Three conjectures of Ostrander on digraph Laplacian eigenvectors*

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Abstract

Ostrander proposed three conjectures on the connections between topological properties of a weighted digraph and combinatorial properties of its Laplacian eigenvectors. We verify one of his conjectures, give counterexamples to the other two, and then seek for related valid connections and generalizations to Schrödinger operators on countable digraphs. We suggest the open question of deciding if the countability assumption can be dropped from our main results.

Keywords: Alexandrov topology, harmonic function, nonnegative eigenvector.

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1 Background

An eigenfunction of a linear operator can be viewed as a fixed point, namely a time-invariant point, of the operator in a corresponding projective space. Many dynamical processes on a geometric domain, including diffusion processes and consensus processes [18, 24, 29, 40, 42, 44, 57, 58], are driven by a Laplacian operator that reflects the local connectivity scenarios of the space. It is natural to expect that the shape of an eigenfunction of a Laplacian operator should somehow follow the shape of the underlying space; That is, you may be able to tell/predict the shape of space from some time-invariant data. Classical Fourier analysis provides deep understanding of signals over regular domains, to process graph-supported data we should accordingly develop spectral graph theory or the theory of graph Fourier transforms [3, 9, 12, 21, 27, 39, 41, 46, 50]. There are already quite some

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results and questions on the relationship between the oscillations of Laplacian eigenvector and the landscape of the underlying digraph, say those related to Courant's nodal line theorem [55], those related to Fiedler's monotonicity theorem [32, 48, 50], those related to Rauch's hot spots conjecture [16, 22] and others [52, 56]. To describe the influence of eigenstructures on connectivity patterns, various concepts on the digraphs, say diffusion distance [16] or Alexandrov topology [45], have been considered. Especially, Ostrander listed three conjectures on digraph Laplacians and Alexandrov topology [45], which is the starting point of this work.

2 Alexandrov topologies and Laplacian eigenvectors

We write $\mathbb{R}_+$ for the set of positive reals, $\mathbb{R}_{\geq 0}$ for the set of nonnegative reals, $\mathbb{N}_0$ for the set of nonnegative integers, $\mathbb{N}$ for the set of positive integers, and $\mathbb{K}$ for a field being either $\mathbb{R}$ or $\mathbb{C}$. For each $n \in \mathbb{N}$, $[n]$ refers to the set of first n positive integers. Let S be a set and let M be a nonnegative function on $S \times S$, namely $M : S \times S \rightarrow \mathbb{R}_{\geq 0}$. We naturally view this function M as a weighted digraph on S and say that S is the vertex set of M , denoted by $V(M)$, and let the arc set of M be $A(M) := \{(v, w) : M(v, w) > 0\} \subseteq V(M) \times V(M)$. For each $v \in V(M)$, its out-degree in M is $\deg_M^+(v) := \sum_{w \in S} M(v, w)$ and its in-degree in M is $\deg_M^-(v) := \sum_{w \in S} M(w, v)$. If the summation does not converge, it is regarded as $+\infty$. We say that a digraph M is a finite-out-degree (FOD) digraph if every vertex of M has a finite out-degree. Moreover, if the out-degrees of vertices of M is bounded, we say that M is a bounded-out-degree (BOD) digraph. The out-neighbors of a vertex v in M is $N_M^+(v) := \{w : M(v, w) > 0\}$ and the in-neighbors of v in M is $N_M^-(v) := \{w : M(w, v) > 0\}$. We call a digraph M a finite-out-neighbor (FON) digraph if every vertex of M has a finite set of out-neighbors. An FON digraph is surely a BOD digraph, and a BOD digraph is trivially an FOD digraph. We name a digraph a finite/countable digraph if its vertex set is a finite/countable set. Surely, every finite digraph is FON, and hence FOD. A path of length ℓ from a vertex v to a vertex w in M is a sequence of vertices $v = v_0, v_1, \dots, v_\ell = w$ such that $(v_{i-1}, v_i) \in A(M)$ for $i \in [\ell]$. We denote by $v \rightarrow w$ that $(v, w) \in A(M)$ and by $v \twoheadrightarrow w$ that there is a path from v to w in M . The future of v in M is defined to be

$$v\uparrow_M := \{w : v \twoheadrightarrow w\}$$

while the history of v in M is defined as

$$v\downarrow_M := \{w : w \twoheadrightarrow v\}.$$

In general, for any subset T of $V(M)$, let $T\uparrow_M := \bigcup_{v \in T} v\uparrow_M$ and $T\downarrow_M := \bigcup_{v \in T} v\downarrow_M$. If M takes value in $\{0, 1\}$, we call M transitive provided $M(u, v) = M(v, w) = 1$ implies $M(u, w) = 1$. In general, we refer to a digraph M as a transitive digraph provided $M(v, w) > 0$ implies that $M(u, v) \leq M(u, w)$ for all $u, v, w \in V(M)$. The Alexandrov topology on $S = V(M)$ induced by M has a set $T \subseteq V(M)$ as an open set if and only if $T\downarrow_M = T$. We will simply speak of an open set or a closed set of M to mean such a set in the Alexandrov topology induced by M . To see the naturalness of the concept of Alexandrov topology, you can recall that basically the step from a digraph to its lattice of open sets is a Birkhoff transform and this operation produces all distributive lattices [47]. A strongly connected component of M is a minimal nonempty subset of $V(M)$ which is the intersection of a closed set and an open set of M . A strongly connected component of

M is called a sink component if it is a closed set in M . We write the union of the set of sink components of a weighted digraph M by Ξ_M .

