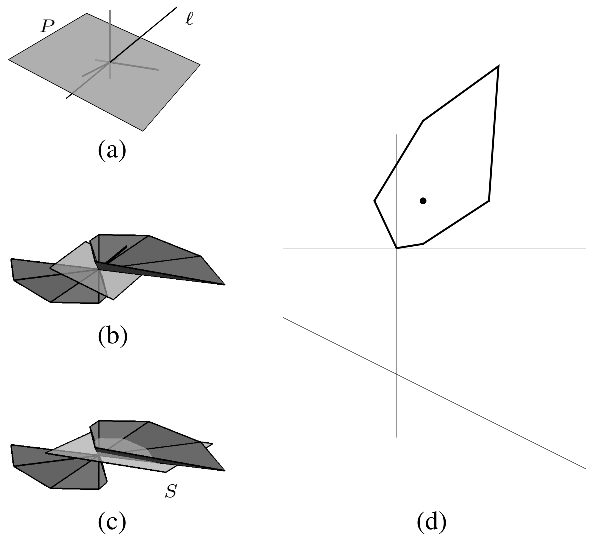


Figure 2: Given a line ℓ and plane P (a), a cone is produced (b). This cone is sliced by a coordinate plane S (c), and the resulting intersection is represented in $\mathbb{R}^2$ (d). In the final image, the point inside the conic section and the line below it indicate the intersections of ℓ and P with S respectively.

To ensure that our conic sections are objects in taxicab two-space, we restrict our slicing plane to be a coordinate plane, while allowing the cone to vary. See Figure 2 for a representation of the process under consideration.

Taxicab conic sections via sliced cones have been explored in [5]. In that setting, a single cone, defined using one coordinate axis for ℓ and the complementary coordinate plane for P, is used. An arbitrary slicing plane is chosen, and the resulting intersection is projected onto P. This process reproduces the conic sections defined using the focus-directrix method. The exploration in this paper considers more general cones and eliminates the need to project since the slicing plane will already be a coordinate plane.

The nature of the taxicab distance leads to objects that are piecewise linear and, while a purely algebraic description could be established, this would prove to be rather unenlightening. Our goal here is to develop the results more geometrically. A complete characterization of the conic sections comprises three main steps. In Theorems 4.1 and 5.1 we compute the vertices of a conic section. Then, Theorems 4.2 and 4.4 show how to connect these vertices by segments and rays to form the conic section. Finally, Theorems 4.6 and 5.3 characterize when the resulting conic sections are ellipses, parabolas, or hyperbolas. Two theorems are needed for each step because the analysis when ℓ is horizontal is somewhat different from when it is not horizontal.

This paper is organized as follows: in Section 2 we introduce the taxicab distance in $\mathbb{R}^2$ and $\mathbb{R}^3$, and discuss a number of its properties. In Section 3 we discuss taxicab cones, paying special attention to strategies for measuring the distance between a point and a plane or a point and a line. While technical, this leads to insight into how to think about the set of parameters that best characterize planes and lines in taxicab space. Also in this section, we establish our conventions about the slicing plane and discuss its many roles. The analysis of conic sections themselves when the defining line is non-horizontal and when it is horizontal are in Sections 4 and 5 respectively. This analysis leads to the discovery of rich geometric relationships that in turn provide useful shortcuts for drawing the wide variety of conic sections that arise. Finally, with methods for producing and understanding conic sections established, we explore some special cases in Section 6.

2 The taxicab distance

In this section, we establish our notation and review some basic facts about the taxicab distance in $\mathbb{R}^2$ and $\mathbb{R}^3$ that motivate many of the choices that are useful for our analysis.

2.1 Definitions

The taxicab distance between points $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$ in $\mathbb{R}^n$ is

$$d(x, y) = |x_1 - y_1| + |x_2 - y_2| + \dots + |x_n - y_n|.$$

More generally, the distance between a point x and set T is

$$d(x, T) = \inf_{y \in T} d(x, y).$$

We will also want to measure partial distances between a point x and set T, so we define

$$d_i(x, T) = \inf_{y \in T : y_j = x_j \forall j \neq i} |x_i - y_i|$$

and

$$d_{i,j}(x, T) = \inf_{y \in T : y_k = x_k \forall k \neq i,j} |x_i - y_i| + |x_j - y_j|.$$

A sphere centered at the point y with radius $r \geq 0$ in $(\mathbb{R}^n, d)$ is

$$\sigma_r(y) = \{x \in \mathbb{R}^n : d(x, y) = r\}.$$

When $n = 2$, this is a square with vertices on the coordinate lines passing through the center. When $n = 3$, this is an octahedron with vertices on the coordinate lines passing through the center. When $n = 3$, the subsets of $\sigma_r(y)$ determined by restricting to a coordinate slice passing through y will be useful, and we call these three sets *great circles* of the taxicab sphere. Finally, the particular sphere $\sigma_1\big((0, 0, 1)\big)$ will make an appearance regularly and we denote it simply as σ.

2.2 Isometries

The group of isometries of two- and three-dimensional taxicab space that fix a point are both finite, unlike for Euclidean space. While a full justification takes a bit of work, it is not surprising that the group of isometries that fix a point in the taxicab plane is isomorphic to the symmetry group of a square [3, 7], while for taxicab 3-space, it is isomorphic to the symmetry group of an octahedron.

For the purposes of this paper, it is enough to know that arbitrary translations, rotations about coordinate lines by multiples of $\frac{\pi}{2}$, and reflections across coordinate planes are all isometries. The richer structure of the isometry group for $(\mathbb{R}^3, d)$ will not be necessary.

2.3 Subspaces and induced distances

The taxicab distance in $\mathbb{R}^3$ naturally induces a taxicab distance on coordinate planes. The induced distance on planes containing one coordinate line is a taxicab distance but appropriate coordinates need to be chosen to see this. These coordinates cannot be extended to coordinates in $\mathbb{R}^3$ that are related isometrically to the ambient coordinates since there is no isometry that moves a non-coordinate plane to a coordinate plane. Planes containing no coordinate lines do not inherit a taxicab distance at all. For example, see [8] for an analysis of the induced distance on the plane containing the points $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$.

In light of these issues, we make the choice to restrict to coordinate planes when we want the geometry in that plane to be a taxicab geometry that is consistent with the ambient taxicab structure. As such, we do not restrict the planes that define the cone, but we do restrict the planes that slice the cone to produce a section. These issues are also what motivate our definition of great circles.

2.4 Angles

While various notions of angle could be introduced, none enjoy all the properties and relationships we are used to in Euclidean geometry since the taxicab distance does not arise from an inner product. See [9] for further discussion of this. In this paper, we avoid the use of angle for any of the constructions performed. In the Euclidean setting, while a circular cone can be defined in terms of a plane and a line that are perpendicular to one annother, and relaxing the perpendicularity constraint does change the shape of the cone, it does not increase the types of objects that might be called conic sections. The cones explored in this paper will be of this more general type.

3 Taxicab cones

In this section, we develop the structure necessary to understand the cones under consideration. This involves a careful analysis of the distance between a point and a plane and the distance between a point and a line. We also establish conventions for our slicing plane, and discuss its many uses.

3.1 Cones

Let ℓ be a line and let P be a plane where, to avoid degenerate cases, $\ell \cap P$ consists of a single point. Also, let $\kappa \in (0, \infty)$. We define a cone to be

$$C(\ell, P, \kappa) = \{x \in \mathbb{R}^3 : d(x, \ell) = \kappa\, d(x, P)\}.$$

We call ℓ the defining line and P defining plane for the cone. Without loss of generality, we restrict our attention to the cases where $\ell \cap P = \{(0,0,0)\}$.

We represent the defining plane by choosing $\boldsymbol{A} \in \mathbb{R}^3 \backslash \{\boldsymbol{0}\}$ as follows:

$$P = P_{\boldsymbol{A}} = \{y \in \mathbb{R}^3 : A_1 y_1 + A_2 y_2 + A_3 y_3 = 0\}.$$

We represent the defining line by choosing $\{\boldsymbol{a} \in \mathbb{R}^3 \backslash \{\boldsymbol{0}\}$ such that

$$\ell = \ell_{\boldsymbol{a}} = \{\boldsymbol{a}t : t \in \mathbb{R}\}$$

and we denote the point corresponding to the value t by $\ell(t)$.

With these conventions in place, we note that a cone is degenerate if and only if $A_1 a_1 + A_2 a_2 + A_3 a_3 = 0$.

3.2 The slicing plane

As mentioned above, we choose to restrict the slicing plane to be a coordinate plane and without loss of generality, we restrict our choice of slicing plane further to the plane $S = \{x \in \mathbb{R}^3 : x_3 = 1\}$. While this choice is arbitrary, we will use it to establish terminology that will clarify the variety of cases under consideration. In particular, we say the slicing plane, and any plane parallel to it, is horizontal.

3.3 The defining plane

The formula for the distance between a point and a plane motivates parameter spaces for planes that capture some of the emergent geometric features.

3.3.1 Distance between a point and a plane

Theorem 3.1. *Given a point x and a plane $P = P_{\boldsymbol{A}}$,*

$$d(x, P) = \frac{|A_1 x_1 + A_2 x_2 + A_3 x_3|}{\max\{|A_1|, |A_2|, |A_3|\}}. \tag{3.1}$$

Proof. Let r be the smallest radius such that $\sigma_r(x)$ makes contact with the plane. At this radius, the point(s) of contact will include a vertex. The distance from x to that vertex is the distance to the plane so

$$d(x, P) = \min_{i \in \{1,2,3\}} d_i(x, P).$$

Let y^i be the point in the plane sharing two coordinates with x, and where the i coordinate differs, noting that if $A_i = 0$, then this point does not exist and the corresponding partial distance is infinite. Then, since $y^i \in P$,

$$A_1 y_1^i + A_2 y_2^i + A_3 y_3^i = 0$$

so that

$$y_i^i = \frac{-A_j y_j - A_k y_k}{A_i} = \frac{-A_j x_j - A_k x_k}{A_i}$$

and so

$$d(x, y^i) = |x_i - y_i^i| = \frac{1}{|A_i|} |A_1 x_1 + A_2 x_2 + A_3 x_3|.$$

From this, it follows that the distance to the plane is given by Equation (3.1). □

In light of this result, we say a plane is shallow if $\max\{|A_1|, |A_2|, |A_3|\}$ is $|A_3|$ alone, we say a plane is transitional if $\max\{|A_1|, |A_2|, |A_3|\}$ is $|A_3|$ along with at least one of $\{|A_1|, |A_2|\}$, and we say a plane is steep otherwise. In particular, we say a plane is vertical if $A_3 = 0$.

3.3.2 Parameter space for defining planes

Since the set of planes passing through the origin can be identified with the projective plane $\mathbb{R}P^2$, the cube

$$\mathscr{P} = \{\boldsymbol{A} \in \mathbb{R}^3 : \max\{|A_1|, |A_2|, |A_3|\} = 1\}.$$

is a natural double cover so that, in light of Theorem 3.1, for $\boldsymbol{A} \in \mathscr{P}$,

$$d(x, P_{\boldsymbol{A}}) = |A_1x_1 + A_2x_2 + A_3x_3|.$$

Moreover, thinking again about the sphere centered at x that makes first contact with a given plane, the open faces of $\mathscr{P}$ correspond to planes which make contact with this sphere at just a vertex, the edges of $\mathscr{P}$ correspond to planes that make contact along an edge of the sphere, and the vertices of $\mathscr{P}$ correspond to the planes that make contact along an entire face of the sphere. Additionally, the face of the cube in which a given parameter lies identifies the vertex that will make first contact.

In $\mathscr{P}$, shallow planes correspond to the top and bottom faces, transitional planes correspond to the top and bottom square edges, and steep planes correspond to the sides.

