THE ART OF DISCRETE AND APPLIED MATHEMATICS

ISSN 2590-9770
The Art of Discrete and Applied Mathematics 6 (2023) #P1.03
https://doi.org/10.26493/2590-9770.1507.dc5
(Also available at http://adam-journal.eu)

Coupon coloring of lexicographic product of graphs*

Reji Thankachan, Pavithra Rajamani†

Government College, Chittur, Palakkad,
Kerala-678104, India

Received 20 November 2021, accepted 5 June 2022, published online 27 September 2022

Abstract

A k-coupon coloring of a graph G without isolated vertices is an assignment of colors from $[k] = \{1, 2, \ldots, k\}$ to the vertices of G such that the neighborhood of every vertex of G contains vertices of all colors from $[k]$. The maximum k for which a k-coupon coloring exists is called the *coupon coloring number* of G. In this paper, we have studied the coupon coloring number of Lexicographic product of graphs G and H if G has a Hamiltonian path. We have found a sharp bound for the coupon coloring number of Lexicographic product of connected graphs.

Keywords: Coupon coloring number, Total domatic number, Lexicographic product.

Math. Subj. Class.: 05C15, 05C69, 05C76

1 Introduction

Coupon coloring is an improper vertex coloring and it has been studied in relation to large multi-robot networks. In [1], there is an example of a network of robots that monitor different statistics in an environment (e.g. temperature, humidity, barometric pressure, etc.). A graph can be constructed with robots in the network as vertices and two vertices are adjacent if the corresponding robots are able to communicate with each other. Each robot must monitor different statistics, but due to power limitations it is only equipped with a single sensor (thermometer, barometer, etc.). In order to obtain the remaining data, each robot must communicate with its neighbors. So, by finding the coupon coloring number of

*The authors would like to thank the referees for their invaluable suggestions and comments which greatly contributed in the improvement of the paper

†Corresponding author. For the second author, this research was supported by Junior Research Fellowship, University Grants Commission, India.

E-mail addresses: rejiaran@gmail.com (Reji Thankachan), pavithrajeni94@gmail.com (Pavithra Rajamani)

the network, we can find the maximum number of sensors that can be given to the robots to get the complete data from all the robots in the network.

In this paper, we study the coupon coloring number of the Lexicographic product of graphs. We have studied the coupon coloring of $G \circ H$ when G contains a Hamiltonian path and when G and H are connected. We get the exact coupon coloring number of $P_n \circ P_m$, $K_n \circ P_2$, etc. Lexicographic product of graphs can be used in multi-robot network. So the study of coupon coloring number of Lexicographic product of graphs is useful.

2 Preliminaries

Chen et al. introduced the concept of coupon coloring number in [4]. Let G be a graph without isolated vertices. A k-coupon coloring of G is an assignment of colors from $[k] = \{1, 2, \ldots, k\}$ to the vertices of G such that the neighborhood of every vertex of G contains vertices of all colors from $[k]$. The maximum k for which a k-coupon coloring exists is called the *coupon coloring number* of G and it is denoted by $\chi_c(G)$. Clearly, coupon coloring is an improper coloring and $\chi_c(G) \leq \delta(G)$, where $\delta(G)$ is the minimum degree of G.

The coupon coloring number is also referred as the total domatic number introduced in [2], which is the maximum number of disjoint total dominating sets. Let $G = (V, E)$ be a graph without isolated vertices. $D' \subseteq V$ is a total dominating set if every vertex of G is adjacent to at least one vertex in D'. The minimum cardinality among all the total dominating sets in G is called the total domination number, $\gamma_t(G)$.

In [9] Y Shi et al. determined coupon coloring number of complete graphs, complete k-partite graphs, wheels, cycles, unicyclic graphs and bicyclic graphs. Coupon coloring is also studied in [5, 8]. P Francis and Deepak Rajendraprasad studied the coupon coloring of Cartesian product of some graphs in [6].

All graphs considered in this paper are simple, finite and undirected. Consider some basic concepts and symbols, which will be used in our proof. As usual K_n, P_n and C_n denote the complete graph, path and cycle with n vertices respectively. The minimum and maximum degrees of vertices in a graph G are denoted by $\delta(G)$ and $\Delta(G)$ respectively. Let $|V(G)|$ denote the number of vertices of the graph G. A Hamiltonian path of a graph is a path that contains every vertex of the graph. The Lexicographic product of two graphs G and H, denoted $G \circ H$, is a graph whose vertex set is $V(G) \times V(H) = \{(x, y) : x \in V(G) \text{ and } y \in V(H)\}$ and two vertices (x_1, y_1) and (x_2, y_2) of $G \circ H$ are adjacent if and only if either $x_1x_2 \in E(G)$ or $x_1 = x_2$ and $y_1y_2 \in E(H)$. Lexicographic product is a non-commutative graph product. The Lexicographic product is well studied in [7].