The Laplacian $\mathcal{L}_M$ of a digraph M is a linear operator from a suitable linear subspace U of $\mathbb{K}^{V(M)}$ to some other subspace of $\mathbb{K}^{V(M)}$ such that

$$(\mathcal{L}_M f)(v) = \sum_{w \in V(M)} M(v, w)(f(v) - f(w)) \quad (2.1)$$

for every $f \in U$. In general, the summation in Equation (2.1) may not converge, and so, in order to make Equation (2.1) well-defined, we will specify the domain of the Laplacian for various digraph classes later. In particular, if the digraph M is viewed as an FOD digraph, the domain will be chosen as $\ell^\infty(V(M))$; If the digraph M is viewed as a BOD digraph, the domain will be chosen as $\ell^1(V(M))$; If the digraph M is viewed as an FON digraph, the domain will be chosen as all $\mathbb{K}$ -valued functions on $V(M)$. Additionally, when the digraph is finite, the Laplacian can be represented by the matrix $\mathcal{L}_M = D_M - M$, where D_M is the diagonal matrix on $V(M)$ whose v -th diagonal element is $\deg_M^+(v)$.

For any linear map L from U to U' , $\text{Ker } L$ stands for the right null (kernel) space $\{x \in U : L(x) = \mathbf{0} \in U'\}$. For each weighted digraph M , we refer to each $f \in \text{Ker } \mathcal{L}_M$ as a harmonic function with respect to M . We say that f is harmonic at $v \in V(M)$ with respect to M if $(\mathcal{L}_M f)(v) = 0$.

Figure 1: A transitive digraph and its reverse.

Example 2.1. Let M be the transitive digraph as shown on the left of Figure 1 and let $M^\top$ be its transpose, also called its reverse digraph, which is the transitive digraph as shown on the right of Figure 1.

We display a basis for the left eigenspaces of $\mathcal{L}_M$ and $\mathcal{L}_{M^\top}$ as follows:

$$\mathcal{L}_M : \begin{bmatrix} 0 & 1 & 1 \\ 2 & -1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{matrix} u \\ v \\ w \end{matrix} \begin{bmatrix} 2 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & & \\ & 2 & \\ & & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 2 & -1 & -1 \\ 0 & 1 & -1 \end{bmatrix};$$

$$\mathcal{L}_{M^\top} : \begin{bmatrix} 2 & 0 & 0 \\ 2 & -1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{matrix} u \\ v \\ w \end{matrix} \begin{bmatrix} 0 & 0 & 0 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & & \\ & 1 & \\ & & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 2 & -1 & -1 \\ 0 & 1 & -1 \end{bmatrix}.$$

We display a basis for the right eigenspaces of $\mathcal{L}_M$ and $\mathcal{L}_{M^\top}$ as below:

$$\mathcal{L}_M : \begin{matrix} u \\ v \\ w \end{matrix} \begin{bmatrix} 2 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & & \\ & 2 & \\ & & 2 \end{bmatrix};$$

$$\mathcal{L}_M^\top : \begin{matrix} u & v & w \\ \begin{bmatrix} 0 & 0 & 0 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \end{matrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & & \\ & 1 & \\ & & 3 \end{bmatrix}.$$

In general, for a matrix A , its Jordan canonical form J and an invertible matrix P , it surely holds

$$PA = JP \text{ if and only if } AP^{-1} = P^{-1}J.$$

Therefore, if we can say something on the relationship between the combinatorial patterns of a matrix and its inverse matrix, we may be able to somehow link the patterns of the left eigenspace and the right eigenspace of that matrix.

Ostrander proposed three conjectures in [45]. He stated each conjecture in two parts but one can easily check that the two parts are equivalent to each other. We summarize his conjectures as below and invite the readers verify the conjecture for Example 2.1.

Conjecture 2.2. *Let M be a finite digraph.*

- (1) ([45, Conjecture 1]) *If M is transitive and f is a nonnegative right eigenvector of $\mathcal{L}_M$, then $f(v) \geq f(w)$ for every $v, w \in V(M)$ such that $v \rightarrow w$.*
- (2) ([45, Conjecture 2]) *If f is a nonnegative right eigenvector of $\mathcal{L}_M$, then $\text{supp}(f)$ is open in M .*
- (3) ([45, Conjecture 3]) *A set $P \subseteq V(M)$ is the support of a nonnegative harmonic function with respect to M if and only if $P = S \downarrow_M$ for some sink component S of M .*

As with Conjecture 2.2 (2), it has a very simple proof.