3.3.3 An alternative parameter space

If a plane $P_{\widetilde{\boldsymbol{A}}}$ is not vertical, we can choose an alternative parameterization by dividing the parameter vector by $\widetilde{A}_3$. This is a gnomonic projection of sorts for the upper half of $\mathscr{P}$. The map is $p\colon \mathscr{P} \cap \{x \in \mathbb{R}^3 : x_3 > 0\} \to \mathbb{R}^3$, $(A_1, A_2, A_3) = p(\widetilde{\boldsymbol{A}}) = \left(\frac{\widetilde{A}_1}{\widetilde{A}_3}, \frac{\widetilde{A}_2}{\widetilde{A}_3}, 1\right)$ and provides us with a unique parameter vector for each non-vertical plane.

In this setting, vertical planes can be parameterized by a circle at infinity, represented by nonzero vectors with $A_3 = 0$. The fact that this does not produce a unique parameter for each plane does not significantly impact the analysis. As such, we have the alternative parameter space

$$\mathscr{P}' = \{\boldsymbol{A} \in \mathbb{R}^3 : A_3 = 1\} \cup \{\boldsymbol{A} \in \mathbb{R}^3 \backslash \{0\} : A_3 = 0\}.$$

We will find that this alternative parameter space will serve well in light of our choice of a horizontal slicing plane. In this setting, P is shallow if $\boldsymbol{A}$ lies in the open central square $(-1, 1) \times (-1, 1)$, transitional if $\boldsymbol{A}$ lies in the boundary of this square, and steep otherwise. See Figure 3 for visualizations of these parameter spaces.

3.4 The defining line

We find that the computation of the distance between a point and a line proves to be somewhat more complicated than the distance between a point and a plane. After introducing some technical facts about sums of absolute values, we then explore the geometry in order to establish a general method for computing the distance between a point and a line. Finally, we use this to develop appropriate parameter spaces for lines.

3.4.1 Sums of absolute values

Given a set of numbers $a_1, \ldots, a_N$, we say a_j dominates the other values if

$$|a_j| > \sum_{i=1, i\neq j}^{N} |a_i|$$

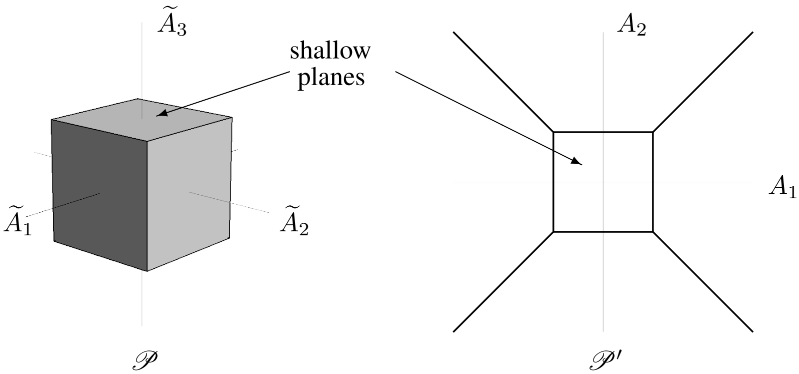


Figure 3: Two parameter spaces for planes. The cube $\mathscr{P}$ is a natural choice because of the way distance between a point and a plane is measured. The planar parameter space $\mathscr{P}'$ is a good choice in the context of conic sections because of our choice of slicing plane. Note that for $\mathscr{P}'$ the circle at infinity is not shown.

and we say a_j transitionally dominates the other values if

$$|a_j| = \sum_{i=1, i\neq j}^{N} |a_i|.$$

We will find that measuring the distance to a line $\ell_{\boldsymbol{a}}$ requires knowing whether or not a component of the parameter $\boldsymbol{a}$ dominates the other components. The following technical lemma, the proof of which is left to the reader, captures the essential reason for this.

Lemma 3.2. *Let*

$$f(t) = \sum_{i=1}^{N} |b_i - m_i t| = \sum_{i=1}^{N} |m_i| \left| \frac{b_i}{m_i} - t \right|$$

where the m_i are all nonzero. Note that if $\frac{b_i}{m_i} = \frac{b_j}{m_j}$, the corresponding terms can be merged. Wth this in mind, suppose, without loss of generality, that the indexing is such that the $\frac{b_i}{m_i}$ are in strictly increasing order. Then:

- *If for all $j \in \{1, \ldots, N-1\}$*

$$\sum_{i=1}^{j} |m_i| - \sum_{i=j+1}^{N} |m_i| \neq 0,$$

then the critical points for f are exactly the points $t = \frac{b_i}{m_i}$.

- *If there exists a (necessarily unique) $j \in \{1, \ldots, N-1\}$ such that*

$$\sum_{i=1}^{j} |m_i| - \sum_{i=j+1}^{N} |m_i| = 0,$$

then the critical points for f are exactly the points $t = \frac{b_i}{m_i}$ and the interval $\left[\frac{b_j}{m_j}, \frac{b_{j+1}}{m_{j+1}}\right]$.

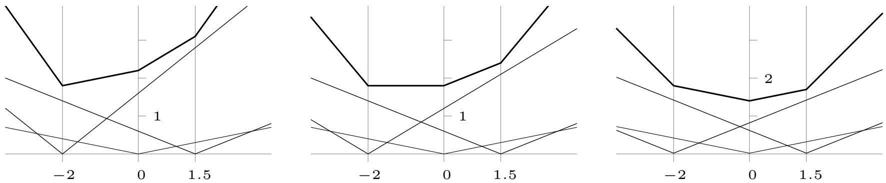


Figure 4: Graphs of $f(t) = |b_1 - m_1 t| + |b_2 - m_2 t| + |b_3 - m_3 t|$ where the darker function f is the sum of the three lighter functions. In all three cases, $b_2 = 0$, $m_2 = .2$, $b_3 = .6$, and $m_3 = .4$. In (a), $b_1 = -1.6$ and $m_1 = .8$, m_1 dominates the other two slopes, and the global minimum occurs at $\frac{b_1}{m_1}$. In (b), $b_1 = -1.2$ and $m_1 = .6$, m_1 transitionally dominates the other two slopes, and the global minimum is achieved along the interval $\left[\frac{b_1}{m_1}, \frac{b_2}{m_2}\right]$. In (c), $b_1 = -.8$ and $m_1 = .4$, there is no dominant or transitionally dominant slope and the global minimum is achieved at the middle critical point.

- *The function f is convex and as such has no local maxima. It achieves one local, and hence global, minimum value at either a unique point or an interval.*
- *Let j be the index where*

$$\sum_{i=1}^{j-1} |m_i| < \sum_{i=j}^{N} |m_i|$$

and

$$\sum_{i=1}^{j} |m_i| \geq \sum_{i=j+1}^{N} |m_i|.$$

Then the local minimum occurs at $\frac{b_j}{m_j}$ or, in the case of equality in the second line, on the interval $\left[\frac{b_j}{m_j}, \frac{b_{j+1}}{m_{j+1}}\right]$.

- *Suppose for some $j \in \{1, \ldots, N\}$, m_j dominates the other m_i. Then the global minimum is realized at $t = \frac{b_j}{m_j}$.*
- *In the special case where $N = 3$,*
  - *if there is no dominant slope, then the minimum occurs the middle value in the set $\left\{\frac{b_1}{m_1}, \frac{b_2}{m_2}, \frac{b_3}{m_3}\right\}$;*
  - *if m_j is transitionally dominant, then the minimum also occurs at $\frac{b_j}{m_j}$, and if this value is not the middle value, then the minimum occurs along an interval.*

See Figure 4 for an illustration of the main results from Lemma 3.2.

3.4.2 Distance between a point and a line

Given a point x and a line $\ell = \ell_{\boldsymbol{a}}$,

$$d(x,\ell) = \inf_{t\in\mathbb{R}} d(x,\ell(t)) = \inf_{t\in\mathbb{R}} |x_1 - a_1 t| + |x_2 - a_2 t| + |x_3 - a_3 t|.$$

By Lemma 3.2, the infimum is realized at (at least) one of the values $t = \frac{x_1}{a_1}$, $t = \frac{x_2}{a_2}$, and $t = \frac{x_3}{a_3}$, and the particular minimizing value depends on whether or not one of the components a_i dominates the others. Geometrically, we can think about the distance to ℓ in a way that is similar to that for a plane. Specifically, let r be the minimum radius such that $\sigma_r(x)$ makes contact with ℓ. In this case, the first contact will involve a great circle of $\sigma_r(x)$. As such, the point on ℓ that is closest to x will always share at least one coordinate with x:

$$d(x,\ell) = \min\{d_{1,2}(x,\ell), d_{1,3}(x,\ell), d_{2,3}(x,\ell)\}.$$

We seek a geometric way to determine which critical point minimizes the given function. Let P^i be the plane containing the ith coordinate axis and ℓ:

$$P^i = \{x \in \mathbb{R}^3 : a_k x_j = a_j x_k \text{ where } i \neq j \neq k\}.$$

Note that if ℓ is a coordinate line, the corresponding plane is not uniquely defined, and in fact we leave it undefined in this case. Suppose all three components of $\boldsymbol{a}$ are nonzero. These three planes all contain ℓ, and as such, they subdivide $\mathbb{R}^3$ into six wedge-shaped regions. Let W^i be the union of the two wedges, including the boundary planes, that do not have P^i as part of their boundary.

Lemma 3.3. *Suppose the components of $\boldsymbol{a}$ are all nonzero. The point x is an element of W^i if and only if $\frac{x_i}{a_i}$ is the middle value or a duplicated value in the set $\left\{\frac{x_1}{a_1}, \frac{x_2}{a_2}, \frac{x_3}{a_3}\right\}$.*

Proof. First, since P^i is defined by the equation $\frac{x_j}{a_j} = \frac{x_k}{a_k}$, duplicated values occur exactly on the planes P^i. For all other points, if the lemma is true for one point in W^i, then it must be true for all points in W^i since the inequalities defining the middle value cannot reverse without crossing one of the planes P^j.

Next, consider the transformation $\varphi\colon \mathbb{R}^3 \to \mathbb{R}^3$, $y = \varphi(x)$ where $y_i = \frac{x_i}{a_i}$. Note that $\frac{x_i}{a_i}$ is the middle value if and only if y_i is the middle value. Moreover the coordinate axes are preserved (although their orientation may be reversed) and $\varphi(\ell_{\boldsymbol{a}}) = \ell_{(1,1,1)}$. Hence, $\varphi(P^i) = Q^i$ where Q^i is the plane containing the y_i-axis and $\ell_{(1,1,1)}$.

Then confirm directly that points whose coordinates are permutations of 0, 1, and 2 establish the result. For example, the point $(1,0,2)$ lies in one of the wedges defined by Q^2 and Q^3, and the first coordinate is the middle value. □

With Lemma 3.3 we establish an explicit way to determine the distance between a point and a line.

Theorem 3.4. *Given a line $\ell = \ell_{\boldsymbol{a}}$ and a point x,*

- *if there is a permutation (i,j,k) of $(1,2,3)$ such that a_i dominates or transitionally dominates a_j and a_k, then $d(x,\ell) = d_{j,k}(x,\ell)$;*
- *otherwise when x lies in the double-wedge W^i, then $d(x,\ell) = d_{j,k}(x,\ell)$, where again (i,j,k) is a permutation of $(1,2,3)$.*

In either case,

$$d_{j,k}(x,\ell) = d\left(x, \ell\left(\frac{x_i}{a_i}\right)\right) = \left|x_j - \frac{a_j}{a_i}x_i\right| + \left|x_k - \frac{a_k}{a_i}x_i\right|. \tag{3.2}$$

To clarify the subtlety in this result, note that in the first case $\boldsymbol{a}$ determines which partial distance function to use independent of x, while in the second case $\boldsymbol{a}$ determines the double-wedges and then with that structure in place, the location of x determines which partial distance function to use.