Theorem 2.1 ([9]). (1) *Let G be a complete graph with n vertices. Then $\chi_c(G) = \lfloor \frac{n}{2} \rfloor$.*

(2) *Let $G = K_{n_1,n_2,\ldots,n_k}$ be a complete k-partite graph where $k \geq 3$ and $n_1 \leq n_2 \leq \cdots \leq n_k$ such that $s = \sum_{i=1}^{k-1} n_i$ and $n = \sum_{i=1}^{k} n_i$. Then*

$$\chi_c(G) = \begin{cases} \lfloor \frac{n}{2} \rfloor & \text{if } s \geq \frac{n}{2}, \\ s & \text{otherwise.} \end{cases}$$

3 Some bounds for the coupon coloring number of $G \circ H$

Consider some observations on the degree of the Lexicographic product of G and H.

Observation 3.1. Let $G \circ H$ be the Lexicographic product of G and H. Then

1. $\deg((x, y)) = \deg(y) + |V(H)| \deg(x)$ for any $(x, y) \in G \circ H$,
2. $\delta(G \circ H) = \delta(G) + |V(H)|\delta(H)$,
3. $\delta(G \circ H) \geq 1 + |V(H)|$,
4. If G is a disconnected graph with components $G_1, \ldots, G_k$, then the coupon coloring number, $\chi_c(G) = \min\{\chi_c(G_1), \ldots, \chi_c(G_k)\}$.

We can imagine $G \circ H$ as $|V(G)|$ copies of H such that each vertex of a copy of H is adjacent to every vertices of another copy of H if the corresponding vertices of G are adjacent. Let G and H be any two graphs without isolated vertices. Enumerate vertices of H as $h_1, h_2, \ldots, h_{|V(H)|}$ and define $c(g, h_i) = i$ for all $g \in G$. Then clearly c is a coupon coloring of $G \circ H$. For let $(x, y) \in G \circ H$. Then there exists $x \neq g \in G$ such that x is adjacent to g, since G has no isolated vertices. So (x, y) is adjacent to (g, h_i) for all $i = 1, 2, \ldots, |V(H)|$. Hence, $\chi_c(G \circ H) \geq |V(H)|$.

Theorem 3.2. *Let G and H be two graphs without isolated vertices with $|V(G)| \geq 4$ and $|V(H)| \geq 3$. If G contains a Hamiltonian path and K_2 is not a component of the graph H, then $\chi_c(G \circ H) \geq |V(H)| + 1$.*

Proof. Let G and H be two graphs with n and m vertices respectively. Let the Hamiltonian path in G be $x_1 - x_2 - \cdots - x_n$.

Case 1: $n \equiv 0 \pmod 4$

Define $c_1 \colon V(G \circ H) \to [m+1]$ by

$$c_1(x_i, y_j) = \begin{cases} m+1, & \text{if } i \equiv 0, 1 \pmod 4 \\ j, & \text{if } i \equiv 2, 3 \pmod 4). \end{cases}$$

For proving that it is a coupon coloring, let $(x_i, y_j) \in G \circ H$. Since x_i is adjacent to x_{i-1} and x_{i+1}, (x_i, y_j) is adjacent to vertices with colors $1, 2, \ldots, m$ and also (x_i, y_j) is adjacent to vertices with color $m+1$ except when $i = 1$ and $i = n$. But in that case, since H has no isolated vertices y_j is adjacent to y_k in H and so (x_i, y_j) is adjacent to vertex (x_i, y_k) with color $m+1$.

Case 2: $n \equiv 1 \pmod 4$

Now $n - 5 \equiv 0 \pmod 4$ and so define the coloring c_2 as in Case 1 for the vertices (x_i, y_j) for $i \geq 6$. That is, $c_2(x_i, y_j) = c_1(x_{i-1}, y_j)$. For $i = 1, 2, \ldots, 5$, define the coloring

$$c_2(x_i, y_j) = \begin{cases} m+1, & \text{if } (x_i, y_j) = (x_4, y_1) \text{ or } (x_5, y_1) \text{ or } (x_1, y_j) \text{ for all } j \\ 1, & \text{if } (x_i, y_j) = (x_5, y_j) \text{ for all } j \geq 2 \\ j, & \text{otherwise.} \end{cases}$$

