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On the automorphisms of a family of small q-regular graphs of girth 8^*

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Abstract

In this paper we investigate the automorphisms of a family of small $(q, 8)$-graphs of order $2q^3 - 2q$ which are obtained as induced subgraphs of the incidence graph of the classical generalized quadrangle of order q. We show that for q an odd prime power, the automorphism group has four orbits on the set of vertices, thus the investigated graphs cannot be Cayley graphs or lifts of a dipole.

Keywords: Cages, girth, voltage assignments.

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1 Introduction

For given integers $k \geq 3$ and $g \geq 3$ a regular graph of degree k and girth g is called a *(k, g)-graph*. The (k, g)-graphs of the smallest possible order $n(k, g)$ are called *(k, g)-cages*. The problem of determining $n(k, g)$ and finding the corresponding (k, g)-cages is a very attractive problem in extremal graph theory. For the current state-of-art in the cage problem we refer the reader to the dynamic survey [10]. A geometric approach for

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constructions of 'near-cages' for girth $6, 8$ and 12 was presented in [11]. An algebraic approach for girth 6 was given in [15], where the authors showed that the near-cages of girth 6 are very symmetric in the sense of transitivity of the automorphism group on the set of vertices. As a kind of continuation, in this paper we investigate a family of small $(q, 8)$-graphs of order $2q^3 - 2q$ (where q is an odd prime power) constructed in [1], and show that the group of automorphisms has precisely 4 orbits on the set of vertices. A direct consequence is that the members of this family of graphs are neither Cayley graphs, nor lifts of a dipole, what might be slightly surprising regarding the facts that the near-cages of girth 6 are very symmetric, and a related family of $(q, 8)$-graphs of order $2q^3$ appears as a lift of a dipole, see [14].

The paper is organized as follows. In the first two sections we provide the necessary context, definitions and basic information about the topic. In Section 3 we revisit the construction of the examined family Υ_q of $(q, 8)$-graphs. In Section 4 we count a few distances in Υ_q in order to show that the automorphism groups of these graphs have at least four different orbits on the set of vertices, and, finally, in Section 5 using descriptions of several automorphism we prove that the number of orbits is equal to 4.

2 Preliminaries

2.1 Basic definitions

Let Γ be a graph with vertex set $V = V(\Gamma)$ and edge set $E = E(\Gamma)$. The *distance* $d_\Gamma(u, v)$ between two vertices u and v is the length of a shortest path joining vertices u and v. The *girth* $g(\Gamma)$ of a graph Γ is the length of the shortest cycle in it. For every $u \in V$, denote by $N_\Gamma(u)$ the *(open) neighbourhood* of u, i.e. the set of neighbours of u, while $N_\Gamma[u]$ denotes the *closed neighbourhood* of u, i.e. $N_\Gamma[u] = \{u\} \cup N_\Gamma(u)$. For an integer k under the *k-th closed neighbourhood* of u we mean the set of all vertices v, which are at distance at most k from u, i.e. $N^k_\Gamma[u] = \{v \in V : d_\Gamma(u, v) \leq k\}$. A subset $U \subseteq V(\Gamma)$ of vertices is a *perfect dominating set*, if for each vertex $v \in V(\Gamma)$ we have that v is either in U, or it has precisely one neighbour in U.

The *degree* (or *valency*) of a vertex u is the number of its neighbours. A graph is called *regular* when all its vertices have the same degree k. In such a case we call the graph *k-regular* (or *regular of valency k*). A k-regular graph of girth g is called a (k, g)-graph. For a given pair (k, g) of integers, the existence of a (k, g)-graph is not immediate, and it was established by Erdős and Sachs [8]. The number of vertices of a (k, g)-graph is bounded below by the so-called *Moore bound*, which is obtained by counting the number of vertices in a distance partition with respect to a vertex:

$$n(k,g) = \begin{cases} 1 + k + k(k-1) + k(k-1)^2 + \ldots + k(k-1)^{(g-3)/2}, & \text{if } g \text{ is odd;} \\ 2 \cdot \left(1 + (k-1) + (k-1)^2 + \ldots + (k-1)^{g/2-1}\right), & \text{if } g \text{ is even.} \end{cases}$$

The problem of construction of a (k, g)-graph with the smallest possible order has received a lot of attention and it is quite a challenging problem in extremal graph theory in general. As stated earlier, this problem is known as the *cage problem*, and the (k, g)-graph with the smallest possible order is called a *(k, g)-cage*. Families of $(q + 1, g)$-cages are known to exist when q is a prime power and $g \in \{6, 8, 12\}$. They are arising as the incidence graphs of the classical projective plane, the generalized quadrangle and the generalized hexagon, respectively. An interested reader is referred to the dynamic survey on cages by Exoo and Jajcay [10].

2.2 Voltage graphs, lifts and Cayley graphs

There are several constructions producing graphs rich in symmetries and the most frequent one appears to rely on Cayley (di)graphs.

Let G be a group. For any subset X of G, the *(directed) Cayley graph* $\mathrm{Cay}(G, X)$ is the graph with vertex set G; and there is a dart from g to h if and only if $g^{-1}h \in X$. If the set X is closed under inverses, then the resulting Cayley graph becomes undirected. In order to obtain Cayley graphs without loops, one needs to exclude the identity element of G from X. On the other hand, there is a nice algebraic criterion to figure out whether a given graph Γ can be obtained as a Cayley graph for a suitable group G and its subset X, provided by *Sabidussi's theorem* [17]:

Theorem 2.1. *A graph Γ is a Cayley graph if and only if its automorphism group contains a regular subgroup.*

Another construction of graphs related to groups, called *voltage assignments* (or *lifts*) appeared first in topological graph theory, see [13]. The following short introduction to the method of voltage graphs is a compilation of [7, 9, 13] and [16]. The method is also known as *lifting construction*.