Proof. Suppose first that there is a permutation (i,j,k) of $(1,2,3)$, such that a_i dominates or transitionally dominates the other two coordinates. Then by Lemma 3.2 the distance function $d(x,\ell(t))$ is minimized when $t = \frac{x_i}{a_i}$.

If there is no dominant or transitionally dominant coordinate of $\boldsymbol{a}$, then none of the coordinates can be zero and $d(x,\ell)$ is determined by the middle value or duplicated value in $\left\{\frac{x_1}{a_1}, \frac{x_2}{a_2}, \frac{x_3}{a_3}\right\}$. By Lemma 3.3, the middle or duplicate value is determined by the double-wedge in which x resides. If $x \in W^i$, the middle or duplicate value is $\frac{x_i}{a_i}$.

In either case, plugging in directly yields Equation (3.2). □

In light of Theorem 3.4, we say ℓ is *steep* if a_3 is the dominant coordinate, we say ℓ is *shallow* if a_1 or a_2 is dominant, we say ℓ is transitional if any coordinate is transitionally dominant, and we say ℓ is *intermediate* if there is no dominant or transitionally dominant value in the set $\{a_1, a_2, a_3\}$. Finally, we say ℓ is horizontal if $a_3 = 0$.

3.4.3 Parameter space for defining lines

As for planes, the set of lines through the origin can be identified with $\mathbb{R}P^2$ and in light of Theorem 3.4 a natural parameter space for lines is the cuboctahedron

$$\mathscr{L} = \partial\big\{\boldsymbol{a} \in \mathbb{R}^3 : \max\{|a_1|, |a_2|, |a_3|\} \leq 1 \text{ and } d(\boldsymbol{a}, 0) \leq 2\big\}.$$

In this setting, the steep lines correspond to the top and bottom square faces, the shallow lines correspond to the other square faces, the intermediate lines correspond to the triangular faces, and the transitional lines correspond to the edges and vertices

3.4.4 An alternative parameter space

If a defining line $\ell_{\widetilde{\boldsymbol{a}}}$ is not horizontal, we can choose an alternative parameterization by dividing the parameter vector by $\widetilde{a}_3$. As with the situation for the plane, this is similar to a gnomonic projection. Our map is $l\colon \mathscr{L} \cap \{x \in \mathbb{R}^3 : x_3 > 0\} \to \mathbb{R}^3$, $(a_1, a_2, a_3) = l(\widetilde{\boldsymbol{a}}) = \left(\frac{\widetilde{a}_1}{\widetilde{a}_3}, \frac{\widetilde{a}_2}{\widetilde{a}_3}, 1\right)$. See Figure 5. As with the alternate parameter space for planes, this space misses horizontal lines. Again, these are represented using a circle at infinity and we have

$$\mathscr{L}' = \{\boldsymbol{a} \in \mathbb{R}^3 : a_3 = 1\} \cup \{\boldsymbol{a} \in \mathbb{R}^3 \backslash \{0\} : a_3 = 0\}.$$

In this setting, ℓ is steep if $\boldsymbol{a}$ lies in the central square, shallow if $\boldsymbol{a}$ lies in one of the four quadrants, intermediate if $\boldsymbol{a}$ lies in a semi-infinite strip, and transitional if $\boldsymbol{a}$ lies in a boundary between these regions.

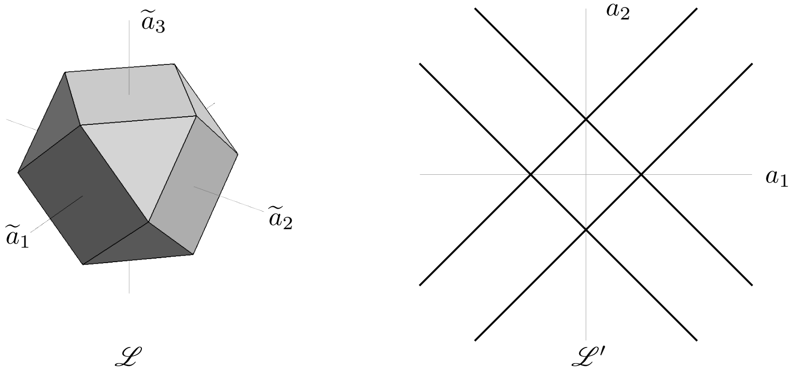


Figure 5: Two parameter spaces for lines. The cuboctahedron $\mathscr{L}$ is a natural choice because of the way distance between a point and a line is measured. The planar parameter space $\mathscr{L}'$ is a good choice in the context of conic sections because of how we will deal with the slicing plane. Note that for $\mathscr{L}'$ the circle at infinity is not shown.

3.5 The slicing plane revisited

The slicing plane S will serve a number of distinct but related purposes. First, it will be the plane that contains the conic section resulting from a given cone C. We denote the conic section by $C^S = C \cap S$. Second, as long as the defining line ℓ is not horizontal, the point $\boldsymbol{a} = \ell \cap S = (a_1, a_2, 1)$ can serve as the parameter for ℓ from $\mathscr{L}'$. As such, we will identify S with the set of non-horizontal parameters in $\mathscr{L}'$. Third, as long as the defining plane P is not horizontal, it will intersect S along a line. If $\boldsymbol{A} \in \mathscr{P}'$ defines P, then this line is

$$P^S = P \cap S = \{A_1 x_1 + A_2 x_2 + \delta = 0\},$$

which will prove to be a geometrically useful way to identify P. For example, a cone is degenerate if and only if $\boldsymbol{a} \in P^S$. Observe that P is steep if P intersects the open taxicab ball defined by σ, it is transitional if P intersects σ but not the open ball, and it is shallow otherwise. We can see this in S by looking at how P^S intersects $\sigma^S = \sigma \cap S$. Fourth, as long as P^i is defined and not horizontal, it will intersect S along a line ρ^i which we call a reference line. Note that ρ^1 and ρ^2 are the coordinate lines through $\boldsymbol{a}$ and ρ^3 is the line through $(0, 0, 1)$ and $\boldsymbol{a}$.

These lines identify transitions in the formula for the distance between a point and a line. In this role, they serve two subtly different purposes. First, when ℓ is intermediate, the double-wedges defined by ℓ determine which partial distance to use to measure the distance between a point and ℓ. Since the reference lines ρ^i are the boundaries of these double-wedges restricted to S, they identify transitions in which partial distance formula is to be used. In this case, we say all three reference lines are active. Second, if ℓ is not intermediate, the distance between a point and ℓ is determined by a single partial distance, and two of the reference lines indicate transitions in how the absolute value expressions in the partial distance resolve. When $d_{j,k}(x, \ell)$ is being used, the reference lines ρ^j and ρ^k indicate the transitions and we say these two lines are active, while the third is inactive.

Finally, when ℓ is horizontal, the point $\boldsymbol{a}$ does not lie in S, nor do ρ^1 or ρ^2. However, ρ^3 is defined, and serves as a representative for the point at infinity that parameterizes ℓ.

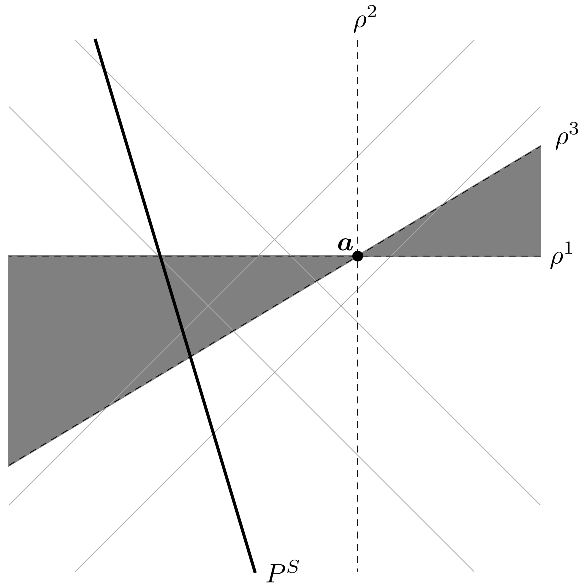


Figure 6: The slicing plane S and its intersections with ℓ at $\boldsymbol{a}$, P at P^S, the planes P^i at the reference lines ρ^i, and the double-wedges W^i (W^2 shaded). Using the identification of $\mathscr{L}'$ with S, the gray lines indicate transitionally dominant parameters. From this we can see that ℓ is intermediate since $\boldsymbol{a}$ lies in a semi-infinite strip, and P is shallow since P^S does not intersect the central taxicab circle σ^S.

3.5.1 Wedges revisited

While double-wedges W^i are defined whenever all three a_i are non-zero, they are most useful when ℓ is intermediate. If ℓ is not intermediate and only two reference lines are active, the corresponding planes divide $\mathbb{R}^3$ into four wedges similar to the six wedges described previously, and it should be understood that when wedges are discussed, the ones being used depend on whether or not ℓ is intermediate. We indicate this by referring to the wedges as active. The intersections of these 3-dimensional wedges and S are 2-dimensional wedges separated by active reference lines.

See Figure 6 for the various ways that information can be encoded in the slicing plane S.

4 Conic sections when ℓ is non-horizontal

With the structure developed above, we are now ready to understand and characterize conic sections. We explore the case where ℓ is not horizontal here. Using $\boldsymbol{A} \in \mathscr{P}'$, and $\boldsymbol{a} \in \mathscr{L}'$, we begin by finding the vertices of the conic section, which will lie on the active reference lines determined by $\ell_{\boldsymbol{a}}$. Once these vertices are known, the conic section will consist of line segments or rays connecting these vertices.

4.1 Vertices of the sections

In all cases, we are analyzing the equation $d(x, \ell) = \kappa\, d(x, P)$, where $x \in S$. For the distance to the plane, from Equation (3.1) we have

$$d(x, P) = \frac{|A_1 x_1 + A_2 x_2 + \delta|}{M}$$

where $\delta = 1$ if P is not vertical and 0 if P is vertical, and $M = \max\{|A_1|, |A_2|, \delta\}$.

For the distance to the line, from Equation (3.2) we have

$$\begin{aligned} d_{1,2}(x, \ell) &= |x_1 - a_1| + |x_2 - a_2|, \\ d_{1,3}(x, \ell) &= \left| x_1 - \frac{a_1}{a_2} x_2 \right| + \left| 1 - \frac{1}{a_2} x_2 \right|, \\ d_{2,3}(x, \ell) &= \left| x_2 - \frac{a_2}{a_1} x_1 \right| + \left| 1 - \frac{1}{a_1} x_1 \right| \end{aligned}$$

Measuring the distance between a point on a reference line and ℓ reduces to a single absolute difference, so determining the points in the conic section on the reference lines is relatively simple. Moreover, these points are important because they are the vertices of the conic section.

Theorem 4.1. *Given a cone $C(\ell_{\boldsymbol{a}}, P_{\boldsymbol{A}}, \kappa)$ where $\boldsymbol{a} = (a_1, a_2, 1)$ and $\boldsymbol{A} = (A_1, A_2, \delta)$, and slicing plane S,*

- *if the reference line ρ^1 is active, then the vertices lying on it are*

$$v^{1\pm} = \left(a_1 + \frac{A_1 a_1 + A_2 a_2 + \delta}{\pm \frac{M}{\kappa} - A_1}, a_2, 1 \right); \tag{4.1}$$

- *if the reference line ρ^2 is active, then the vertices lying on it are*

$$v^{2\pm} = \left(a_1, a_2 + \frac{A_1 a_1 + A_2 a_2 + \delta}{\pm \frac{M}{\kappa} - A_2}, 1 \right); \tag{4.2}$$

- *if the reference line ρ^3 is active, then the vertices lying on it are*

$$v^{3\pm} = (r^{\pm} a_1, r^{\pm} a_2, 1) \tag{4.3}$$

where

$$r^{\pm} = 1 + \frac{A_1 a_1 + A_2 a_2 + \delta}{\pm \frac{M}{\kappa} - (A_1 a_1 + A_2 a_2)}.$$

Proof. If ρ^1 is active, then on this line $d(x, \ell) = |x_1 - a_1|$ so the points in the conic section on ρ^1 are the solutions to

$$|x_1 - a_1| = \kappa \frac{|A_1 x_1 + A_2 a_2 + \delta|}{M}.$$

Resolving the absolute values results in two different equations, and the two solutions produce Equation (4.1). The calculation for the reference line ρ^2 is similar.