Theorem 2.3. [45, Conjecture 2] is correct. That is, for any nonnegative square matrix M and any nonnegative right eigenvector f of $\mathcal{L}_M$, it holds $\text{supp}(f) = \text{supp}(f) \downarrow_M$.

Proof. By the definition of open set, our task is to deduce $f(v) > 0$ from the assumption that $f(w) > 0$ and $M(v, w) > 0$. We assume that $\mathcal{L}_M f = \lambda f$. Evaluating both sides at the vertex v yields

$$(\deg_M^+(v) - \lambda)f(v) = \sum_{u \in V(M)} M(v, u)f(u) \geq M(v, w)f(w) > 0.$$

This implies $f(v) \neq 0$ and so, since f is nonnegative, $f(v) > 0$ follows. $\square$

Unfortunately, Conjecture 2.2 (1) does not hold true.

Example 2.4. Let $m \geq 3$ and let M be the matrix on $\{v_1, \dots, v_m\}$ with

$$M(v_i, v_j) = \begin{cases} 1, & \text{if } i = 1 \text{ and } j > 1; \\ 0, & \text{otherwise.} \end{cases}$$

Then the function f with $f(v_1) = 1$, $f(v_2) = m - 1$ and $f(v_i) = 0$ for $i = 3, \dots, m$, is a nonnegative harmonic function with respect to M ; See Figure 2. Note that $f(v_1) < f(v_2)$ while $(v_1, v_2) \in A(M)$.

Figure 2: A digraph and a harmonic function with respect to it.

Let us further demonstrate that one can even see arbitrary oscillation pattern in a Laplacian eigenvector of a transitive digraph along a path of the digraph.

Figure 3: The Hasse diagram of a poset.

Example 2.5. Note that a poset is a transitive acyclic digraph. Consider a digraph M which induces a poset whose Hasse diagram is as depicted in Figure 3. In other words, $M(v, w) = 1$ if and only if $v \neq w$ and there is a path from v to w in the Hasse diagram. We show that, for every element $\chi \in \{0, \pm 1\}^m$, there is a nonnegative harmonic function f on M such that $\chi = (\text{sgn}(f(t_m) - f(t_{m-1})), \text{sgn}(f(t_{m-1}) - f(t_{m-2})), \dots, \text{sgn}(f(t_1) - f(t_0)))$, where sgn is the sign function. For each $j \in [m]$, we choose $f_j \in \mathbb{R}^{V(M)}$ by setting

$$f_j(v) = \begin{cases} 2j, & \text{if } v = \ell_j; \\ 1, & \text{if } v = \ell_k, k > j; \\ 1, & \text{if } v = t_k, k \geq j; \\ 0, & \text{otherwise.} \end{cases}$$

It is obvious that $f_j(\ell_j) = 2j > 1 = f_j(t_j)$ and that f_j is harmonic with respect to M . It follows that a convex combination of such f_j and the constant function can give us a required harmonic function.

Example 2.6. Consider a digraph M which has more than one sink component. The all-ones function on $V(M)$ is surely harmonic with respect to M . But its support, namely $V(M)$, is not the history of any single sink component of M .

In order to exclude Example 2.6 as a counterexample, we have to modify the statement of Conjecture 2.2 (3). Let us propose the following variant of Conjecture 2.2 (3), which will be proved in Section 3.

Theorem 2.7. *Let M be a finite digraph and let $P \subseteq V(M)$. Then the following are equivalent.*

- (1) The set P is the support of a nonnegative harmonic function with respect to M .
- (2) The set P is the history of the union of several sink components of M .
- (3) The set P is the support of a nonnegative right eigenvector of $\mathcal{L}_M$ and $P \cap \Xi_M \neq \emptyset$.

Here is a left eigenvector counterpart of Conjecture 2.2, which we will also prove in Section 3.

Theorem 2.8. *Let M be a finite digraph and let f be a nonnegative left eigenvector of the Laplacian matrix $\mathcal{L}_M$. Then the following hold.*

- (1) $\text{supp}(f)$ is closed in M .
- (2) If M is transitive, then for every $v, w \in V(M)$ with $v \rightarrow w$ it holds $f(v) \leq f(w)$.
- (3) A set $P \subseteq V(M)$ is the support of a nonnegative left eigenvector of M if and only if P is the union of several sink components of M .