We need only to show that c_2 is an $(m+1)$-coupon coloring of the vertices (x_i, y_j) of $G \circ H$ with $i = 1, 2, \ldots, 5$ and $j = 1, 2, \ldots, m$. The color j is given to the vertices $(x_2, y_j), (x_3, y_j)$ for all j and (x_4, y_j) for all $j \geq 2$. The following representation gives the coloring of the $5m$ vertices of $G \circ H$ described above.

$$\begin{array}{ccccc} m+1 & m+1 & m+1 & \cdots & m+1 \\ 1 & 2 & 3 & \cdots & m \\ 1 & 2 & 3 & \cdots & m \\ m+1 & 2 & 3 & \cdots & m \\ m+1 & 1 & 1 & \cdots & 1 \end{array}$$

The first row gives the coloring of the vertices (x_1, y_j), second row gives the coloring of the vertices (x_2, y_j) and so on. Since x_1 is adjacent to x_2, all the vertices (x_1, y_j) are adjacent to (x_2, y_j). Similarly, Since x_2 is adjacent to x_3, all the vertices (x_2, y_j) are adjacent to (x_3, y_j). Hence, the vertices (x_2, y_j), (x_3, y_j) and (x_4, y_j) are adjacent to vertices with all the $m+1$ colors, the vertices (x_1, y_j) are adjacent to vertices with colors $1, 2, \ldots, m$ and the vertices (x_5, y_j) are adjacent to vertices with colors $2, 3, \ldots, m+1$. Since H has no isolated vertices, (x_1, y_j) is adjacent to (x_1, y_k) for some $j \neq k$ with color $m+1$. Also H does not contain K_2 as a component and thus we can choose y_1 as a vertex which is not adjacent to any of the leaves in H. This implies that (x_5, y_j) is adjacent to a vertex with color 1.

Case 3: $n \equiv 2 \pmod 4$

Now $n - 6 \equiv 0 \pmod 4$. Define the coloring c_3 as $c_3(x_i, y_j) = c_1(x_{i-2}, y_j)$ for the vertices (x_i, y_j) for $i \geq 7$. For $i = 1, 2, \ldots, 6$, define the coloring

$$c_3(x_i, y_j) = \begin{cases} m+1, & \text{if } (x_i, y_j) = (x_1, y_1) \text{ or } (x_2, y_1) \text{ or } (x_5, y_1) \text{ or } (x_6, y_1) \\ 1, & \text{if } (x_i, y_j) = (x_1, y_j) \text{ or } (x_6, y_j) \text{ for all } j \geq 2 \\ j, & \text{otherwise.} \end{cases}$$

We need only to show that c_3 is an $(m+1)$-coupon coloring of the vertices (x_i, y_j) of $G \circ H$ with $i = 1, 2, \ldots, 6$ and $j = 1, 2, \ldots, m$. The following representation gives the coloring of the $6m$ vertices of $G \circ H$ described above.

$$\begin{array}{ccccc} m+1 & 1 & 1 & \cdots & 1 \\ m+1 & 2 & 3 & \cdots & m \\ 1 & 2 & 3 & \cdots & m \\ 1 & 2 & 3 & \cdots & m \\ m+1 & 2 & 3 & \cdots & m \\ m+1 & 1 & 1 & \cdots & 1 \end{array}$$

The proof is similar as in Case 2.

Case 4: $n \equiv 3 \pmod 4$

Now $n - 7 \equiv 0 \pmod 4$ and so define the coloring c_4 as in case 1 for the vertices (x_i, y_j) for $i \geq 8$. That is, $c_4(x_i, y_j) = c_1(x_{i-3}, y_j)$. For $i = 1, 2, \ldots, 7$, define the coloring

$$c_4(x_i, y_j) = \begin{cases} m+1, & \text{if } i = 1, 4, 7 \\ j, & \text{otherwise.} \end{cases}$$

To show that c_4 is an $(m+1)$-coupon coloring of the vertices (x_i, y_j) of $G \circ H$ with $i = 1, 2, \ldots, 7$ and $j = 1, 2, \ldots, m$, consider the following representation of the c_4-coloring of these $7m$ vertices of $G \circ H$.

$$\begin{array}{ccccc} m+1 & m+1 & m+1 & \cdots & m+1 \\ 1 & 2 & 3 & \cdots & m \\ 1 & 2 & 3 & \cdots & m \\ m+1 & m+1 & m+1 & \cdots & m+1 \\ 1 & 2 & 3 & \cdots & m \\ 1 & 2 & 3 & \cdots & m \\ m+1 & m+1 & m+1 & \cdots & m+1 \end{array}$$

In all the above cases, we get a coupon coloring with $m+1$ colors. Thus $\chi_c(G \circ H) \geq m+1$. □

The bound proved in the above theorem is sharp. The graph that attains this lower bound is given in Corollary 3.3.