Let Γ be a multigraph. Let us think of the edges of Γ as being formed by pairs of oppositely directed *darts*; if e is a dart then e^{-1} denotes its reverse. The set $D(\Gamma)$ of all darts of Γ satisfies $|D(\Gamma)| = 2|E(\Gamma)|$. For a finite group G, a mapping $f\colon D(\Gamma) \to G$ is a *voltage assignment* if $f(e^{-1}) = (f(e))^{-1}$ for any dart $e \in D(\Gamma)$. The pair (Γ, f) determines the *lift* Γ_f of Γ. The graph Γ_f is often called the *derived graph*, and Γ the *base graph*. A base graph with a voltage assignment is a *voltage graph*. The vertex set and the dart set of the lift are $V(\Gamma_f) = V(\Gamma) \times G$ and $D(\Gamma_f) = D(\Gamma) \times G$, respectively. In the lift, (e, g) is a dart from the vertex (u, g) to the vertex (v, h) if and only if e is a dart from u to v in the voltage graph Γ, and, at the same time, $h = gf(e)$. Since the darts (e, g) and $(e^{-1}, gf(e))$ form an undirected edge of Γ_f, the lifted graph Γ_f can be considered to be an undirected graph. Briefly, we call the graph Γ_f a *voltage lift*. In order to specify a voltage assignment in a pictorial representation of a graph, we usually fix an orientation of the multigraph G in advance and assign voltages to the obtained darts; the reverse darts are assumed to carry the corresponding inverse voltages. The set $\pi_f^{-1}(u)$ is called a *fiber* above the vertex u, where $\pi_f\colon \Gamma_f \to \Gamma$ is the projection defined by $\pi_f(v, g) = v$ for each $v \in V(\Gamma)$ and $g \in G$.

Clearly, Cayley graphs are a special case of lifts, when the base graph of a Cayley graph is a bouquet, i.e., a multigraph having one vertex with several loops incident to it.

Since we are here interested in the lifts of dipoles (graphs of order 2), we formulate an analogue to the Sabidussi theorem for recognizing which graphs can be obtained as lift of dipoles. This result is a consequence of Theorem 2.2.2 in [13].

Theorem 2.2. *A graph Γ is isomorphic to a voltage lift of a dipole if and only if the group of automorphisms $\mathrm{Aut}(\Gamma)$ contains a semiregular subgroup with two orbits.*

We remind that a nontrivial permutation group is *semiregular*, if all of its orbits have equal size and the stabilizer of all the elements is trivial.

3 The family of $(q, 8)$-graphs Υ_q

In general, not much is known about the (k, g)-cages for $k \geq 3$, $g \geq 3$. However, very nice exceptions appear when $(k, g) \in \{(q + 1, 6), (q + 1, 8), (q + 1, 12)\}$, where q is a prime power. It is a well-known fact that the $(q+1,6)$, $(q+1,8)$, and the $(q+1,12)$-cages arose as the incidence graphs of the classical projective plane, generalized quadrangles and generalized hexagons of order q, respectively. By the nature of the construction, these graphs are bipartite and highly symmetric. There were several attempts to construct (k,g)-cages, or at least small (k,g)-graphs from these graphs as induced subgraphs (e.g., see [1, 2, 3, 4, 5]), when $k-1$ is not a prime power and $g \in \{6,8,12\}$, because in these cases no (k,g)-cages are known. Trivially, the induced subgraphs are also necessarily bipartite, so there is some interest to investigate, whether these graphs are vertex-transitive, or at least transitive on both parts, in the sense, that the two parts are formed by two fibers of a graph obtained as a lift from a dipole.

Let P be a chosen point and ℓ a chosen line of the generalized quadrangle of order q. It is known that those vertices in the $(q+1,8)$-cage that are at distance at least 3 from both objects P and ℓ are inducing a $(q,8)$-graph of order $2q^3$, if (P,ℓ) is a flag (an incident point-line pair); and of order $2q^3-2q$, if (P,ℓ) is an anti-flag (a non-incident point-line pair), cf. [11].

As it was shown in [14], the family of graphs of order $2q^3$ can also be obtained as a lift of dipole with a suitable voltage group of order q^3. Here we show that the analogous result is not true for the family of $(q,8)$-graphs of order $2q^3-2q$. In order to do this we show that the group of automorphisms of these graphs has precisely 4 orbits. We would like to stress once more that we are considering just the case when q is an odd prime power, since some of our argumentations are not valid when q is even.

In this section we start from an alternative description of the $(q+1,8)$-cages according to [3], since we will use for our computations the coordinates defined in that description. With their aid we will describe the inspected family Υ_q of $(q,8)$-graphs.

Starting from this section we use *division* x/y, or $\frac{x}{y}$ in the field $\mathbb{F}_q$ to indicate xy^{-1}.

3.1 A description of the initial graph Γ_q

Let q be a prime power. The following description of the $(q+1,8)$-cages was given by Abreu et al. in [3] and [1].

Definition 3.1. Let $\mathbb{F}_q$ be a finite field with $q \geq 2$ a prime power and ϱ be a symbol not in $\mathbb{F}_q$. Let $\Gamma_q = \Gamma_q[W_0, W_1]$ denote a bipartite graph with vertex sets

$$W_i = \{(a,b,c)_i \mid a,b,c \in \mathbb{F}_q\} \cup \{(\varrho,b,c)_i, (\varrho,\varrho,c)_i : b,c \in \mathbb{F}_q\} \cup \{(\varrho,\varrho,\varrho)_i\},$$

where $i \in \{0,1\}$, and edge set defined as follows:

For all $a,b,c \in \mathbb{F}_q$

$$\begin{aligned}
N_{\Gamma_q}((a,b,c)_1) &= \{(w, aw+b, a^2w+2ab+c)_0 : w \in \mathbb{F}_q\} \cup \{(\varrho,a,c)_0\} \\
N_{\Gamma_q}((\varrho,b,c)_1) &= \{(c,b,w)_0 : w \in \mathbb{F}_q\} \cup \{(\varrho,\varrho,c)_0\} \\
N_{\Gamma_q}((\varrho,\varrho,c)_1) &= \{(\varrho,c,w)_0 : w \in \mathbb{F}_q\} \cup \{(\varrho,\varrho,\varrho)_0\} \\
N_{\Gamma_q}((\varrho,\varrho,\varrho)_1) &= \{(\varrho,\varrho,w)_0 : w \in \mathbb{F}_q\} \cup \{(\varrho,\varrho,\varrho)_0\}.
\end{aligned}$$