3 Eigenfunctions of finite digraph Laplacians

To prove Theorems 2.7 and 2.8, we need to recall some facts on the right and left null spaces of digraph Laplacians. These facts are in the folklore in various guises for many years. For the purpose of our subsequent proof and for the convenience of readers, we will briefly outline a proof of these facts; Whenever possible, we will try to avoid the most well-known treatments. But let us also point out here several references where such facts and their proofs can be found explicitly. Agaev and Chebotarev [1, Theorem 4] obtained the rank of a Laplacian matrix. The right null space of a Laplacian matrix was determined by Caughman and Veerman [13, Theorem 3.3], by Gunawardena [25, p. 5], and by Mirzaev and Gunawardena [42, Proposition 5]. Ostrander [45, Proposition 16] described the structure of the right null space of the Laplacian of a transitive digraph. Veerman and Kummel [57, Theorem 4.5] presented the structure of the left null space of a Laplacian matrix.

The spectral radius of a matrix A is denoted by $\rho(A)$. An $n \times n$ matrix A is irreducible if for every $i, j \in [n]$ there exists a nonnegative integer t such that $A^t(i, j) \neq 0$. A nonnegative matrix is irreducible if and only if it is strongly connected. It turns out that Perron eigenvalues and Perron eigenvectors of nonnegative matrices [36] have close relation with Laplacian spectrum and Laplacian eigenvectors.

Theorem 3.1 (Perron-Frobenius Theorem [28, p. 10–4]). *Let A be an irreducible nonnegative $n \times n$ matrix with spectral radius r . Then the followings hold.*

- (1) The number r is an eigenvalue of A . If $n \geq 2$, we must have $r > 0$.
- (2) The eigenvalue r is simple.
- (3) The matrix A has a positive right eigenvector x and a positive left eigenvector $y^\top$ associated to the eigenvalue r .
- (4) The only positive eigenvectors, left or right, are those associated to the eigenvalue r .
- (5) The eigenvalue r satisfies $\min_{i \in [n]} \sum_{j \in [n]} A(i, j) \leq r \leq \max_{i \in [n]} \sum_{j \in [n]} A(i, j)$.

Corollary 3.2. *Let M be a finite strongly connected weighted digraph. Then the following hold.*

- (1) *The number 0 is an eigenvalue of $\mathcal{L}_M$.*
- (2) *The eigenvalue 0 is simple. With the exception of the zero eigenvalue, all eigenvalues of M have a positive real part.*
- (3) *$\mathcal{L}_M$ has a positive right eigenvector $x = \mathbf{1}_{V(M)}$ and a positive left eigenvector $y^\top$ associated to the eigenvalue 0.*
- (4) *The only positive eigenvectors of $\mathcal{L}_M$, left or right, are those associated to the eigenvalue 0.*

Proof. Let Δ be the maximum out-degree of M . Note that $\mathcal{L}_M = \Delta I - (\Delta I + M - D_M)$ while $\Delta I + M - D_M$ is an irreducible nonnegative square matrix. It is clear that λ is an eigenvalue of $\mathcal{L}_M$ if and only if $\Delta - \lambda$ is an eigenvalue of $\Delta I + M - D_M$. By the last claim in Theorem 3.1, $\rho(\Delta I + M - D_M) = \Delta$. We thus see that everything follows from Theorem 3.1. $\square$

Let us fix M to be a finite weighted digraph throughout the remainder of this section. Let C be a strongly connected component of M , let $H = C \downarrow_M \setminus C$, and let $R = V(M) \setminus C \downarrow_M$. We can represent the matrix $\mathcal{L}_M$ as follows:

$$\mathcal{L}_M = \begin{matrix} & \begin{matrix} H & C & R \end{matrix} \\ \begin{matrix} H \\ C \\ R \end{matrix} & \begin{bmatrix} D|_H - W_{H,H} & -W_{H,C} & -W_{H,R} \\ \mathbf{0} & D|_C - W_{C,C} & -W_{C,R} \\ \mathbf{0} & \mathbf{0} & L_R \end{bmatrix} \end{matrix}, \quad (3.1)$$

where L_R is the Laplacian on the induced subgraph $M(R, R)$, $D|_H$ and $D|_C$ are the $H \times H$ diagonal matrix and the $C \times C$ diagonal matrix whose v -th diagonal element is equal to $\deg_M^+(v)$, and $W_{P,Q}$ records the weights of arcs from P to Q for $P, Q \in \{H, C, R\}$. Note that $W_{C,R} = \mathbf{0}$ and $D|_C - W_{C,C}$ coincides with the Laplacian of $M(C, C)$ if C is a sink component of M . We will write $D|_C - W_{C,C}$ as L_C in the case that C is a sink component of M .

Observation 3.3. For each sink component C of M , there exists $y_C \in \mathbb{R}_{\geq 0}^{V(M)} \cap \text{Ker } \mathcal{L}_M^\top$ with $\text{supp}(y_C) = C$.