If the reference line ρ^3 is active, then without loss of generality, suppose $|a_1| \geq |a_2|$. Then on ρ^3, $d(x,\ell) = d_{2,3}(x,\ell) = \left|1 - \frac{1}{a_1}x_1\right|$ so the points in the conic section on this line are the solutions to

$$\left|1 - \frac{1}{a_1}x_1\right| = \kappa\,\frac{\left|A_1x_1 + A_2\frac{a_2}{a_1}x_1 + \delta\right|}{M}.$$

Resolving the absolute values results in two different equations for x_1. Once x_1 is found, x_2 can be computed resulting in Equation (4.3). □

If a vertex is undefined due to a zero in the denominator, we say the vertex lies at infinity.

Equations (4.1), (4.2), and (4.3) can be rewritten in terms of deviations from $\boldsymbol{a}$ as follows:

$$v^{1\pm} = \boldsymbol{a} + \left(\frac{A_1a_1 + A_2a_2 + \delta}{\pm\frac{M}{\kappa} - A_1}, 0, 0\right), \tag{4.4}$$

$$v^{2\pm} = \boldsymbol{a} + \left(0, \frac{A_1a_1 + A_2a_2 + \delta}{\pm\frac{M}{\kappa} - A_2}, 0\right), \tag{4.5}$$

$$v^{3\pm} = \boldsymbol{a} + \tilde{r}^{\pm}(a_1, a_2, 0) \tag{4.6}$$

where

$$\tilde{r}^{\pm} = \frac{A_1a_1 + A_2a_2 + \delta}{\pm\frac{M}{\kappa} - (A_1a_1 + A_2a_2)}.$$

4.2 Constructing the sections: connecting the dots

With the vertices known, the conic sections can be constructed by connecting certain vertices by straight lines. We first establish some terminology. We say two rays γ and λ are adjacent if together they are the boundary of an active two-dimensional wedge. We say two rays γ and η are anti-adjacent if η and λ form a line and γ and λ are adjacent. See Figure 7(a). Note that if there are only two active reference lines, then a pair of adjacent rays are also anti-adjacent.

We say two vertices are adjacent if they are on adjacent rays and they are on the same side of P^S. We say two vertices are anti-adjacent if they are on anti-adjacent rays and are on opposite sides of P^S. See Figure 7(b). If a vertex lies at infinity, it is interpreted as lying on both sides of P^S.

Let γ be the line through two points v and w. We say the line segment with endpoints v and w is the segment associated to v and w. We say the points in γ that are not part of the segment are the complementary rays associated to v and w. Given a point v and a line ρ, we say the two rays based at v and parallel to ρ are the rays associated to v and ρ. See Figure 7(c).

To more easily discuss the vertices and the lines they define, let $s_i \in \{+,-\}$ so that v^{is_i} is one of the two vertices on ρ^i. Also let $\gamma^{is_ijs_j}$ be the line defined by v^{is_i} and v^{js_j}.

Theorem 4.2. *Given a cone $C = C(\ell, P, \kappa)$ the conic section C^S consists of the vertices on the active reference lines, the line segments associated to adjacent vertices, the complementary rays associated to anti-adjacent vertices, and, if a vertex v is adjacent to a vertex at infinity on reference line ρ, the ray associated to v and ρ that does not cross P^S.*

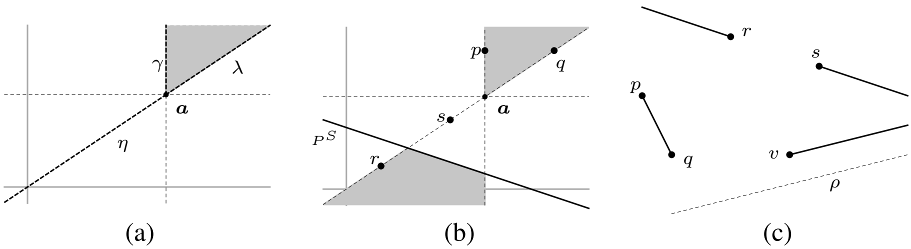


Figure 7: In (a), the rays γ and λ are adjacent, while γ and η are anti-adjacent. In (b), the point p is adjacent to q and anti-adjacent to r. If ρ^1 is active, then p is neither adjacent nor anti-adjacent to s. If ρ^1 is not active, then p and s are adjacent. In (c), the segment associated to p and q, the complementary rays associated to r and s, and one of the rays associated to v and ρ.

See Figure 8 for an illustration of this theorem.

Proof. Let $d_{j,k}(x,\ell)$ be the partial distance being used in an active double wedge W comprising $\hat{W}$ and $\check{W}$. The absolute value in $d_{j,k}(x,\ell)$ resolves in two opposite ways, depending on which wedge is being considered. At the same time, $d(x,P)$ resolves in two opposite ways, depending on which side of P^S is being considered. The resulting linear equation valid in $\hat{W}$ and on one side of P^S is also valid in $\check{W}$ and on the other side of P^S. Let $\widetilde{W}$ be this region, an example of which is shown shaded in Figure 7(b). Any point that satisfies the resulting linear equation and that lies in $\widetilde{W}$ is part of the conic section.

Let v^{is_i} and v^{js_j} be adjacent or anti-adjacent vertices on the boundary of $\widetilde{W}$, with neither at infinity. These points are already known to solve the desired linear equation, so $\gamma^{is_ijs_j}$ is the set of all points solving the linear equation. If v^{is_i} and v^{js_j} are adjacent, then the segment associated to v^{is_i} and v^{js_j} lies in $\widetilde{W}$. Similarly, if v^{is_i} and v^{js_j} are anti-adjacent, then the complementing rays lie in $\widetilde{W}$ with one ray in $\hat{W}$ and the other ray in $\check{W}$.

Finally, if v^{is_i} and v^{js_j} are adjacent and v^{js_j} lies at infinity, then $\gamma^{is_ijs_j}$ must intersect $\widetilde{W}$ through v^{is_i} but cannot cross ρ^j since, if it did, the resulting intersection v^{js_j} would not be at infinity. Therefore, $\gamma^{is_ijs_j}$ is parallel to ρ^j and the intersection of $\gamma^{is_ijs_j}$ and $\widetilde{W}$ is the ray associated to v^{is_i} and ρ^j. □

4.3 Auxiliary points on P^S

The vertices found above enjoy a geometric relationship with P^S. Given two reference lines ρ^i and ρ^j, select one vertex on ρ^i and one on ρ^j. This defines a line. The complementary pair of points also forms a line. It turns out that these lines intersect at a point on P^S that we call an auxiliary point. This property was observed for the special case of $C(\ell_{0,0,1}, P_{0,0,1}, \kappa)$ in [5].

Theorem 4.3. *Given the conic section C^S for cone $C = C(\ell_{\boldsymbol{a}}, P_{\boldsymbol{A}}, \kappa)$ where $\ell_{\boldsymbol{a}}$ is non-horizontal, for $i \neq j$*

- *the lines γ^{i+j+} and γ^{i-j-} meet at a point on $P^S_{\boldsymbol{A}}$;*

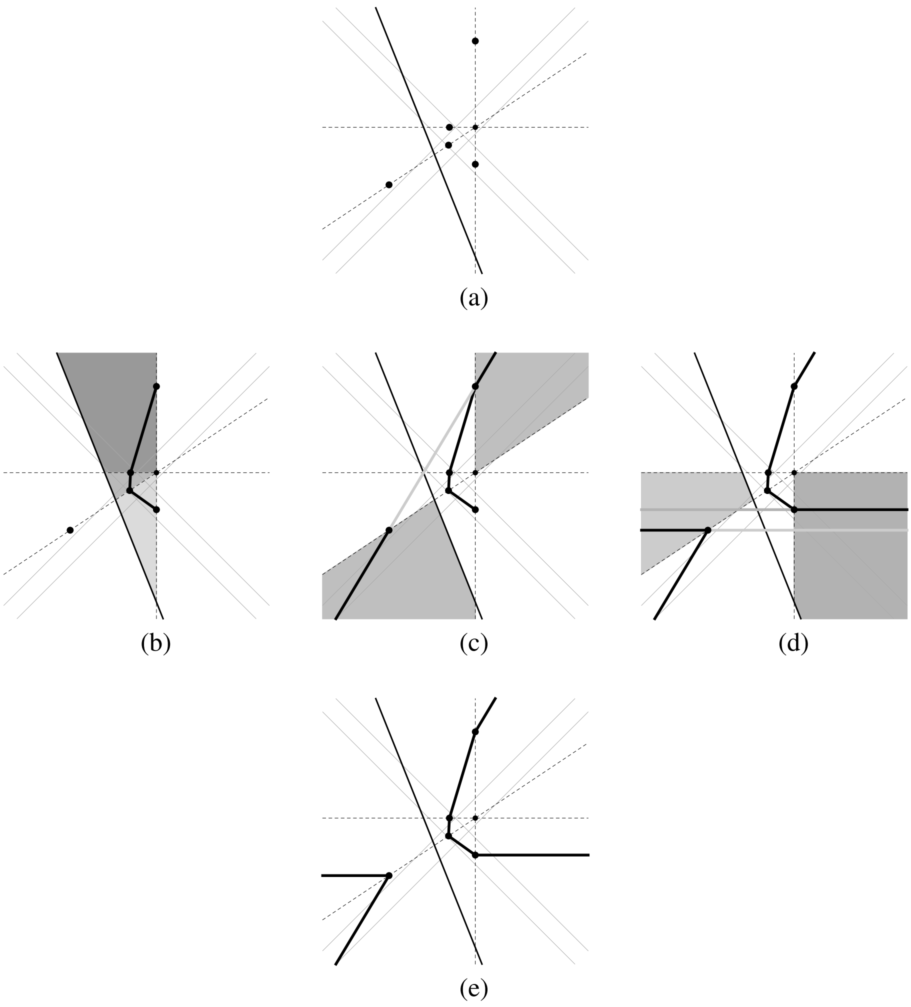
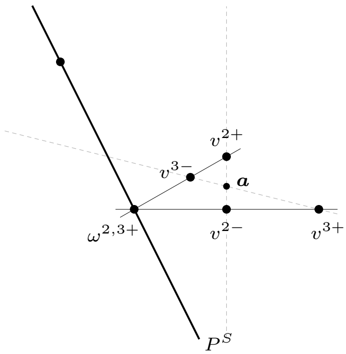


Figure 8: Connecting the dots to produce a conic section. In this example, $\boldsymbol{A} = \left(\frac{1}{2}, \frac{1}{5}, 1\right)$, $\boldsymbol{a} = \left(\frac{3}{2}, 1, 1\right)$, and $\kappa = 2$. Starting with the vertices (a), connect adjacent vertices using their associated segments (b), then add complementing rays associated to anti-adjacent vertices (c), and finally, if there are vertices at infinity, add rays associated to vertices and reference lines (d) to produce the complete conic section (e).