Equivalently, for all $i, j, k \in \mathbb{F}_q$

$$
\begin{aligned}
N_{\Gamma_q}((i,j,k)_0) &= \{(w, j-wi, w^2 i - 2wj + k)_1 : w \in \mathbb{F}_q\} \cup \{(\varrho, j, i)_1\} \\
N_{\Gamma_q}((\varrho,j,k)_0) &= \{(j, w, k)_1 : w \in \mathbb{F}_q\} \cup \{(\varrho, \varrho, j)_1\} \\
N_{\Gamma_q}((\varrho,\varrho,k)_0) &= \{(\varrho, w, k)_1 : w \in \mathbb{F}_q\} \cup \{(\varrho, \varrho, \varrho)_1\} \\
N_{\Gamma_q}((\varrho,\varrho,\varrho)_0) &= \{(\varrho, \varrho, w)_1 : w \in \mathbb{F}_q\} \cup \{(\varrho, \varrho, \varrho)_1\}.
\end{aligned}
$$

As it was proved in [3], the graph Γ_q described in the above definition is isomorphic to the incidence graph $\mathcal{I}(q)$ of the classical generalized quadrangle $W(q)$, hence it has girth 8 and is regular of valency $q+1$. As we already mentioned, the graph $\mathcal{I}(q)$ (or Γ_q in our notation) served as an initial graph in many constructions of $(k,8)$-graphs for $k < q+1$. See, for example [1, 4, 5, 11].

The authors in [1] constructed a $(q,8)$-graph from Γ_q by removing a perfect dominating set consisting from the union of the second closed neighbourhood of vertices u and v such that their distance in Γ_q is 3. Equivalently, in the terms of the incidence graph $\mathcal{I}(q)$, one has to choose a non-incident point-line pair (P,ℓ) in the generalized quadrangle of order q, and to remove all the vertices from $\mathcal{I}(q)$ which are in distance at most 2 from P or ℓ. Following this idea, we describe the $(q,8)$-graphs Υ_q and will investigate their group of automorphisms.

3.2 Definition of the graph Υ_q

From now on, we will focus on a family of $(q,8)$-graphs Υ_q, for q an *odd* prime power. We will not consider the case, when q is even, since, in general, most of the results being stated are not valid in that case.

Let us choose the point $P = (\varrho,\varrho,\varrho)_0 \in V(\Gamma_q)$ and the line $\ell = (0,0,0)_1 \in V(\Gamma_q)$. It is easy to check, that P is not incident to ℓ, and thus $d_{\Gamma_q}(P,\ell) = 3$, since the diameter of Γ_q is 4, and the distance between any point and line is odd.

In fact, the choice of a non-incident point-line pair (P,ℓ) does not depend on the concrete choice of P and ℓ, since the collineation group of the corresponding generalized quadrangle is anti-flag transitive.

The second closed neighbourhoods are:

$$
\begin{aligned}
N^2_{\Gamma_q}[P] &= \{(\varrho,\varrho,\varrho)_0, (\varrho,x,y)_0, (\varrho,\varrho,x)_1 : x \in \mathbb{F}_q \cup \{\varrho\}, y \in \mathbb{F}_q\} \\
N^2_{\Gamma_q}[\ell] &= \{(x,0,0)_0, (\varrho,0,0)_0, (0,x,0)_1, (\varrho,0,x)_1, (\varrho,\varrho,0)_1, (y,-yx,y^2x)_1 : x,y \in \mathbb{F}_q\}.
\end{aligned}
$$

It is an easy routine-job to find the cardinalities $|N^2_{\Gamma_q}[P]| = |N^2_{\Gamma_q}[\ell]| = q^2+2q+2$ and also the intersection $N^2_{\Gamma_q}[P] \cap N^2_{\Gamma_q}[\ell] = \{(\varrho,0,0)_0, (\varrho,\varrho,0)_1\}$. Thus $|N^2_{\Gamma_q}[P] \cup N^2_{\Gamma_q}[\ell]| = 2(q+1)^2$.

Let Υ_q be the subgraph of Γ_q induced by $V(\Upsilon_q) = V(\Gamma_q) \setminus (N^2_{\Gamma_q}[P] \cup N^2_{\Gamma_q}[\ell])$. Thus, the graph Υ_q contains q^3-q points and q^3-q lines corresponding to V_0 and V_1, respectively:

$$
\begin{aligned}
V_0 &= \{(x,y,z)_0, x,y,z \in \mathbb{F}_q\} \setminus \{(x,0,0)_0 : x \in \mathbb{F}_q\}, \\
V_1 &= \{(\varrho,j,k)_1 : j,k \in \mathbb{F}_q, j \neq 0\} \cup \{(i,j,k)_1 : i,j,k \in \mathbb{F}_q, ij+k \neq 0\}.
\end{aligned}
$$

The edge set of Υ_q can be described using the neighbourhoods as follows:

$$N_{\Upsilon_q}((x,0,z)_0) = \{(r, -rx, r^2x+z)_1, r \in \mathbb{F}_q\},\ \text{for all } x, z \in \mathbb{F}_q,\ z \neq 0;$$
$$N_{\Upsilon_q}((x,y,z)_0) = \{(r, y-rx, r^2x - 2ry + z)_1, r \in \mathbb{F}_q, r \neq y^{-1}z\} \cup$$
$$\cup \{(\varrho, y, x)_1\},\ \text{for all } x, y, z \in \mathbb{F}_q,\ y \neq 0.$$

As well as equivalently:

$$N_{\Upsilon_q}((i,j,k)_1) = \{(x, ix+j, i^2x + 2ij + k)_0, x \in \mathbb{F}_q\},\ \text{for all } i, j, k \in \mathbb{F}_q \text{ and } ij + k \neq 0;$$
$$N_{\Upsilon_q}((\varrho, j, k)_1) = \{(k, j, x)_0, x \in \mathbb{F}_q\},\ \text{for all } j, k \in \mathbb{F}_q,\ j \neq 0.$$

Thus, the graph Υ_q is regular of valency q.