Proof. By Corollary 3.2, there exists $y \in \mathbb{R}_+^C$ such that $y^\top L_C = \mathbf{0}_C^\top$. Let

$$y_C = \begin{matrix} H \\ C \\ R \end{matrix} \begin{bmatrix} \mathbf{0} \\ y \\ \mathbf{0} \end{bmatrix}. \quad (3.2)$$

It is easy to see that $y_C^\top \mathcal{L}_M = \mathbf{0}^\top$. $\square$

Observation 3.4 (Local minimum principle). Let M be a finite weighted digraph. Suppose f is harmonic at v with respect to M , namely $f(v)$ is the weighted average of the values of f at the out-neighbors of v in M . If f takes local minimum value at v in the sense that $f(v) \leq f(w)$ for every $w \in N_M^+(v)$, then f takes the same value at its out-neighbors.

Observation 3.5. If $H \neq \emptyset$, then

$$\text{Ker}(D|_H - W_{H,H}) = \{\mathbf{0}\}. \quad (3.3)$$

In particular, $D|_H - W_{H,H}$ is of full rank.

Proof. Suppose $g \in \text{Ker}(D|_H - W_{H,H})$. Define $f \in \mathbb{K}^{V(M)}$ by setting

$$f(v) = \begin{cases} g(v), & \text{if } v \in H; \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that f vanishes on all sink components of M and $\mathcal{L}_M f = \mathbf{0}$. By Observation 3.4, f must be the zero function and hence $g = \mathbf{0}$ follows. $\square$

A perturbed Laplacian is the sum of a Laplacian and a nonzero nonnegative diagonal matrix. Observation 3.5 essentially says that a perturbed Laplacian of any strongly connected digraph is nonsingular. This fact has been proved many times in the literature. Caughman and Veerman [13, Lemma 2.4] as well as Mirzaev and Gunawardena [42, Lemma 2, Fig. 4] used the same trick of embedding a perturbed Laplacian into a Laplacian, which is somehow a disguised version of our deduction in Observation 3.5; Note that Gunawardena [25, p. 5] made use of the matrix-tree Theorem (indeed, Markov chain tree Theorem [2, 30]) to obtain the same result.

For any digraph M , a basic open set in the Alexandrov topology induced by M is a set of the form $S \downarrow_M$ for some finite sink component S of M .

Observation 3.6. For each sink component C of M , there exists $x_C \in \mathbb{R}_{\geq 0}^{V(M)} \cap \text{Ker } \mathcal{L}_M$ with $\text{supp}(x_C) = C \downarrow_M$. That is, each basic open set of M is the support of a nonnegative harmonic function on M .

Proof. If $H = \emptyset$, we can simply take

$$x_C := \frac{C}{R} \begin{bmatrix} \mathbf{1}_C \\ \mathbf{0}_R \end{bmatrix}.$$

Assume that $H \neq \emptyset$. By Observation 3.5, $D|_H - W_{H,H}$ is nonsingular. This allows us set

$$x_C := \frac{H}{R} \begin{bmatrix} (D|_H - W_{H,H})^{-1} W_{H,C} \mathbf{1}_C \\ \mathbf{1}_C \\ \mathbf{0}_R \end{bmatrix}. \quad (3.4)$$

It is straightforward to check that $\mathcal{L}_M x_C = \mathbf{0}$. Note that x_C , a real harmonic function with respect to M , is determined by its values at the sink component C .

By Observation 3.4, the minimum value of x_C is achieved inside $R \cup C$ and so x_C is nonnegative. It then follows from Theorem 2.3 that $\text{supp}(x_C)$ is open. Since $C \subseteq \text{supp}(x_C)$, we must have $\text{supp}(x_C) \subseteq C \cup H = C \downarrow_M \subseteq \text{supp}(x_C)$, as wanted. $\square$

For each $v \in H$ and $w \in C$, there is a path from v to w . This implies

$$\sum_{\ell=0}^{\infty} ((D|_H)^{-1} W_{H,H})^{\ell} (D|_H)^{-1} W_{H,C} \mathbf{1}_C \in \mathbb{R}_+^H.$$

Therefore, another way to get Observation 3.6 is to show that $(D|_H - W_{H,H})^{-1} = (I - D|_H^{-1}W_{H,H})^{-1}D|_H^{-1} = \sum_{\ell=0}^{\infty} ((D|_H)^{-1}W_{H,H})^\ell (D|_H)^{-1}$. This is not as trivial as it may seem: Note that $-\frac{1}{3} = (1-4)^{-1} \neq 1 + 4 + 4^2 + 4^3 + \dots$. Anyway, we can derive it by appealing to the fact that $\rho(A) > \rho(B)$ if B is an irreducible matrix and $A \succeq B \geq 0$. If one thinks of C as one absorbing state, $I - (D|_H^{-1})W_{H,H}$ plays a role as the fundamental matrix in the study of Markov chains and the argument in the proof above parallels the usual deduction there [58]. The inverse of $I - (D|_H^{-1})W_{H,H}$, which exposes lots of geometric information of the Laplacian, can be said to be a discrete analogue of Green's function in analysis [4, 31, 35, 37, 59].