Corollary 3.3. *Let G and H be two graphs without isolated vertices with $|V(G)| \geq 4$ and let K_2 is not a component of H. If G contains a Hamiltonian path and if both G and H has a leaf, then $\chi_c(G \circ H) = |V(H)| + 1$. Moreover, if $n \geq 4$ and $m \geq 3$, then $\chi_c(P_n \circ P_m) = m + 1$.*

Proof. By Theorem 3.2, $\chi_c(G \circ H) \geq |V(H)| + 1$. If u and v are the leaf vertices of G and H, then $\deg((u, v)) = 1 + |V(H)|$ and so $\delta(G \circ H) = 1 + |V(H)|$ by Observation 3.1. Hence $\chi_c(G \circ H) \leq \delta(G \circ H) = |V(H)| + 1$. □

Corollary 3.4. *Let H be a graphs without isolated vertices and K_2 is not a component of H. If H has a leaf, then $\chi_c(P_n \circ H) = |V(H)| + 1$ for all $n \geq 4$.*

Corollary 3.5. *Let H be a graphs without isolated vertices nd K_2 is not a component of H. Then $\chi_c(C_n \circ H) \geq |V(H)| + 1$ for all $n \geq 4$.*

Theorem 3.6. *Let G and H be two graphs without isolated vertices with $n \geq 4$ and $m \geq 3$ vertices respectively and c be the k-coupon coloring of H with the colors $\{1, 2, \ldots, k\}$. If G contains a Hamiltonian path and $n \equiv 0, 3 \pmod 4$, then $\chi_c(G \circ H) \geq |V(H)| + \chi_c(H)$.*

Proof. Let G and H be two graphs with n and m vertices respectively and let $\chi_c(H) = k$ and c be a k-coupon coloring of H. Let the Hamiltonian path in G be $x_1 - x_2 - \cdots - x_n$.

If $n \equiv 0 \pmod 4$, define $c_1 \colon V(G \circ H) \to [m + k]$ by

$$c_1(x_i, y_j) = \begin{cases} c(y_j), & \text{if } i \equiv 0, 1 \pmod 4 \\ k + j, & \text{if } i \equiv 2, 3 \pmod 4. \end{cases}$$

If $n \equiv 3 \pmod 4$, then $n - 7 \equiv 0 \pmod 4$ and so define the coloring c_1 for the vertices (x_i, y_j) for $i \geq 8$. For $i = 1, 2, \ldots, 7$, define the coloring define $c_2 \colon V(G \circ H) \to [m + k]$ by

$$c_2(x_i, y_j) = \begin{cases} c(y_j), & \text{if } i = 1, 4, 7 \\ k + j, & \text{otherwise.} \end{cases}$$

Clearly, c_1 and c_2 are coupon colorings of $G \circ H$. □

4 Coupon coloring of Lexicographic product of connected graphs

Let G and H be connected nontrivial graphs. We can write any subset C of $V(G) \times V(H)$ as $C = \cup_{x \in S}(\{x\} \times T_x) \subseteq V(G \circ H)$, where $S \subseteq V(G)$ and $T_x = \{a \in V(H) : (x, a) \in C\}$ for all $x \in S$. Cris L. Armada et. al. [3], proved a necessary and sufficient condition for a subset of $V(G) \times V(H)$ to be a total dominating set of $V(G \circ H)$.

Theorem 4.1 ([3]). *Let G and H be both nontrivial connected graphs. Then $C = \cup_{x \in S}(\{x\} \times T_x) \subseteq V(G \circ H)$ where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for every $x \in S$ is a total dominating set of $(G \circ H)$ if and only if either*

(i) *S is a total dominating set of G or*

(ii) *S is a dominating set of G and T_x is a total dominating set of H for every $x \in S \setminus N_G(S)$.*

Corollary 4.2 ([3]). *If G and H be both nontrivial connected graphs, then $\gamma_t(G \circ H) = \gamma_t(G)$.*

Using these ideas we have the following theorem. Corollary 4.4, shows that the upper bound in Theorem 4.3 is sharp.