4 Distances in Υ_q

Let us define a few subsets of vertices of Υ_q, which will be important later on.

$$O_1 = \{(x,y,0)_0 : x, y \in \mathbb{F}_q, y \neq 0\}$$
$$O_2 = \{(x,y,z)_0 : x, y, z \in \mathbb{F}_q, z \neq 0\}$$
$$O_3 = \{(0,j,k)_1 : j, k \in \mathbb{F}_q, k \neq 0\}$$
$$O_4 = \{(i,j,k)_1 : i, j, k \in \mathbb{F}_q, i \neq 0, ij + k \neq 0\} \cup \{(\varrho, j, k)_1, j, k \in \mathbb{F}_q, j \neq 0\}.$$

Clearly, V_0 is a disjoint union of O_1 and O_2, while V_1 is a disjoint union of O_3 and O_4.

Let us start with a useful lemma.

Lemma 4.1. *Let $q > 4$ be an odd prime power. Let $u, v \in V_0$ be two different vertices corresponding to points. Then the following hold:*

(i) *If $d_{\Gamma_q}(u,v) = 4$, then $d_{\Upsilon_q}(u,v) = 4$.*

(ii) *If $d_{\Gamma_q}(u,v) = 2$, then $d_{\Upsilon_q}(u,v) \in \{2,6\}$.*

Proof. Since the graph Γ_q is the incidence graph of a generalized quadrangle of order q, then the diameter of Γ_q is equal to 4, and the pairs of vertices at distance 4 correspond to two non-collinear points, or dually, to two non-concurrent lines. Moreover, from the axioms of generalized quadrangles it is not hard to derive that each pair of vertices at distance 4 is joined by $q+1$ internally disjoint paths of length 4 in Γ_q.

(i) We will prove that at least one of these internally disjoint uv-paths of length 4 in Γ_q will be preserved in the induced subgraph Υ_q, when u and v correspond to points. (Here we also notice that, maybe surprisingly, this is not true when u and v correspond to lines.)

Let us choose two points $u = (u_1, u_2, u_3)_0$ and $v = (v_1, v_2, v_3)_0$ which are vertices in Υ_q and are at distance 4 in the overgraph Γ_q. Consider all the uv-paths of length 4 in Γ_q. Since exactly one neighbour of u in Γ_q was deleted, which was not a vertex in Υ_q, this results in destroying of one path of length 4. Similary, a deleted neighbour of v can destroy one more such path. Thus there are at least $q-1$ paths of length 4

in Γ_q of form $u - \ell_1 - P - \ell_2 - v$ such that ℓ_1 and ℓ_2 are vertices in Υ_q. Such a path is not a path in Υ_q if and only if $P \notin V(\Upsilon_q)$. From the definition of adjacencies in Υ_q it follows that this is possible only in two cases: if $P = (\varrho, \varrho, w)_0$ for some $w \in \mathbb{F}_q$, or if $P = (\varrho, a, b)_0$ for some $a, b \in \mathbb{F}_q$ and $(a, b) \neq (0, 0)$. The first case can appear just in the case when $w = u_1 = v_1$. Thus, if $u_1 = v_1$, then one more path of length 4 is destroyed. In the second case we find that the destroyal of a path of length 4 can happen when the value of a satisfies the following equation in $\mathbb{F}_q$:

$$a^2(u_1 - v_1) - 2a(u_2 - v_2) + (u_3 - v_3) = 0.$$

Clearly, this equation has at most one solution when $u_1 = v_1$, and at most two solutions when $u_1 \neq v_1$. Whenever the value of a exists, then the value b is uniquely determined. We omit the detailed computations.

Thus, in both cases at most 2 more uv-paths of length 4 can be destroyed in Γ_q. Altogether this means that between points u and v in Υ_q there are at least $q-3$ paths of length 4. Thus, the distance between them is exactly 4, since for $q > 4$ we have a uv-path of length 4 and $4 = d_{\Gamma_q}(u, v) \leq d_{\Upsilon_q}(u, v)$, because Υ_q is an induced subgraph of Γ_q.

(ii) Since the diameter of Γ_q is 4, the case $d_{\Upsilon_q}(u, v) > 4$ can appear only for those $u, v \in V_0$, for which $d_{\Gamma_q}(u, v) = 2$, according to part (i). Thus u and v have precisely one common neighbour in Γ_q. If this unique neighbour belongs to $V(\Upsilon_q)$, then $d_{\Upsilon_q}(u, v) = 2$. Otherwise $d_{\Upsilon_q}(u, v) \geq 6$, because the girth of Γ_q is 8. We will prove by contradiction that in this case $d_{\Upsilon_q}(u, v) = 6$. Suppose that there are $u, v \in V_0$ such that $d_{\Gamma_q}(u, v) = 2$ and $d_{\Upsilon_q}(u, v) > 6$. Since Υ_q is bipartite, we can equivalently say that $d_{\Upsilon_q}(u, v) \geq 8$. Consider a shortest uv-path in Υ_q. Denote by x_i the vertex at distance i from u on this path. Since $d_{\Upsilon_q}(u, x_6) > 4$ and $d_{\Upsilon_q}(u, x_8) > 4$, we have $d_{\Gamma_q}(u, x_6) = 2$ and $d_{\Gamma_q}(u, x_8) = 2$, according to (i). Thus, there exist vertices $y, z \in V(\Gamma_q)$ such that $u - y - x_6$ and $u - z - x_8$ are paths of length 2 in Γ_q. But in this case $u - y - x_6 - x_7 - x_8 - z - u$ is a cycle of length 6 in Γ_q, which contradicts the girth of Γ_q. □

We prove a similar lemma for the distances between vertices corresponding to lines.