Remark 3.7. $\text{Ker } \mathcal{L}_M^\top$ governs the asymptotic behavior of the diffusion process given by $\mathcal{L}_M$ while $\text{Ker } \mathcal{L}_M$ records all data about the asymptotic behavior of the consensus process given by $\mathcal{L}_M$ [5, 57].

Theorem 3.8. *Let M be a finite weighted digraph. Then the following hold.*

- (1) *A basis of $\text{Ker } \mathcal{L}_M$ is given by nonnegative vectors x_S in Observation 3.6 where S ranges over all sink components of M .*
- (2) *A basis of $\text{Ker } \mathcal{L}_M^\top$ is given by nonnegative vectors y_S in Observation 3.3 where S ranges over all sink components of M .*
- (3) *Both the algebraic multiplicity and the geometric multiplicity of the eigenvalue 0 of $\mathcal{L}_M$ are equal to the number of sink components of M .*

Proof. In light of Observation 3.4, a harmonic function with respect to M is uniquely determined by its restriction on Ξ_M . But for each $f \in \text{Ker } \mathcal{L}_M$ and each sink component C of M , $f|_C$ is harmonic with respect to $M(C, C)$ and so, by Observation 3.4 again, can only be a constant function. It then follows from Observation 3.6 that the first reading is valid.

Since $\text{Ker } \mathcal{L}_M$ and $\text{Ker } \mathcal{L}_M^\top$ share the same dimension, the second reading follows from the first one and Observation 3.3.

For the third reading, it remains to verify that the number of nonzero eigenvalues of $\mathcal{L}_M$ plus the dimension of $\text{Ker } \mathcal{L}_M$ equals $|V(M)|$. When M has only one sink component, Corollary 3.2 together with Observation 3.5 proves the claim. In general, in view of Equation (3.1) and employing Theorem 3.2 and Observation 3.5, we can apply induction on the number of sink components to complete the proof. $\square$

Proof of Theorem 2.7. Consider a nonnegative right eigenvector f of $\mathcal{L}_M$, say $\mathcal{L}_M f = \lambda f$. If $\text{supp}(f) \cap C \neq \emptyset$ for some sink component C of M , then $y_C^\top f > 0$, according to Observation 3.3. This gives

$$0 = 0f = y_C^\top \mathcal{L}_M f = \lambda y_C^\top f,$$

and hence $\lambda = 0$. An application of the first part of Theorem 3.8 now concludes the proof. $\square$

Proof of Theorem 2.8. Consider a nonnegative left eigenvector $f^\top$ of $\mathcal{L}_M$, say $f^\top \mathcal{L}_M = \lambda f^\top$. From Observation 3.6 we derive that

$$\bigcup_{C: \text{sink components of } M} \text{supp}(x_C) = \bigcup_{C: \text{sink components of } M} C \downarrow_M = V(M).$$

Therefore, there exists a sink component C of M such that $\text{supp}(f) \cap C \downarrow_M \neq \emptyset$. It then follows

$$0 = f^\top \mathcal{L}_M x_C = \lambda f^\top x_C, \quad (3.5)$$

and hence $\lambda = 0$. By the second part of Theorem 3.8, the left null space of $\mathcal{L}_M$ has a basis y_S , where S runs through all sink components of M . This implies that f lies in the cone generated by this basis. Note that $\text{supp}(y_S) = S$ for any sink component S of M , and that any union of finitely many sink components of M is a closed set of M . We thus arrive at Theorem 2.8 (1) and Theorem 2.8 (3).

If M is even transitive, we see that M takes a constant value on $C \times C$ for each strongly connected component C of M . Especially, for every sink component S of M , from Equation (3.2) we see that y_S satisfies the condition asserted in the second reading. Since f lies in the cone generated by such y_S , Theorem 2.8 (2) is proved. $\square$

4 Schrödinger operators on countable digraphs

Trofimov [54] proved that every infinite locally finite vertex-transitive graph has a nonconstant harmonic function, which grows at most exponentially with respect to the distance to a base point. Tointon gave a lovely treatment of a qualitative version of this result on general weighted graphs [53, Proposition 1.4]. The next example indicates a difference between graphs and digraphs.

Example 4.1. Let M be the infinite directed path, namely $V(M) = \mathbb{Z}$, $M(i, i+1) = 1$ and $M(i, j) = 0$ for $j \neq i+1$. The digraph M is an infinite locally finite vertex-transitive digraph whose harmonic functions are all constant functions.