Theorem 4.3. *Let G and H be two graphs without isolated vertices. If H is connected, then $\chi_c(G \circ H) \geq |V(H)|\chi_c(G)$.*

Proof. Let $|V(G)| = n, |V(H)| = m$ and $\chi_c(G) = k$. Suppose that G is connected. Put $C_{ij} = \cup_{x \in S_i}\{(x, h_j)\}$, for all $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, m$, where S_i is a total dominating set of G. Then by Theorem 4.1, C_{ij} is a total dominating set of $G \circ H$ for all i and j. Define $c\colon V(G \circ H) \to [mk]$ by

$$c(x, y) = \begin{cases} (i-1)m + j, & \text{if } x \in C_{ij}, \text{ for some } i \text{ and } j \\ 1, & \text{otherwise.} \end{cases}$$

Clearly, c is a coupon coloring of $G \circ H$. For let (x, y) be any vertex of $G \circ H$. Since each C_{ij} is a total dominating set of $G \circ H$, (x, y) is adjacent to at least one vertex of each C_{ij}, $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, m$. Hence, $\chi_c(G \circ H) \geq mk$.

If G is a disconnected graph with components $G_1, \ldots, G_k$, then $G \circ H$ is a disconnected graph with components $G_1 \circ H, \ldots, G_k \circ H$. Since $G_1, \ldots, G_k$ and H are connected graphs, we can proceed as above. So, by Observation 3.1, $\chi_c(G \circ H) = \min\{\chi_c(G_1 \circ H), \ldots, \chi_c(G_k \circ H)\} \geq \min\{|V(H)|\chi_c(G_1), \ldots, |V(H)|\chi_c(G_2)\} \geq |V(H)|\chi_c(G)$. □

Corollary 4.4. *If n is an even positive integer, then $\chi_c(K_n \circ K_2) = n$.*

Proof. By Theorem 4.3 and Theorem 2.1, $\chi_c(K_n \circ P_2) \geq 2(\frac{n}{2}) = n$. Note that in a coupon coloring each color should appear at least two times. So $\chi_c(K_n \circ P_2) \leq n$. □

ORCID iDs

Reji Thankachan https://orcid.org/0000-0003-2712-3775
Pavithra Rajamani https://orcid.org/0000-0001-7895-5091

References

[1] W. Abbas, M. Egerstedt, C. Liu, R. Thomas and P. Whalen, Deploying robots with two sensors in $k_{1,6}$-free graphs, *J. Graph Theory* **82** (2016), 236–252, doi:10.1002/jgt.21898, https://doi.org/10.1002/jgt.21898.

[2] H. Aram, S. M. Sheikholeslami and L. Volkmann, On the total domatic number of regular graphs, *Trans. Comb.* **1** (2012), 45–51, doi:10.22108/toc.2012.760, https://doi.org/10.22108/toc.2012.760.

[3] C. L. Armada, J. R. Canoy, Sergio and C. E. Go, Forcing subsets for γ_c-sets and γ_t-sets in thelexicographic product of graphs, *Eur. J. Appl. Math.* **12** (2019), 1779–1786, doi:10.29020/nybg.ejpam.v12i4.3485, https://doi.org/10.29020/nybg.ejpam.v12i4.3485.

[4] B. Chen, J. H. Kim, M. Tait and J. Verstraete, On coupon colorings of graphs, *Discret. Appl. Math.* **193** (2015), 94–101, doi:10.1016/j.dam.2015.04.026, https://doi.org/10.1016/j.dam.2015.04.026.

[5] H. Chen and Z. Jin, Coupon coloring of cographs, *Appl. Math. Comput.* **308** (2017), 90–95, doi:10.1016/j.amc.2017.03.023, https://doi.org/10.1016/j.amc.2017.03.023.

[6] P. Francis and D. Rajendraprasad, On coupon coloring of cartesian product of some graphs, in: A. Mudgal and C. R. Subramanian (eds.), *Algorithms and Discrete Applied Mathematics*, Springer International Publishing, Cham, 2021 pp. 309–316, doi:10.1007/978-3-030-67899-9_25, https://doi.org/10.1007/978-3-030-67899-9_25.

[7] R. H. Hammack, W. Imrich, S. Klavžar, W. Imrich and S. Klavžar, *Handbook of Product Graphs*, volume 2, CRC press Boca Raton, 2011, doi:10.1201/b10959, https://doi.org/10.1201/b10959.

[8] Z. L. Nagy, Coupon-coloring and total domination in hamiltonian planar triangulations, *Graphs Comb.* **34** (2018), 1385–1394, doi:10.1007/s00373-018-1945-1, https://doi.org/10.1007/s00373-018-1945-1.

[9] Y. Shi, M. Wei, J. Yue and Y. Zhao, Coupon coloring of some special graphs, *J. Comb. Optim.* **33** (2017), 156–164, doi:10.1007/s10878-015-9942-2, https://doi.org/s10878-015-9942-2.