Lemma 4.2. *Let $q > 4$ be an odd prime power. Then the followings hold:*

(i) *If $u = (u_1, u_2, u_3)_1$ and $v = (v_1, v_2, v_3)_1$ are vertices in V_1 such that $u_1 v_2 - u_2 v_1 \neq 0$, then $d_{\Gamma_q}(u, v) = 4 \implies d_{\Upsilon_q}(u, v) = 4$.*

(ii) *For each $u = (u_1, u_2, u_3)_1 \in V_1 \subseteq V(\Upsilon_q)$ there exists $v = (v_1, v_2, v_3)_1 \in V_1 \subseteq V(\Upsilon_q)$ such that $d_{\Gamma_q}(u, v) = 4$ and $d_{\Upsilon_q}(u, v) > 4$.*

Proof. (i) In a way similar to the proof of Lemma 4.1, here also at most two uv-paths of length 4 are destroyed because of deletion of a neighbour of u and v. The remaining paths of length 4 in Γ_q, which are destroyed in Υ_q have necessarily the form $u - P_1 - \ell - P_2 - v$, where $\ell \in \{(\varrho, 0, w)_1, (i, j, -ij)_1\}$ for some $w, i, j \in \mathbb{F}_q$, $w \neq 0$, $(i, j) \neq (0, 0)$.

The first possibility $\ell = (\varrho, 0, w)_1$ may appear only when $P_1 = (w, 0, x)_0$ and $P_2 = (w, 0, y)_0$ for some $x, y \in \mathbb{F}_q - \{0\}$, $x \neq y$. But in this case the adjacencies lead to the condition $w = -u_2/u_1 = -v_2/v_1$, contradicting to $u_1v_2 - u_2v_1 \neq 0$.

The second possibility $\ell = (i, j, -ij)_1$ leads to the following system of linear equations over $\mathbb{F}_q$:

$$u_2 \cdot i - u_1 \cdot j = u_1u_2 + u_3$$
$$v_2 \cdot i - v_1 \cdot j = v_1v_2 + v_3$$

having a unique solution if $u_1v_2 - u_2v_1 \neq 0$.

Thus, we conclude that there is at least one uv-path of length 4 in Υ_q.

(ii) Let us choose $\alpha, u_1, u_2 \in \mathbb{F}_q - \{0\}$, $\alpha \neq 1$. Let $u_3 \in \mathbb{F}_q$ such that $u_1u_2 + u_3 \neq 0$. Then the pair of vertices $u = (u_1, u_2, u_3)_1$ and $v = (\alpha u_1, \alpha u_2, v_3)_1$ has the required property, where $v_3 = (\alpha - \alpha^2)u_1u_2 + \alpha u_3$. One can see this with the help of the conditions derived in part (i). The uv-path of length 4 in Γ_q with middle vertex $(\varrho, 0, -u_2/u_1)_1$ is not in Υ_q, since $(\varrho, 0, -u_2/u_1)_1 \notin V(\Upsilon_q)$, while the remaining q paths of length 4 between them are also not in Υ_q, since in these cases the middle vertices of these paths have coordinates

$$((u_1u_2 + u_3 + u_1t)/u_2, t, -t(u_1u_2 + u_3 + u_1t)/u_2)_1,$$

where t runs through $\mathbb{F}_q$. Thus $d_{\Upsilon_q}(u, v) > 4$.

Similarly, one can show the existence of such vertices in the remaining cases.

- If $u_1 = 0$ then for all $u_2, u_3, \alpha \in \mathbb{F}_q$, $u_3 \neq 0$, $\alpha \notin \{0, 1\}$ we have that

$$d_{\Gamma_q}((0, u_2, u_3)_1, (0, \alpha u_2, \alpha u_3)_1) = 4 \text{ and}$$
$$d_{\Upsilon_q}((0, u_2, u_3)_1, (0, \alpha u_2, \alpha u_3)_1) > 4.$$

- If $u_2 = 0$ and $u_1, u_3 \neq 0$, then for all $\alpha \in \mathbb{F}_q - \{0, 1\}$ we have that

$$d_{\Gamma_q}((u_1, 0, u_3)_1, (\alpha u_1, 0, \alpha u_3)_1) = 4 \text{ and}$$
$$d_{\Upsilon_q}((u_1, 0, u_3)_1, (\alpha u_1, 0, \alpha u_3)_1) > 4.$$

□

Lemma 4.3. *For the number of vertices at distance* 6 *from a chosen vertex* $v \in V(\Upsilon_q)$ *the following holds, when* $q > 3$ *is an odd prime power:*

$$|N^6_{\Upsilon_q}(v)| = \begin{cases} q-1, & \text{if } v \in O_1; \\ q-2, & \text{if } v \in O_2; \\ 2q-3, & \text{if } v \in O_3 \cup O_4. \end{cases}$$

Proof. According to Lemma 4.2, the distance between two vertices corresponding to points is 6 in Υ_q, if they had a common neighbour in Γ_q which is not in Υ_q.

Considering a vertex $v = (x, y, 0)_0 \in O_1$, $x, y \in \mathbb{F}_q$, $y \neq 0$, we can say that all the vertices of form $u = (z, y, 0)_0$, where $z \in \mathbb{F}_q$, $z \neq x$ are at distance 6 from v in Υ_q, since their common neighbour $(0, y, 0)_0$ in Γ_q is not a vertex in Υ_q. Moreover, all other neighbours of v in Γ_q are also vertices of Υ_q, thus the number of vertices at distance 6 from v in Υ_q is $q-1$.

Now let us consider $v \in O_2$. In this case denote x the only deleted neighbour of v. Since x has precisely two such neighbours which are not in Υ_q, the number of vertices at distance 6 from v in Υ_q is equal to $q-2$.