Let M be an out-finite digraph and let $\Lambda \in \mathbb{R}^{V(M)}$. The Schrödinger operator on M with potential function Λ , denoted by $\mathcal{S}_{M,\Lambda}$, is the linear map from $\mathbb{K}^{V(M)}$ to itself such that $(\mathcal{S}_{M,\Lambda} f)(v) := (\mathcal{L}_M f)(v) + \Lambda(v)f(v)$ for all $f \in \mathbb{K}^{V(M)}$ [8, 10, 17, 20, 43, 51]. A Schrödinger operator with a nonnegative potential on a finite digraph is either a Laplacian matrix or a perturbed Laplacian matrix. We call any element from $\text{Ker } \mathcal{S}_{M,\Lambda}$ a harmonic function with respect to $\mathcal{S}_{M,\Lambda}$. Here is an easy example to demonstrate the different behaviors of Schrödinger operators on finite and infinite digraphs.

Figure 4: A strongly connected out-finite digraph which is not in-finite.

Example 4.2. Let $(a_i)_{i=1}^\infty$ be a sequence of increasing positive numbers. We assign weights

to the arcs in Figure 4 by

$$M(x, y) = \begin{cases} \frac{1}{2}, & \text{if } y \in \{v_{1,1}, v_{2,1}\}, x = v_0; \\ \frac{a_j}{a_{j+1}}, & \text{if } y = v_{i,j+1}, x = v_{i,j}, i \in \{1, 2\}, j \in \mathbb{N}; \\ 1 - \frac{a_j}{a_{j+1}}, & \text{if } y = v_0, x = v_{i,j}, i \in \{1, 2\}, j \in \mathbb{N}. \end{cases}$$

For the resulting weighted digraph M , and for every $\Lambda \in \mathbb{R}_{\geq 0}^{V(M)}$, we have $f \in \text{Ker } \mathcal{S}_{M,\Lambda}$, where

$$f(v) = \begin{cases} 0, & \text{if } v = v_0; \\ (-1)^i a_j, & \text{if } v = v_{i,j}, i \in \{1, 2\}, j \in \mathbb{N}. \end{cases}$$

In particular, if we let $a_i = i$ for $i \in \mathbb{N}$, then we obtain an unbounded harmonic function; If we let $a_i = \frac{i}{i+1}$ for $i \in \mathbb{N}$, then we obtain a bounded harmonic function without maximum value. Moreover, we can add new vertices $v_{i,j}$ with $i = 3, 4, \dots, m, j \in \mathbb{N}$, and set

$$M(x, y) = \begin{cases} \frac{1}{m}, & \text{if } y \in \{v_{1,1}, v_{2,1}, \dots, v_{m,1}\}, x = v_0; \\ \frac{a_j}{a_{j+1}}, & \text{if } y = v_{i,j+1}, x = v_{i,j}, i \in [m], j \in \mathbb{N}; \\ 1 - \frac{a_j}{a_{j+1}}, & \text{if } y = v_0, x = v_{i,j}, i \in [m], j \in \mathbb{N}. \end{cases}$$

For every $\Lambda \in \mathbb{R}_{\geq 0}^{V(M)}$, the right null space of the Schrödinger operator $\mathcal{S}_{M,\Lambda}$ is of dimension at least $m - 1$. Indeed, for every pair of integers i, i' such that $1 \leq i < i' \leq m$, we have $f_{i,i'} \in \text{Ker } \mathcal{S}_{M,\Lambda}$, where

$$f_{i,i'}(v) = \begin{cases} a_j, & \text{if } v = v_{i,j}, j \in \mathbb{N}; \\ -a_j, & \text{if } v = v_{i',j}, j \in \mathbb{N}; \\ 0, & \text{otherwise.} \end{cases}$$

Let us try to seek some counterparts of results about finite Laplacians in the infinite setting. Here is an obvious parallel to Theorem 2.3.

Lemma 4.3. *Let M be a digraph and let $\Lambda \in \mathbb{R}_{\geq 0}^{V(M)}$ be a potential function on M . Let X be a subset of $V(M)$ and let $M' = M(X, X)$. Suppose $f \in \mathbb{R}_{\geq 0}^{V(M)}$ satisfy $(\mathcal{S}_{M,\Lambda} f)|_X = \lambda f|_X$. Then $\text{supp}(f) \cap X$ is open in the Alexandrov topology induced by M' . In particular, if $X = V(M)$, namely f is a nonnegative right eigenvector of $\mathcal{S}_{M,\Lambda}$, then $\text{supp}(f)$ is open in the Alexandrov topology induced by M .*

Proof. Assume that $M'(u, v) > 0$ for $u \in X$ and $v \in \text{supp}(f) \cap X$. It follows that

$$(\deg_M^+(u) + \Lambda(u) - \lambda)f(u) = \sum_{w \in V(M)} M(u, w)f(w) \geq M(u, v)f(v) = M'(u, v)f(v) > 0,$$

and hence $u \in \text{supp}(f) \cap X$. □

We now arrive at the main result of this paper.