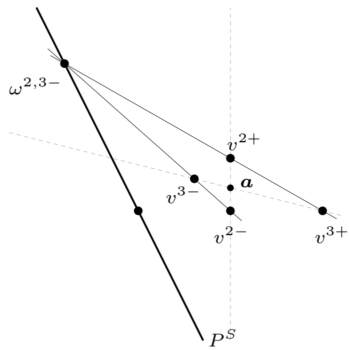


Figure 9: Auxiliary points. In both images, $\boldsymbol{A} = \left(\frac{1}{2}, \frac{1}{4}, 1\right)$, $\boldsymbol{a} = \left(2, -\frac{1}{2}, 1\right)$, and $\kappa = \frac{1}{2}$. On the left, γ^{2+3-} and γ^{2-3+} meet at $\omega^{2,3+}$. On the right,γ^{2+3+} and γ^{2-3-} meet at $\omega^{2,3-}$.

- *the lines γ^{i+j-} and γ^{i-j+} meet at a point on $P^S_{\boldsymbol{A}}$.*

The auxiliary points corresponding to the vertices $v^{1\pm}$ and $v^{2\pm}$ are

$$w^{1,2\pm} = \left(\frac{\pm A_2 a_1 + A_2 a_2 + \delta}{-A_1 \pm A_2}, \frac{A_1 a_1 \pm A_1 a_2 + \delta}{\pm A_1 - A_2}, 1 \right).$$

The auxiliary points corresponding to the vertices $v^{1\pm}$ and $v^{3\pm}$ are

$$w^{1,3\pm} = \left(\frac{-(\delta a_1 \pm A_2 a_2 \pm \delta)}{A_1 a_1 + A_2 a_2 \pm A_1}, \frac{\pm A_1 a_2 - \delta a_2}{A_1 a_1 + A_2 a_2 \pm A_1}, 1 \right).$$

The auxiliary points corresponding to the vertices $v^{2\pm}$ and $v^{3\pm}$ are

$$w^{2,3\pm} = \left(\frac{\pm A_2 a_1 - \delta a_1}{A_1 a_1 + A_2 a_2 \pm A_2}, \frac{-(\pm A_1 a_1 + \delta a_2 \pm \delta)}{A_1 a_1 + A_2 a_2 \pm A_2}, 1 \right).$$

See Figure 9 for an illustration of this result. Note that the auxiliary points do not depend on κ.

Proof. The vertices are on the lines defined by resolving the absolute values in

$$d_{i,j}(x, \ell) = \kappa\, d(x, P).$$

For a given pair of lines, the potential vertices defining one of the lines are determined by a particular choice of resolving the absolute values, and the potential vertices defining the other line are determined by flipping the choices on one side of the equation, while keeping the choice on the other.

The resulting equations are for the lines in a given pair and these two equations can be written in the form $\alpha = \beta$ and $-\alpha = \beta$ where α is a resolution of $d_{i,j}(x, \ell)$ and β is the resolution of $d(x, P)$. The solution occurs if and only if $\alpha = \beta = 0$. But $\beta = 0$ is equivalent to $A_1 x_1 + A_2 x_2 + \delta = 0$, which is the equation for the line P^S.

For each choice of reference lines, there are two ways to distribute the vertices, resulting in two different auxiliary points, and there are three pairs of reference lines, so there are a total of six auxiliary points on P^S, the formulas for which are found by explicitly solving the resulting system of equations. □

Similarly to the reference lines, the auxiliary points are not always active in the sense that they lie on extensions of segments or rays in the conic section. If $\ell_{\boldsymbol{a}}$ is intermediate, the only active auxiliary points are those that arise from adjacent or anti-adjacent vertices. If $\ell_{\boldsymbol{a}}$ is not intermediate, then only the vertices on active reference lines produce active auxiliary points. As such, only three auxiliary points are active if $\ell_{\boldsymbol{a}}$ is intermediate, and only two auxiliary points are active otherwise.

With the help of the auxiliary points, an alternative method for constructing a conic section can be formulated. Note that auxiliary points can lie at infinity, but if all active auxiliary points are finite, we have the following:

Theorem 4.4. *Given a cone $C = C(\ell, P, \kappa)$, such that all active auxiliary points are finite, consider the rays based at each active auxiliary point and containing the associated vertices. On each ray, consider the set of points between two vertices or, if there is only one vertex, the set of points on the side of the vertex that does not include the auxiliary point. The conic section C^S consists of the union over the rays of these points.*

See Figure 10 for an illustration of this. While this theorem does not provide a particularly useful way to construct conic sections when ℓ is non-horizontal, it will serve as an effective construction method for conic sections resulting from horizontal defining lines.

Proof when ℓ is non-horizontal. Since the active auxiliary points are identified by those lines that have adjacent or anti-adjacent vertices lying on them, the rays described here are parts of these lines and resulting sets of points are just alternate characterizations of the associated segments and rays identified in Theorem 4.2. □

If there are active auxiliary points at infinity, the construction described by Theorem 4.4 can be modified. An auxiliary point lies at infinity precisely when the lines defining it are parallel to P^S. In this case, the resulting part of the conic section is always a segment associated to the two active vertices defining a line producing the auxlilary point at infinity.

4.4 Characteristics of sections

Here, we establish some general facts about where the vertices must be located. These characteristics combined with Theorems 4.2 and 4.4 allow us to sketch conic sections without detailed calculations and allow us to distinguish realistic sketches from non-realistic ones.

As in the Euclidean setting, we say a conic section is an ellipse if it is a bounded set, a parabola if it is unbounded with one component, and a hyperbola if is unbounded with two components. Also as in the Euclidean setting, ellipses arise when the slicing plane completely slices one half of the cone, parabolas arise when the slicing plane slices one half of the cone, but is parallel to at least one line in the cone, and hyperbolas arise when the slicing plane slices both halves of the cone.

Lemma 4.5. *Given the conic section C^S for a cone $C = C(\ell_{\boldsymbol{a}}, P_{\boldsymbol{A}}, \kappa)$, on each active reference line ρ^i,*

- *exactly one of the vertices $v^{i\pm}$ lies on the segment of ρ^i between P^S and $\boldsymbol{a}$;*
- *if neither vertex lies at infinity, then they lie on the opposite sides of $\boldsymbol{a}$ if and only if they lie on the same side of P^S.*

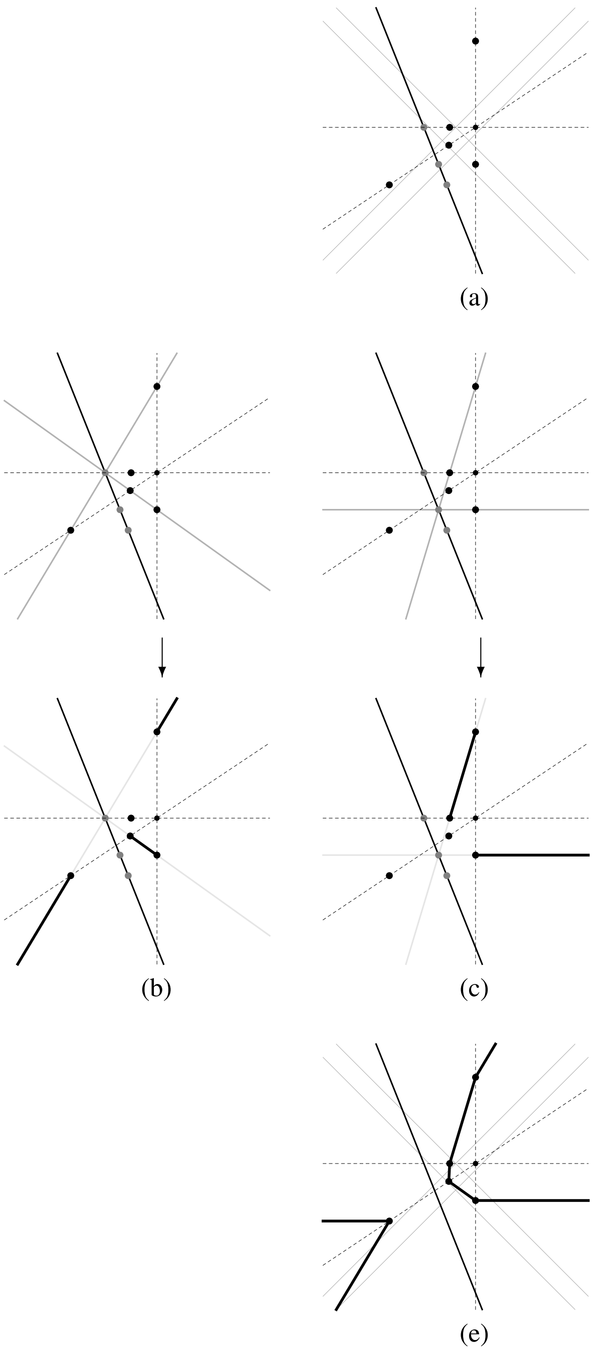


Figure 10: Using the rays through auxiliary points to produce a conic section. In this example, just like in Figure 8, $\boldsymbol{A} = \left(\frac{1}{2}, \frac{1}{5}, 1\right)$, $\boldsymbol{a} = \left(\frac{3}{2}, 1, 1\right)$, and $\kappa = 2$. Starting with the vertices and active auxiliary points (a), for the rays associated to each auxiliary point, identify the points between two vertices or beyond a single vertex on each ray (b), (c), (d) to produce the complete conic section (e).

Proof. The first statement follows immediately from the Intermediate Value Theorem and the monotonicity of the distance functions $d(x, \ell)$ and $d(x, P)$ restricted to the line ρ^i.

The second statement follows from the fact that P^S and $\boldsymbol{a}$ subdivide ρ^i into two rays and a segment. By the first statement, only one vertex v can lie on the segment, so the other vertex must lie on one of the rays. The ray on the opposite side of $\boldsymbol{a}$ as v lies on the same side of P^S, and vice versa. □

With the help of this lemma we can make some immediate observations: On one hand, C^S is an ellipse if and only if all pairs of active vertices $v^{i\pm}$ are finite and lie on opposite sides of $\boldsymbol{a}$. On the other hand, C^S is a hyperbola if an only if for some pair of active vertices, both vertices are finite, lie on the same ray associated to $\boldsymbol{a}$, and, by necessity, P^S separates them.

This lemma and these observations provide some structure. To determine when the conditions are actually met, define the set

$$Q_{\boldsymbol{A},\kappa} = \left\{ x \in S : |A_1 x_1 + A_2 x_2| < \frac{M}{\kappa} \right\}.$$

We call this set the characterizing strip for the cone $C(\ell_{\boldsymbol{a}}, P_{\boldsymbol{A}}, \kappa)$. Also, let $e^{1\pm} = (\pm 1, 0, 1)$ and $e^{2\pm} = (0, \pm 1, 1)$ be the vertices of σ^S. We then have the following:

Theorem 4.6. *Let C^S be the conic section for the cone $C = C(\ell_{\boldsymbol{a}}, P_{\boldsymbol{A}}, \kappa)$ where $\ell_{\boldsymbol{a}}$ is non-horizontal, let $Q = Q_{\boldsymbol{A},\kappa}$ be the characterizing strip for C, and let $i \in \{1, 2\}$.*

- *If σ and $\mathbf{a}$ lie in Q, then C^S is an ellipse.*
- *If σ and $\mathbf{a}$ lie in $\overline{Q}$, and at least one intersects ∂Q, then C^S is a parabola.*
- *If any of the vertices $e^{i\pm}$ of σ or $\mathbf{a}$ lie in $S \backslash \overline{Q}$, then C^S is a hyperbola.*

Proof. In light of Lemma 4.5 it is enough to determine under what conditions the vertices of C^S lie on the same or opposite sides of $\boldsymbol{a}$ or at infinity. This is determined completely by the denominators in Equations (4.4), (4.5), and (4.6), and the positions of $e^{i\pm}$ and $\boldsymbol{a}$ relative to Q simply provide a geometric representation of this. Specifically,