The remaining part can be done similarly by tracking paths of length 6 and we leave it to the reader. □

Corollary 4.4. *There is no automorphism of Υ_q mapping a vertex from $O_1 \cup O_2$ to a vertex in $O_3 \cup O_4$, and vice versa.*

Corollary 4.5. *Let $q > 3$ be an odd prime power. Then the diameter of Υ_q is equal to 6.*

Proof. Since the graph is bipartite and the distance between two vertices corresponding to points can not be larger than 6, we have that the diameter is less than 8. By contradiction suppose that the diameter is equal to 7. Then there is a vertex v corresponding to a point, and a vertex w corresponding to a line such that $d_{\Upsilon_q}(v,w) = 7$. However, in this case all the q neighbours of w must belong to $N^6_{\Upsilon_q}(v)$, but this cannot occur, since $|N^6_{\Upsilon_q}(v)| < q$ according to Lemma 4.3. Thus the diameter of Υ_q can not be 7, thus it is equal to 6. □

Lemma 4.6. *For any vertex $v \in O_3$ we have $N_{\Upsilon_q}(v) \cap O_1 = \emptyset$, while for any $u \in O_4$ we have $N_{\Upsilon_q}(u) \cap O_1 \neq \emptyset$.*

Proof. We need to distinguish three cases.

(i) If $v \in O_3$, then $v = (0,j,k)_1$ for some $j,k \in \mathbb{F}_q$, $k \neq 0$. Further, the set of neighbours of v, according to the definition, is $N_{\Upsilon_q}(v) = \{(x,j,k)_0, x \in \mathbb{F}_q\}$. Since $k \neq 0$, the sets O_1 and $N_{\Upsilon_q}(v)$ are disjoint.

(ii) For all $j,k \in \mathbb{F}_q$, $j \neq 0$ the pair $\{(\varrho,j,k)_1,(k,j,0)_0\}$ is an edge of Υ_q.

(iii) For all $i,j,k \in \mathbb{F}_q$, $ij+k \neq 0$, $i \neq 0$ we have that

$$\{(i,j,k)_1,(-i^{-2}(2ij+k),-i^{-1}(ij+k),0)_0\}$$

is an edge in Υ_q. □

The previous lemmas have an important corollary.

Corollary 4.7. *Let G be the full group of automorphisms of the graph Υ_q. Then G has at least four orbits on $V(\Upsilon_q)$.*

Proof. Let $u \in O_i$ and $v \in O_j$ be two vertices in $V(\Upsilon_q)$ for some $i,j \in \{1,2,3,4\}$ such that $i \neq j$. Then u and v cannot belong to the same orbit under G. Hence, G has at least four orbits on $V(\Upsilon_q)$. □

From Corollary 4.7 and Theorems 2.1, 2.2 the main result of this paper follows as a consequence.

Theorem 4.8. *The graph Υ_q cannot be obtained as a Cayley graph, nor as a lift of a dipole.*

Proof. Since the group of automorphisms of Υ_q has at least four orbits, none of its subgroups can act regularly, or semi-regularly with two orbits on the set of vertices, therefore Υ_q is neither a Cayley graph, nor a lift of a dipole. □

More precise investigation of the full group of automorphisms of the graph Υ_q follows in the next section. We will show that it has precisely 4 orbits, and these coincide with the above-mentioned sets O_1, O_2, O_3 and O_4.

5 Automorphisms of Υ_q

In this section we describe a few mappings on $V(\Upsilon_q)$ and we show that they are automorphisms of Υ_q.

5.1 Mappings $\varphi_{\alpha,\beta}$

For all $\alpha, \beta \in \mathbb{F}_q$, $\alpha \neq 0$ define the mapping $\varphi_{\alpha,\beta}\colon V(\Upsilon_q) \to V(\Upsilon_q)$ as follows:

$$\begin{aligned}
\varphi_{\alpha,\beta}((x,y,z)_0) &= (\alpha x + \beta, \alpha y, \alpha z)_0,\\
\varphi_{\alpha,\beta}((i,j,k)_1) &= (i, \alpha j - \beta i, \alpha k + \beta i^2)_1\\
\varphi_{\alpha,\beta}((\varrho,j,k)_1) &= (\varrho, \alpha j, \alpha k + \beta)_1.
\end{aligned}$$

First, one has to notice that $\varphi_{\alpha,\beta}$ is closed on V_0, since $(y,z) = (0,0) \iff (\alpha y, \alpha z) = (0,0)$. It is also closed on V_1, since $ij + k = 0 \iff i(\alpha j - \beta i) + \alpha k + \beta i^2 = 0$ and $j = 0 \iff \alpha j = 0$. On the other hand, routine computations show that $\varphi_{\alpha,\beta}$ is a bijective mapping on $V(\Upsilon_q)$ and its inverse is $\varphi^{-1}_{\alpha,\beta} = \varphi_{\alpha^{-1},-\beta\alpha^{-1}}$. Moreover, the set

$$H = \{\varphi_{\alpha,\beta} : \alpha, \beta \in \mathbb{F}_q, \alpha \neq 0\}$$

is closed under the operation $\circ$ of composition of mappings. Again, it is a routine-job to check that $\varphi_{\gamma,\delta} \circ \varphi_{\alpha,\beta} = \varphi_{\alpha\gamma,\beta\gamma+\delta}$ for all $\alpha, \beta, \gamma, \delta \in \mathbb{F}_q$ and $\alpha, \gamma \neq 0$. Thus, we conclude that $(H, \circ)$ is a group isomorphic to the affine linear group $AGL_1(q)$.

In the next step we show that, in fact, H is a subgroup of the group of automorphisms of Υ_q. In Υ_q there are just two types of edges.

First, consider an edge e of type $\{(i,j,k)_1, (x, ix+j, i^2x+2ij+k)_0\}$, where $i, j, k, x \in \mathbb{F}_p$ and $ij + k \neq 0$. Its image under $\varphi_{\alpha,\beta}$ is

$$\begin{aligned}
\varphi_{\alpha,\beta}(e) =& \{(i, \alpha j - \beta i, \alpha k + \beta i^2)_1, (\alpha x + \beta, \alpha(ix + j), \alpha(i^2 x + 2ij + k))_0\}\\
=& \{(i, \alpha j - \beta i, \alpha k + \beta i^2)_1, (\alpha x + \beta, i(\alpha x + \beta) + (\alpha j - \beta i),\\
& i^2(\alpha x + \beta) + 2i(\alpha j - \beta i) + (\alpha k + \beta i^2))_0\},
\end{aligned}$$

which is an edge in Υ_q.

On the other hand, an edge of type $\{(\varrho, j, k)_1, (k, j, x)_0\}$, where $j, k, x \in \mathbb{F}_q$ and $j \neq 0$ is mapped to the pair $\{(\varrho, \alpha j, \alpha k + \beta)_1, (\alpha k + \beta, \alpha j, \alpha x)_0\}$ which, again, is an edge of Υ_q.

Thus, $H \leq \mathrm{Aut}(\Upsilon_q)$.

5.2 Mappings $\sigma_{\alpha,\beta}$

For all $\alpha, \beta \in \mathbb{F}_q$, $\alpha \neq 0$ define the mapping $\sigma_{\alpha,\beta}\colon V_0 \to V_0$ by the following:

$$\sigma_{\alpha,\beta}((x,y,z)_0) = \left(x + \frac{2\beta}{\alpha} y + \frac{\beta^2}{\alpha^2} z, \alpha y + \beta z, \alpha^2 z\right)_0.$$

On the set of points V_0 this mapping is equivalent to a linear transformation $T_{\alpha,\beta}$ corresponding to the matrix

$$\begin{bmatrix} 1 & \frac{2\beta}{\alpha} & \frac{\beta^2}{\alpha^2} \\ 0 & \alpha & \beta \\ 0 & 0 & \alpha^2 \end{bmatrix}$$

having non-zero determinant α^3. On the other hand, the linear transformation $T_{\alpha,\beta}$ leaves the set of points $\{(x,0,0)_0, x \in \mathbb{F}_q\}$ invariant, hence the mapping $\sigma_{\alpha,\beta}$ is a bijection on the set V_0.

The definition of $\sigma_{\alpha,\beta}$ on the set V_1 of lines is a little bit more complicated.

For all $j,k \in \mathbb{F}_q$ and $j \neq 0$:

$$\sigma_{\alpha,\beta}((\varrho,j,k)_1) = (\varrho,\alpha j,k)_1, \qquad \text{if } \beta = 0,$$
$$\sigma_{\alpha,\beta}((\varrho,j,k)_1) = \left(\frac{\alpha^2}{\beta}, -\frac{\alpha}{\beta}(\alpha k + \beta j), \frac{k\alpha^4}{\beta^2}\right)_1, \qquad \text{if } \beta \neq 0.$$

Further, for all $i,j,k \in \mathbb{F}_q$ and $ij + k \neq 0$:

$$\sigma_{\alpha,\beta}((i,j,k)_1) = \left(\varrho, \beta(ij+k), \frac{k\beta^2}{\alpha^2}\right)_1, \qquad \text{if } \alpha + \beta i = 0,$$
$$\sigma_{\alpha,\beta}((i,j,k)_1) = \left(\frac{\alpha^2 i}{\alpha+\beta i}, \alpha j + \frac{k\alpha\beta}{\alpha+\beta i}, \frac{k\alpha^4}{(\alpha+\beta i)^2}\right)_1, \qquad \text{if } \alpha + \beta i \neq 0.$$

Using standard techniques one can check that $\sigma_{\alpha,\beta}$ is a bijection on V_1.

Now, we are ready to prove that the set

$$K = \{\sigma_{\alpha,\beta} : \alpha,\beta \in \mathbb{F}_q, \alpha \neq 0\}$$

is a set of automorphism of Υ_q.

We have to check that all the edges e of Υ_q are mapped to edges. We distinguish several cases:

(i) $\beta = 0$ and $e = \{(\varrho,j,k)_1, (k,j,z)_0\}$, where $j,k,z \in \mathbb{F}_q, j \neq 0$.

$$\sigma_{\alpha,0}(e) = \{(\varrho,\alpha j,k)_1, (k,\alpha j,\alpha^2 z)_0\}$$

and it is easy to see that this is really an edge in Υ_q.

(ii) $\beta \neq 0$ and $e = \{(\varrho,j,k)_1, (k,j,z)_0\}$, where $j,k,z \in \mathbb{F}_q, j \neq 0$.

$$\sigma_{\alpha,\beta}(e) = \left\{\left(\frac{\alpha^2}{\beta}, -\frac{\alpha}{\beta}(\alpha k + \beta j), \frac{k\alpha^4}{\beta^2}\right)_1, \left(k + \frac{2\beta}{\alpha}j + \frac{\beta^2}{\alpha^2}z, \alpha j + \beta z, \alpha^2 z\right)_0\right\}.$$

Since

$$\frac{\alpha^2}{\beta} \cdot \left(k + \frac{2\beta}{\alpha}j + \frac{\beta^2}{\alpha^2}z\right) - \frac{\alpha}{\beta} \cdot (\alpha k + \beta j) = \alpha j + \beta z$$

and

$$\left(\frac{\alpha^2}{\beta}\right)^2 \cdot \left(k + \frac{2\beta}{\alpha}j + \frac{\beta^2}{\alpha^2}z\right) + 2 \cdot \frac{\alpha^2}{\beta} \cdot \frac{-\alpha}{\beta}(\alpha k + \beta j) + \frac{k\alpha^4}{\beta^2} = \alpha^2 z,$$

$\sigma_{\alpha,\beta}(e)$ is an edge in Υ_q.

(iii) Similarly, like in the previous two cases, just using algebraic manipulations with coordinates one can show that $\sigma_{\alpha,\beta}(e)$ is an edge in Υ_q, if

$$e = \big\{(i,j,k)_1, (x, ix+j, i^2x+2ij+k)_0\big\}$$

for all feasible $i,j,k,x \in \mathbb{F}_p$ in both cases: when $\alpha+\beta i = 0$ and also when $\alpha+\beta i \neq 0$.

It requires some effort to see that the set K is closed under the operation of composition of mappings. Moreover, for all $\alpha,\beta,\gamma,\delta \in \mathbb{F}_q$ and $\alpha,\gamma \neq 0$ we have

$$\sigma_{\gamma,\delta} \circ \sigma_{\alpha,\beta} = \sigma_{\alpha\gamma,\beta\gamma+\alpha^2\delta}.$$

Hence we have that $K \leq \mathrm{Aut}(\Upsilon_1)$ and $K \cong AGL_1(q)$.