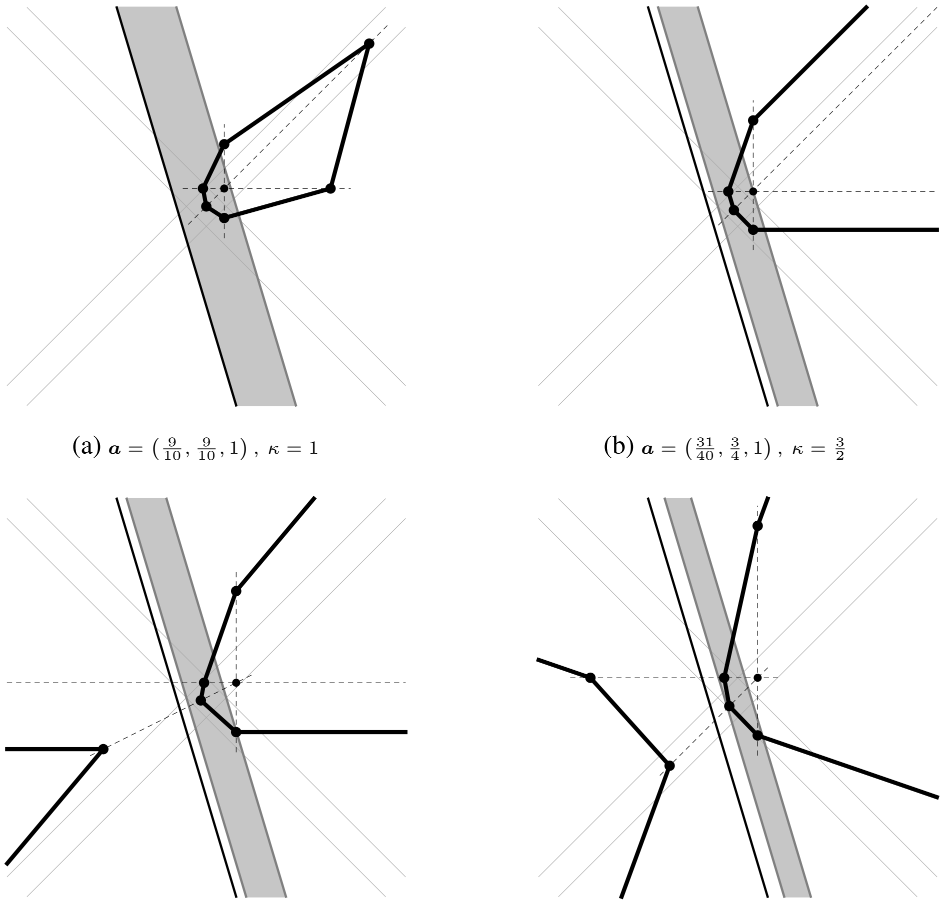
- If $e^{i\pm} \in Q$, then the corresponding vertices $v^{i\pm}$ lie on the same side of P^S. If $\boldsymbol{a} \in Q$, then the vertices $v^{3\pm}$ lie on the same side of P^S.
- If $e^{i\pm} \in \partial Q$, then one of the corresponding vertices $v^{i\pm}$ lies at infinity. If $\boldsymbol{a} \in \partial Q$, then one of the vertices $v^{3\pm}$ lies at infinity.
- If $e^{i\pm} \in S \backslash \overline{Q}$, then the corresponding vertices $v^{i\pm}$ lie on opposite sides of P^S. If $\boldsymbol{a} \in S \backslash \overline{Q}$, then the vertices $v^{3\pm}$ lie on opposite sides of P^S.

With this, the result follows by considering all active vertices at once. □

See Figure 11 for examples illustrating this result.

5 Conic sections when $\boldsymbol{\ell}$ is horizontal

While the approach here starts similarly to the non-horizontal case, there are some significant differences that develop.



(a) $\boldsymbol{a} = \left(\frac{9}{10}, \frac{9}{10}, 1\right)$, $\kappa = 1$

(b) $\boldsymbol{a} = \left(\frac{31}{40}, \frac{3}{4}, 1\right)$, $\kappa = \frac{3}{2}$

(c) $\boldsymbol{a} = \left(\frac{3}{2}, \frac{3}{4}, 1\right)$, $\kappa = \frac{3}{2}$

(d) $\boldsymbol{a} = (1, 1, 1)$, $\kappa = \frac{9}{4}$

Figure 11: The type of conic section depends on the location of $\boldsymbol{a}$ and σ^S relative to Q which is shaded. These are just four of many possibilities. In these cases, $\boldsymbol{A} = \left(\frac{2}{3}, \frac{1}{5}, 1\right)$. The darker line parallel to Q is P^S which coincides in (a) with one edge of Q. In (a), $\boldsymbol{a}$ and σ^S lie inside Q and we have an ellipse. In (b), $\boldsymbol{a}$ and $e^{1\pm}$ lie on ∂Q, $e^{2\pm}$ lies in Q, and we have a parabola. In (c), $\boldsymbol{a}$ lies outside $\overline{Q}$, $e^{1\pm}$ lie on ∂Q, $e^{2\pm}$ lies in Q, and we have a hyperbola. In (d), $\boldsymbol{a}$ and $e^{1\pm}$ lie outside $\overline{Q}$, $e^{2\pm}$ lies in Q, and we have a hyperbola.

5.1 Vertices of the sections

When ℓ is horizontal, $\ell \cap S$ is the empty set, so $\boldsymbol{a}$ is not identified with a point in S. Similarly, the reference lines ρ^1 and ρ^2 do not exist. The reference line ρ^3 does exist, and we can use it as a way to represent $\boldsymbol{a}$.

Theorem 5.1. *Given a cone $C(\ell_{\boldsymbol{a}}, P_{\boldsymbol{A}}, \kappa)$ where $\boldsymbol{a} = (a_1, a_2, 0)$ and $\boldsymbol{A} = (A_1, A_2, \delta)$, and slicing plane S, the reference line ρ^3 is always active and the vertices lying on ρ^3 are*

$$v^{3\pm} = (r^{\pm} a_1, r^{\pm} a_2, 1)$$

where

$$r^{\pm} = 1 + \frac{\pm\frac{M}{\kappa} + A_1 a_1 + A_2 a_2 + \delta}{-(A_1 a_1 + A_2 a_2)}.$$

Proof. Without loss of generality, suppose $|a_1| \geq |a_2|$. Then the partial distance being used is $d(x, \ell) = d_{2,3}(x, \ell) = \left| x_2 - a_2 \frac{x_1}{a_1} \right| + 1$. Restricting attention to points on ρ^3, note that $d(x, \ell) = 1$, so the vertices on ρ^3 are solutions to

$$1 = \kappa \, \frac{|A_1 x_1 + A_2 x_2 + \delta|}{M}.$$

Resolving the absolute values two different ways and solving yields the result. □

5.2 Auxiliary points on P^S and constructing the sections

Here we do not have enough vertices to construct the resulting conic section using the methods of Theroem 4.2, nor can we define the auxiliary points as the intersections of lines defined by vertices. Nonetheless, auxiliary points exist, and once they are found, they can be used to construct the sections using Theorem 4.4.

5.2.1 Auxiliary points

Theorem 5.2. *Given the conic section C^S for cone $C = C(\ell_{\boldsymbol{a}}, P_{\boldsymbol{A}}, \kappa)$ where $\ell_{\boldsymbol{a}}$ is horizontal, there are four auxiliary points on P^S, two of which are active at a time.*

If $|a_1| \geq |a_2|$, then the active auxiliary points are

$$w^{I\pm} = \left(\frac{\pm A_2 a_1 - \delta a_1}{A_1 a_1 + A_2 a_2}, \frac{-(\pm A_1 a_1 + \delta a_2)}{A_1 a_1 + A_2 a_2}, 1 \right).$$

If $|a_2| \geq |a_1|$, then the active auxiliary points are

$$w^{II\pm} = \left(\frac{-(\pm A_2 a_2 + \delta a_1)}{A_1 a_1 + A_2 a_2}, \frac{\pm A_1 a_2 - \delta a_2}{A_1 a_1 + A_2 a_2}, 1 \right).$$

Proof. When $|a_1| \geq |a_2|$, the equation defining the cone is

$$d_{2,3}(x, \ell) = \kappa \, d(x, P)$$

which expands to

$$\left| x_2 - a_2 \frac{x_1}{a_1} \right| + 1 = \kappa \, \frac{|A_1 x_1 + A_2 x_2 + \delta|}{M}.$$

Each absolute value can be resolved in two ways, leading to four different equations. Given a particular resolution, the equation determined by making the opposite choice on both absolute values will have the same slope as the original, so the four lines form a parallelogram. Two of the vertices of this parallelogram are the vertices $v^{3\pm}$. These vertices correspond to where the absolute value on the left is equal to zero. The other two vertices correspond to where the absolute value on the right is equal to zero, but the right hand side is $d(x, P)$ so these points lie on P^S. Moreover, they are the intersections of the lines defined by

$$\pm\left(x_2 - a_2\frac{x_1}{a_1}\right) + 1 = 0$$

and

$$A_1x_1 + A_2x_2 + \delta = 0.$$

Solving this system, we find the two auxiliary points $w^{I\pm}$.

If $|a_2| \geq |a_1|$, then the same analysis as above applies to the equation

$$d_{1,3}(x, \ell) = \kappa\, d(x, P)$$

which results in the auxiliary points $w^{II\pm}$. □

5.2.2 Constructing the sections: connecting the dots

See Figure 12 for an implementation of Theorem 4.4 when ℓ is horizontal, the proof of which follows.

Proof of Theorem 4.4 *when ℓ is horizontal.* The lines that define the auxiliary points arise from the edges of the conic section, so it remains only to determine which parts of these lines to include.

On each line, the auxiliary point ω and the vertex v subdivide the line into the segment and two rays associated to ω and v. The ray and segment terminating at ω cannot be part of the conic section because ω itself is not part of the section. The remaining ray must therefore be the subset of the line that makes up part of the conic section. □

5.3 Characteristics of sections

Compared to the case where ℓ is non-horizontal, this case is much simpler.

Theorem 5.3. *If $\ell_{\boldsymbol{a}}$ is horizontal, the conic section C^S for the cone $C = C(\ell_{\boldsymbol{a}}, P_{\boldsymbol{A}}, \kappa)$ is always a hyperbola.*

Proof. A computation shows that

$$A_1v_1^{3\pm} + A_2v_2^{3\pm} + \delta = \mp\frac{M}{\kappa}$$

which implies that v^{3+} and v^{3-} must lie on opposite sides of P^S. □

Note also that when ℓ is horizontal, $\boldsymbol{a}$ can be thought of as lying at infinity, which is always outside $\overline{Q_{\boldsymbol{A},\kappa}}$ so this result is consistent with Theorem 4.6.

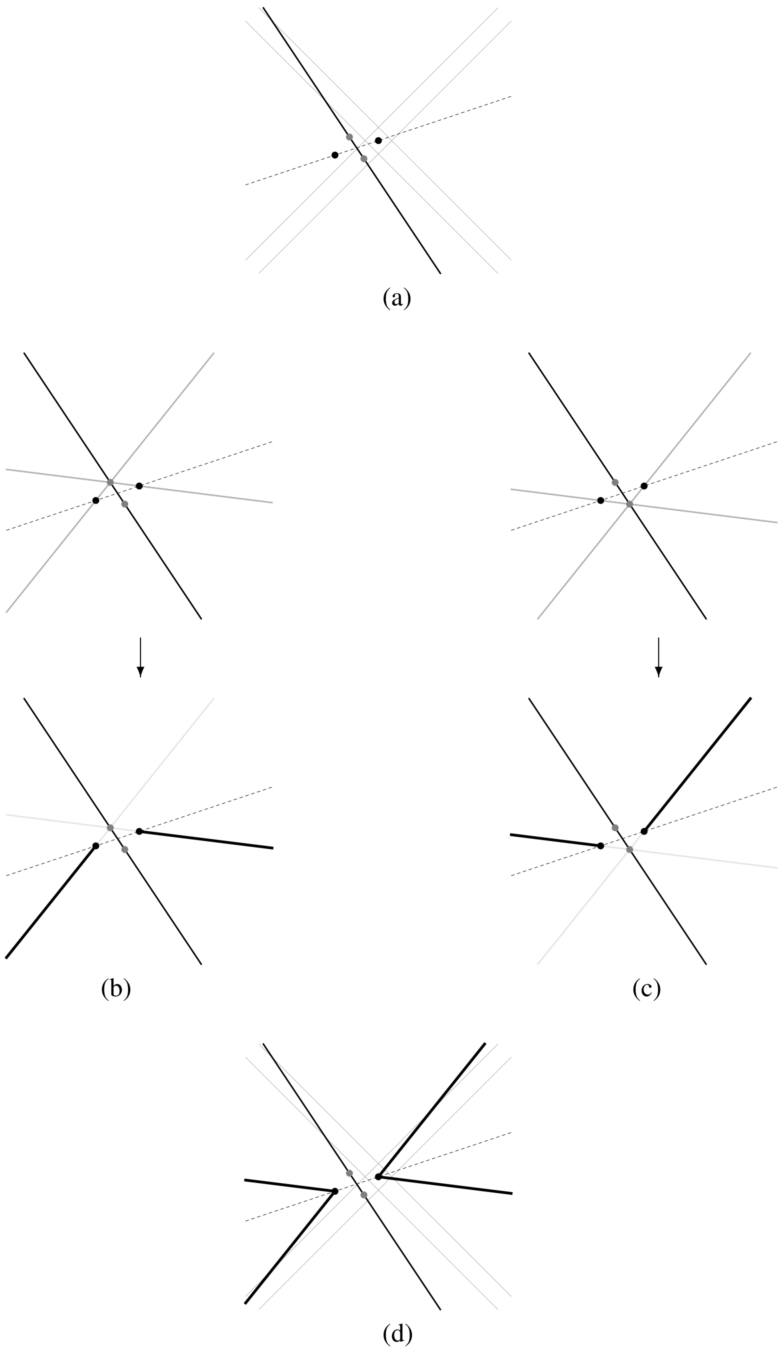


Figure 12: Using the rays through auxiliary points to produce a conic section. In this example, $\boldsymbol{A} = \left(\frac{1}{2}, \frac{1}{3}, 1\right)$, $\boldsymbol{a} = (3, 1, 0)$, and $\kappa = 1$. Starting with the vertices and active auxiliary points (a), for the rays associated to each auxiliary point, identify the points beyond a single vertex on each ray (b), (c), to produce the complete conic section (d). Note that the vertices and auxiliary points are the vertices of a parallelogram.

6 Some special cases

We explore here a few special cases that serve to fill out the picture of taxicab conic sections. This is by no means an exhaustive exploration and we leave it to the reader to discover other interesting cases.

6.1 Horizontal defining plane

This case complements the horizontal line scenario which itself is a special case of sorts. In that setting the conic sections are always hyperbolas. Here we find that all conic sections are ellipses.

Theorem 6.1. *Given a cone* $C = C(\ell_{\boldsymbol{a}}, P_{(0,0,1)}, \kappa)$ *where* $\boldsymbol{a} = (a_1, a_2, 1)$ *and slicing plane* S*, the resulting conic section* C^S *is an ellipse and*

- *if* ρ^1 *is active, then the vertices lying on* ρ^1 *are* $v^{1\pm} = \boldsymbol{a} \pm \kappa(1, 0, 0)$*;*
- *if* ρ^2 *is active, then the vertices lying on* ρ^2 *are* $v^{2\pm} = \boldsymbol{a} \pm \kappa(0, 1, 0)$*;*
- *if* ρ^3 *is active, then the vertices lying on* ρ^3 *are* $v^{3\pm} = \boldsymbol{a} \pm \kappa(a_1, a_2, 0)$*.*

Proof. The conic section is always an ellipse because $Q = S$. The formulas for the vertices follow directly from Equations (4.4), (4.5), and (4.6). □

Note that κ just causes a rescaling of the vertices around $\boldsymbol{a}$ and so does not affect the shape. If ℓ is steep, the resulting conic section is a taxicab circle, if ℓ is shallow or transitional, the resulting conic section is a parallelogram, with rhombi occurring when $\boldsymbol{a}$ lies on a coordinate axis, and when ℓ is intermediate, the resulting conic section is a hexagon with parallel opposite sides. See Figure 13 for some specific examples of the various possibilities.

6.2 "Perpendicular" line and plane

As mentioned earlier, there is no definition of angle that naturally arises from the taxicab distance. Nonetheless, it is worth considering the special case where $\boldsymbol{a}$ is a multiple of $\boldsymbol{A}$.

Note that the definition of angle in [9] is such that an angle is a right angle in the Euclidean setting if and only if it is a right angle in the taxicab setting. They do not explore the notion of taxicab angles in $\mathbb{R}^3$ but it would be reasonable to expect the idea of "perpendicular" to be preserved in any suitably robust definition of angle for $(\mathbb{R}^3, d)$

In this situation, it is not the actual conic sections that are particularly interesting; they share the characteristics discussed already in Sections 4.4 and 5.3. What is interesting is the set of parameters where the resulting conic sections are parabolas.

We restrict parameters to $\mathscr{P}'$ and $\mathscr{L}'$ so that $\boldsymbol{A} = \boldsymbol{a}$. Also, since the particular cases involving vertical planes, and hence horizontal lines, always results in hyperbolas, moving forward we will work with $\delta = a_3 = 1$.

Let $B^E_r(x)$ be the open Euclidean ball of radius r centered at x. In $\mathbb{R}^2$ define

$$U_\kappa = \begin{cases} B^E_{\frac{1}{2\kappa}}\left(\frac{1}{2\kappa}, 0\right) \cup B^E_{\frac{1}{2\kappa}}\left(0, \frac{1}{2\kappa}\right) & \\ \quad \cup B^E_{\frac{1}{2\kappa}}\left(-\frac{1}{2\kappa}, 0\right) \cup B^E_{\frac{1}{2\kappa}}\left(0, -\frac{1}{2\kappa}\right) & \\ \quad \cup B^E_{\frac{1}{\sqrt{\kappa}}}(0,0) & \text{if } 0 < \kappa < 1, \\ \left(-\frac{1}{\kappa}, \frac{1}{\kappa}\right) \times \left(-\frac{1}{\kappa}, \frac{1}{\kappa}\right) \cap B^E_{\frac{1}{\sqrt{\kappa}}}(0,0) & \text{if } \kappa \geq 1. \end{cases}$$

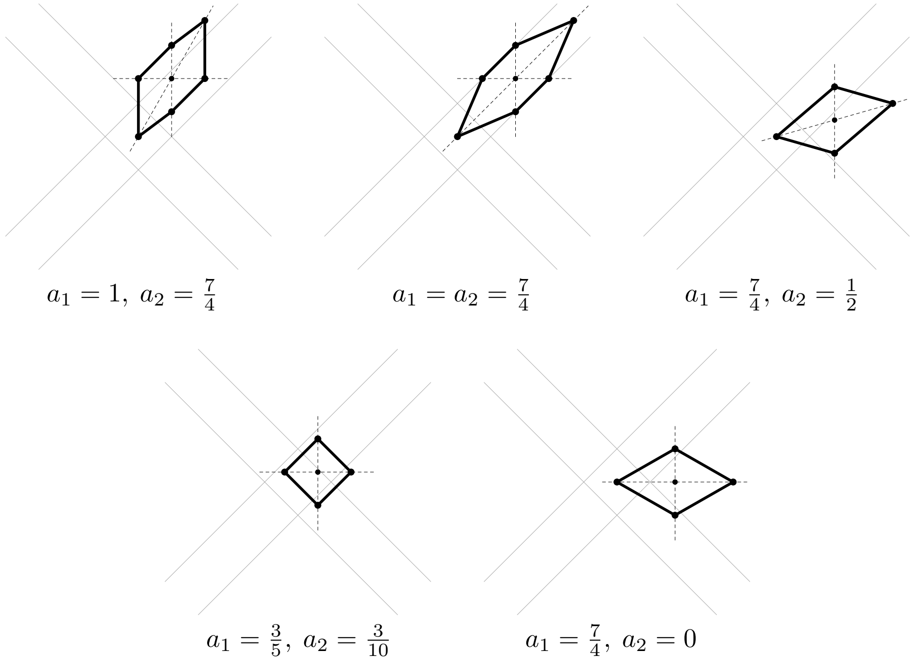


Figure 13: Various conic sections when P is horizontal. In all cases, $\kappa = 1$.

Theorem 6.2. *Let $\boldsymbol{A} \in \mathscr{P}'$ with $\delta = 1$. Then, given a cone $C = C(\ell_{\boldsymbol{A}}, P_{\boldsymbol{A}}, \kappa)$ the resulting conic section C^S is an ellipse when $\boldsymbol{A}$ lies in $U_\kappa \times \{(0,0,1)\}$, a parabola when $\boldsymbol{A}$ lies in $\partial U_\kappa \times \{(0,0,1)\}$, and a hyperbola when $\boldsymbol{A}$ lies outside $\overline{U_\kappa} \times \{(0,0,1)\}$.*

See Figure 14 for examples of ∂U_κ for various values of κ.

Proof. As discussed before, parabolas occur when at least one vertex lies at infinity, and all the remaining vertices lie on the same side of P^S. As such, to find parabolas, noting that $\boldsymbol{A} = \boldsymbol{a} = (0,0,1)$ always results in an ellipse, extend outward until the first non-finite vertex is encountered. To simplify the work, the computations can be performed in the wedge $0 \leq A_2 \leq A_1$, and then extended to the rest of the plane using taxicab isometries. Some care must be taken to account for where various vertices are active. The details are left to the reader. □

Above, we argued that $\boldsymbol{A} = \boldsymbol{a}$ could be a reasonable way to encode a notion of perpendicularity in the taxicab setting. However, in light of Theorem 6.2 and the images in Figure 14, we find some Euclidean structure making an appearance. As such, the condition that $\boldsymbol{A} = \boldsymbol{a}$ seems to be more fundamentally Euclidean in nature, further highlighting the fact that any notion of angle will necessarily introduce structure that is not intrinsic to taxicab space.