5.3 Frobenius automorphisms π

It is well-known from the theory of finite fields that the full group of automorphisms of a finite field of order $q = p^n$, where p is a prime, consists from the so-called *Frobenius automorphisms*. For all $r \in \{0,1,\dots,n-1\}$ the mapping $\pi_r \colon \mathbb{F}_q \to \mathbb{F}_q$ such that $x \mapsto x^{p^r}$ for each $x \in \mathbb{F}_q$ is an automorphism of the field $\mathbb{F}_q$.

We can extend each such automorphism by defining $\pi_r \colon \varrho \mapsto \varrho$. This extension induces naturally (component-wise on the coordinates) a mapping on the vertices of the graph Υ_q. It is not hard to check that these Frobenius automorphisms generate automorphisms of the graph Υ_q.

Let us denote by $\mathcal{F}$ the set of all Frobenius automorphisms of our basic finite field $\mathbb{F}_q$ of order $q = p^n$, for some prime p and natural number n. Then $|\mathcal{F}| = n$ and the group generated by the Frobenius automorphisms is a cyclic group of order n.

5.4 More information about the automorphisms of Υ_q

Let us consider the groups H and K obtained above. The first thing we can notice is that they have trivial intersection. On the other hand, for all $\alpha,\beta,\gamma,\delta \in \mathbb{F}_q$, $\alpha,\gamma \neq 0$ we have that

$$\sigma_{\alpha,\beta} \circ \varphi_{\gamma,\delta} = \varphi_{\gamma,\delta} \circ \sigma_{\alpha,\beta}.$$

In other words, elements of H commute with the elements of K, thus we have the following lemma.

Lemma 5.1. *Let H and K be the groups defined above. Then $H \times K \leq \mathrm{Aut}(\Upsilon_q)$, or equivalently, the full group of automorphisms of Υ_q contains a subgroup isomorphic to $AGL_1(q) \times AGL_1(q)$.*

In addition, if we consider the automorphisms of Υ_q generated by the Frobenius automorphisms of the finite field $\mathbb{F}_q$, we can extend the latter lemma to the following:

Lemma 5.2. *Let $\mathbb{F}_q$ be a finite field of order $q = p^n$, where p is an odd prime and n an integer. Then*

$$\mathrm{Aut}(\Upsilon_q) \geq (H \times K) \rtimes \mathcal{F} \cong (AGL_1(q) \times AGL_1(q)) \rtimes \mathbb{Z}_n.$$

Theorem 5.3. *Let $G = \mathrm{Aut}(\Upsilon_q)$ be the (full) group of automorphisms of Υ_q. Then G has exactly four orbits on $V(\Upsilon_q)$ and these orbits are precisely the sets O_1, O_2, O_3 and O_4. Two of the orbits have length $q(q-1)$ and the remaining two have length $q^2(q-1)$.*

Proof. We already proved that two vertices from different sets O_i cannot be in the same orbit under G, when $q > 3$ is odd. An easy computation using GAP [12] shows that this is the case also when $q = 3$, i.e., the automorphism group of Υ_3 has also 4 orbits coinciding with the sets $O_1, \ldots, O_4$.

We now show that for any two vertices u and v from the same set there is an automorphism $\tau \in H \times K \leq G$ mapping one to the other.

(i) Let $u, v \in O_1$. Then $u = (x, y, 0)_0$ and $v = (x', y', 0)_0$ for some $x, x', y, y' \in \mathbb{F}_q$ and $y, y' \neq 0$. The automorphism $\varphi_{\alpha,\beta} \in H$ maps u to v when $\alpha = y'/y$ and $\beta = x' - \alpha x$.

(ii) Let $u, v \in O_2$. Then $u = (x, y, z)_0$ and $v = (x', y', z')_0$ for some $x, x', y, y', z, z' \in \mathbb{F}_q$ and $z, z' \neq 0$. The automorphism $\varphi_{\gamma,\delta} \circ \sigma_{1,\beta} \in H \times K$ maps u to v when choosing $\gamma = z'/z$, $\beta = (y' - \gamma y)/z'$ and $\delta = x' - \gamma x - 2\beta\gamma y - \beta^2 z$.

(iii) Let $u, v \in O_3$. Then $u = (0, j, k)_1$ and $v = (0, j', k')_1$ for some $j, j', k, k' \in \mathbb{F}_q$ and $k, k' \neq 0$. Then the automorphism $\varphi_{\gamma,0} \circ \sigma_{1,\beta}$ maps u to v when $\gamma = k'/k$ and $\beta = (j' - \gamma j)/\gamma k$.

(iv) Let $u, v \in O_4$.

   (a) The automorphism $\varphi_{\alpha,\beta}$ maps $u = (\varrho, j, k)_1$ to $v = (\varrho, j', k')_1$ for example, when $\alpha = j'/j$ and $\beta = k' - \alpha k$.

   (b) The automorphism $\sigma_{\alpha,\beta} \circ \varphi_{\gamma,\delta}$ maps $u = (\varrho, j, k)_1$ to $v = (i', j', k')_1$, for instance, when $\alpha = \beta = i'$, $\gamma = -(\alpha j' + k')/(\alpha^2 j)$ and $\delta = (k'/\alpha^2) - k\gamma$.

   (c) The automorphism $\sigma_{\alpha,\beta} \circ \varphi_{\gamma,\delta}$ maps $u = (i, j, k)_1$ to $v = (\varrho, j', k')_1$ when one chooses $\alpha = -i$, $\beta = 1$, $\gamma = j'/(ij + k)$ and $\delta = k' - (k\gamma/\alpha^2)$.

   (d) The automorphism $\sigma_{\alpha,0} \circ \varphi_{\gamma,\delta}$ maps $u = (i, j, k)_1$ to $v = (i', j', k')_1$ under the choice $\alpha = i'/i$, $\gamma = (i'j' + k')/(\alpha^2(ij + k))$ and $\delta = (\alpha\gamma j - j')/i'$.

Easy counting arguments show that the lengths of these orbits are as expected. □

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