6.3 Redundancy of conic sections

There is redundancy within the wide variety of conic sections that can arise by considering the different parameters available. The redundancies that result from taxicab isometries are

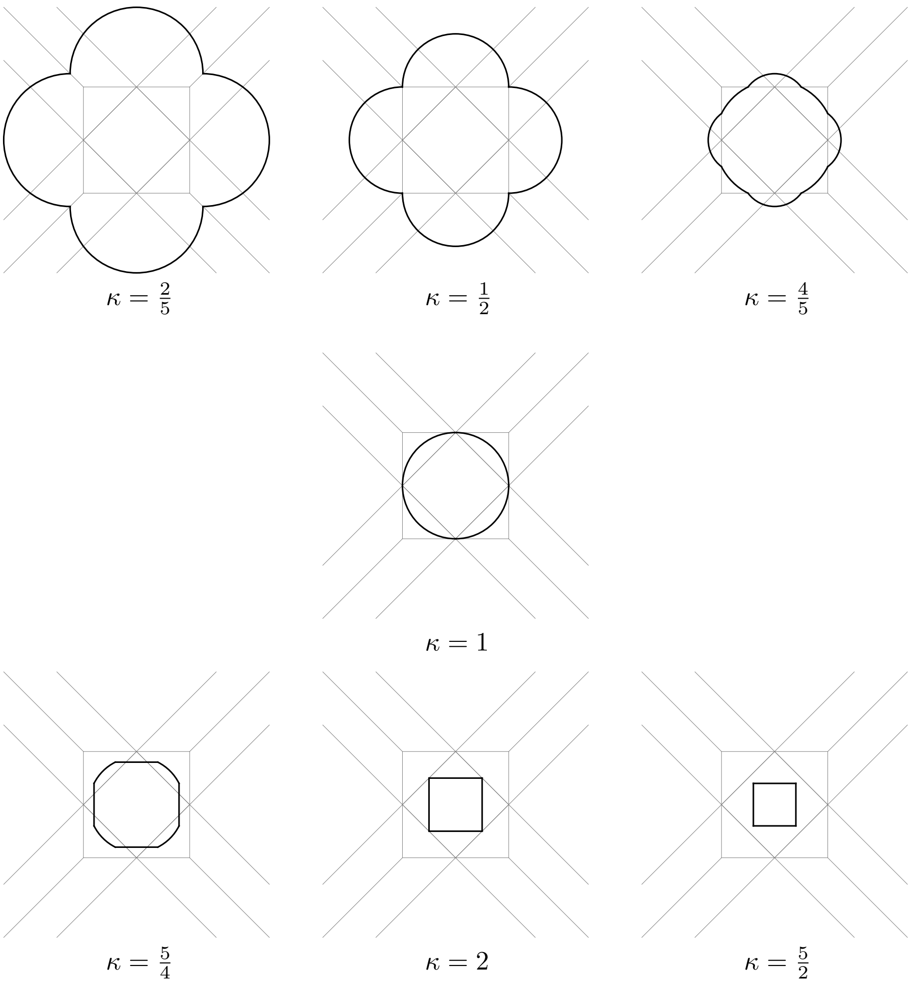


Figure 14: When $\boldsymbol{A} = \boldsymbol{a}$, the parameters that lie in ∂U_κ and produce parabolas are shown. Identifying lines for $\mathscr{P}'$ and $\mathscr{L}'$ are included for reference.

part of this, but they are not surprising. There are additional redundancies that exist through inspired choices of the parameters defining the cones that prove to be more interesting.

6.3.1 Steep defining lines

Theorem 6.3. *Let $\boldsymbol{a} = (a_1, a_2, 1)$, $b = (b_1, b_2, 1) \in \mathscr{L}'$ represent steep lines. Let $C_{\boldsymbol{a}} = C(\ell_{\boldsymbol{a}}, P_{\boldsymbol{A}}, \kappa)$ and let $C_{\boldsymbol{b}} = C(\ell_{\boldsymbol{b}}, P_{\boldsymbol{A}}, \kappa)$. Then, as long as neither cone is degenerate, the resulting conic sections $C^S_{\boldsymbol{a}}$ and $C^S_{\boldsymbol{b}}$ are similar.*

Note that this theorem also applies for "transitionally steep" lines where $|a_1|+|a_2| = 1$, but unsurprisingly not other transitional lines.

Proof. For the vertices corresponding to $C_{\boldsymbol{a}}$, Equations (4.4) and (4.5) can be written

$$v^{1\pm} = \boldsymbol{a} + \left(r^{1\pm}_{\boldsymbol{a}}, 0, 0\right), \; v^{2\pm} = \boldsymbol{a} + \left(0, r^{2\pm}_{\boldsymbol{a}}, 0\right),$$

where for $i \in \{1, 2\}$

$$r^{i\pm}_{\boldsymbol{a}} = \frac{A_1 a_1 + A_2 a_2 + \delta}{\pm\frac{M}{\kappa} - A_i}.$$

The formulas for $v^{3\pm}$ are not necessary since these vertices are inactive when ℓ is steep. The equations for the vertices corresponding to $C_{\boldsymbol{b}}$ are similar.

Let $s \in \{+, -\}$. Then

$$\frac{r^{is}_{\boldsymbol{a}}}{r^{is}_{\boldsymbol{b}}} = \frac{\frac{A_1 a_1 + A_2 a_2 + \delta}{s\frac{M}{\kappa} - A_i}}{\frac{A_1 b_1 + A_2 b_2 + \delta}{s\frac{M}{\kappa} - A_i}} = \frac{A_1 a_1 + A_2 a_2 + \delta}{A_1 b_1 + A_2 b_2 + \delta}$$

This shows that the various ratios, independent of i and the sign choice, are the same. □

6.3.2 Defining planes with parallel intersections with S

Theorem 6.4. *Let $P^S_{\boldsymbol{A}}$ and $P^S_{\boldsymbol{B}}$ be parallel. Let $C_{\boldsymbol{A}} = C(\ell_{\boldsymbol{a}}, P_{\boldsymbol{A}}, \kappa_{\boldsymbol{A}})$. Then, there exists a value $\kappa_{\boldsymbol{B}} \in (0, \infty)$ such that for $C_{\boldsymbol{B}} = C(\ell_{\boldsymbol{a}}, P_{\boldsymbol{B}}, \kappa_{\boldsymbol{B}})$, the resulting conic sections $C^S_{\boldsymbol{A}}$ and $C^S_{\boldsymbol{B}}$ are similar as long as neither cone is degenerate.*

The proof of this result is similar to the proof of Theorem 6.3, but the analysis is more delicate.

Proof. Without loss of generality, suppose $A_1 \neq 0$. Then since $P^S_{\boldsymbol{A}} \| P^S_{\boldsymbol{B}}$, B_1 is also nonzero and

$$\frac{A_1}{B_1} = \frac{A_2}{B_2} \tag{6.1}$$

as long as both are defined.

For $C_{\boldsymbol{A}}$, Equations (4.4), (4.5), and (4.6) can be written

$$v^{1\pm} = \boldsymbol{a} + \left(r^{1\pm}_{\boldsymbol{A}}, 0, 0\right), \; v^{2\pm} = \boldsymbol{a} + \left(0, r^{2\pm}_{\boldsymbol{A}}, 0\right), \; v^{3\pm} = \boldsymbol{a} + r^{3\pm}_{\boldsymbol{A}}(a_1, a_2, 0),$$

where for $i \in \{1, 2\}$

$$r^{i\pm}_{\boldsymbol{A}} = \frac{A_1 a_1 + A_2 a_2 + \delta_{\boldsymbol{A}}}{\pm\frac{M_{\boldsymbol{A}}}{\kappa_{\boldsymbol{A}}} - A_i}$$

and

$$r_{\boldsymbol{A}}^{3\pm} = \frac{A_1 a_1 + A_2 a_2 + \delta_{\boldsymbol{A}}}{\pm \frac{M_{\boldsymbol{A}}}{\kappa_{\boldsymbol{A}}} - (A_1 a_1 + A_2 a_2)}.$$

The equations for $C_{\boldsymbol{B}}$ are similar.

As in the proof of Theorem 6.3, it is sufficient to show that the ratios of corresponding deviations are independent of i or the sign choice. Unlike the proof for Theorem 6.3, the question of which vertices correspond to one another is more subtle. The indices of corresponding vertices will match, but the signs may not.

Consider two cases. First, if both $P_{\boldsymbol{A}}$ and $P_{\boldsymbol{B}}$ are steep or transitional, then

$$M_{\boldsymbol{B}} = \left| \frac{B_1}{A_1} \right| M_{\boldsymbol{A}}$$

Let $f = \frac{\left|\frac{B_1}{A_1}\right|}{\frac{B_1}{A_1}}$ and, noting that $f \in \{1, -1\}$, let $s_{\boldsymbol{A}} \in \{+, -\}$ and let

$$s_{\boldsymbol{B}} = \begin{cases} s_{\boldsymbol{A}} & \text{if } f = 1, \\ \text{opposite of } s_{\boldsymbol{A}} & \text{if } f = -1. \end{cases}$$

For this case, let $\kappa_{\boldsymbol{B}} = \kappa_{\boldsymbol{A}}$. Then for $i \in \{1, 2\}$ a bit of simplification shows that

$$\frac{r_{\boldsymbol{A}}^{i s_{\boldsymbol{A}}}}{r_{\boldsymbol{B}}^{i s_{\boldsymbol{B}}}} = \frac{\frac{A_1 a_1 + A_2 a_2 + \delta_{\boldsymbol{A}}}{s_{\boldsymbol{A}} \frac{M_{\boldsymbol{A}}}{\kappa_{\boldsymbol{A}}} - A_i}}{\frac{B_1 a_1 + B_2 a_2 + \delta_{\boldsymbol{B}}}{s_{\boldsymbol{B}} \frac{M_{\boldsymbol{B}}}{\kappa_{\boldsymbol{B}}} - B_i}} = \left(\frac{A_1 a_1 + A_2 a_2 + \delta_{\boldsymbol{A}}}{B_1 a_1 + B_2 a_2 + \delta_{\boldsymbol{B}}} \right) \frac{B_1}{A_1}$$

which is independent of i and the signs $s_{\boldsymbol{A}}$ and $s_{\boldsymbol{B}}$.

The calculation for $\frac{r_{\boldsymbol{A}}^{3 s_{\boldsymbol{A}}}}{r_{\boldsymbol{B}}^{3 s_{\boldsymbol{B}}}}$ is similar, resulting in

$$\frac{r_{\boldsymbol{A}}^{3 s_{\boldsymbol{A}}}}{r_{\boldsymbol{B}}^{3 s_{\boldsymbol{B}}}} = \left(\frac{A_1 a_1 + A_2 a_2 + \delta_{\boldsymbol{A}}}{B_1 a_1 + B_2 a_2 + \delta_{\boldsymbol{B}}} \right) \frac{B_1}{A_1}.$$

Since all six ratios result in the same quantity, the resulting conic sections must be similar.

For the second case, if both $P_{\boldsymbol{A}}$ and $P_{\boldsymbol{B}}$ are shallow or transitional, then $M_{\boldsymbol{A}} = M_{\boldsymbol{B}} = 1$. For this case, let $\kappa_{\boldsymbol{B}} = \left| \frac{A_1}{B_1} \right| \kappa_{\boldsymbol{A}}$. Then calculations similar to the previous case show that for $i \in \{1, 2, 3\}$

$$\frac{r_{\boldsymbol{A}}^{i s_{\boldsymbol{A}}}}{r_{\boldsymbol{B}}^{i s_{\boldsymbol{B}}}} = \left(\frac{A_1 a_1 + A_2 a_2 + \delta_{\boldsymbol{A}}}{B_1 a_1 + B_2 a_2 + \delta_{\boldsymbol{B}}} \right) \frac{B_1}{A_1}.$$

Again, since all six ratios result in the same quantity, the resulting conic sections must be similar.

To complete the proof, note that similarity is transitive and both cases above include transitional planes, at least one of which is not degenerate. □

6.4 Finding "traditional" taxicab conic sections

Given the variety of conic sections found here as slices of cones, it is illuminating to consider how they relate to the more traditional taxicab conic sections found using the two-foci or focus-directrix definitions as discussed for example in [2].

We find that almost none of the conic sections defined using the two-focus definition, as indicated in the first and last rows of Figure 1, appear among conic sections defined as slices of cones. The only exception is that if P is horizontal and ℓ is steep, the resulting section is a circle. Two-foci ellipses other than circles, and two-foci hyperbolas do not appear in the our slice-formulation. This can most easily be seen by noting that, for the most part, the vertices of two-foci conic sections do not all lie on valid reference lines, or if they do, the resulting segments cannot stem from lines passing through auxiliary points on P^S.

On the other hand, all of the conic sections defined using the focus-directrix definition appear as slices of cones, arising specifically when both P and ℓ are steep. In this case, for $x \in S$,

$$d(x, P) = d(x, P^S) = d_S(x, P^S)$$

and

$$d(x, \ell) = d_{1,2}(x, \ell) = d(x, \ell \cap S) = d_S(x, \ell \cap S)$$

where d_S is the 2-dimensional taxicab distance on S. As such, the formula for the set of points in S satisfying $d(x, \ell) = \kappa\, d(x, P)$ reduces to the focus-directrix definition.

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