Theorem 4.4. *Let M be a countable FON digraph and take $\Lambda \in \mathbb{R}_{\geq 0}^{V(M)}$. Then exactly one of the following holds:*

- (1) *There exists a basic open set of M which is the support of a nonnegative harmonic function for the Schrödinger operator $\mathcal{S}_{M,\Lambda}$.*
- (2) *The Schrödinger operator $\mathcal{S}_{M,\Lambda}$ is a surjective map from $\mathbb{K}^{V(M)}$ to itself.*

If $V(M)$ is finite, basically, Theorem 4.4 will follow from Observations 3.5 and 3.6. For the general case, we need the additional ingredient of taking inverse limit (projective limit). Parallel to the roles of Observations 3.5 and 3.6, we split the proof of Theorem 4.4 into two steps; See Theorems 4.8 and 4.13.

For any $X \subseteq V(M)$, we denote by $\mathbb{K}^{V(M),X}$ the functions in $\mathbb{K}^{V(M)}$ which vanish outside of X . Note that $\mathbb{K}^{V(M),X}$ is isomorphic to $\mathbb{K}^X$ as a linear space. When $V(M)$ is clear from the context, we often write the natural embedding map from $\mathbb{K}^X$ to $\mathbb{K}^{V(M)}$ as ι_X , namely for each $f \in \mathbb{K}^X$ it holds

$$\iota_X(f)(v) = \begin{cases} f(v), & \text{if } v \in X; \\ 0, & \text{if } v \in V(M) \setminus X. \end{cases}$$

Observation 4.5. Let M be a finite digraph and take $\Lambda \in \mathbb{R}_{\geq 0}^{V(M)}$. Let M' be obtained from M by adding a new vertex r and setting $M'(V(M), V(M)) = M$, $M'(u, r) = \Lambda(u)$ for $u \in V(M)$ and $M'(r, v) = 0$ for all $v \in V(M')$. Then the embedding map $\iota_{V(M)}$ from $\mathbb{K}^{V(M)}$ to $\mathbb{K}^{V(M')}$ induces an isomorphism from $\text{Ker } \mathcal{S}_{M,\Lambda}$ to the space of those harmonic functions on M' whose supports do not contain r .

Proof. Take $h \in \text{Ker } \mathcal{S}_{M,\Lambda}$. Then, for all $v \in V(M) = V(M') \setminus \{r\}$ we have

$$\begin{aligned} \mathcal{L}_{M'}(\iota_{V(M)}(h))(r) &= \sum_{w \in V(M')} M'(r, w)(h(r) - h(w)) \\ &= \sum_{w \in V(M')} 0 \times (h(r) - h(w)) \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} \mathcal{L}_{M'}(\iota_{V(M)}(h))(v) &= \sum_{w \in V(M')} M'(v, w)(h(v) - h(w)) \\ &= \sum_{w \in V(M)} M(v, w)(h(v) - h(w)) + M(v, r)(h(v) - 0) \\ &= \sum_{w \in V(M)} M(v, w)(h(v) - h(w)) + \Lambda(v)h(v) \\ &= \mathcal{S}_{M,\Lambda}(h) = 0. \end{aligned}$$

Suppose $h \in \text{Ker } \mathcal{L}_{M'}$ and $h(r) = 0$. Then $h = \iota_{V(M)}(h|_{V(M)})$ and

$$\begin{aligned}
 (\mathcal{S}_{M,\Lambda}(h|_{V(M)}))(v) &= \sum_{w \in V(M)} M(v, w)(h(v) - h(w)) + \Lambda(v)h(v) \\
 &= \sum_{w \in V(M)} M'(v, w)(h(v) - h(w)) + M'(v, r)(h(v) - h(r)) \\
 &= \sum_{w \in V(M')} M'(v, w)(h(v) - h(w)) \\
 &= \mathcal{L}_{M'}(h)(v) = 0
 \end{aligned}$$

for all $v \in V(M)$. $\square$

For any two complex numbers z_1 and z_2 , the real inner product of z_1 and z_2 , denoted by $\langle z_1, z_2 \rangle_{\mathbb{R}}$, is the real part of $\overline{z_1}z_2$. If $z_1 = r_1 \exp\{\sqrt{-1}\theta_1\}$ and $z_2 = r_2 \exp\{\sqrt{-1}\theta_2\}$, where $r_1, r_2 \in \mathbb{R}_{\geq 0}$ and $\theta_1, \theta_2 \in \mathbb{R}$, then $\langle z_1, z_2 \rangle_{\mathbb{R}} = r_1 r_2 \cos(\theta_2 - \theta_1)$. If $|z_2| \geq |z_1|$, the Cauchy-Schwarz inequality says that $\langle z_2, z_2 - z_1 \rangle_{\mathbb{R}} = \langle z_2, z_2 \rangle_{\mathbb{R}} - \langle z_2, z_1 \rangle_{\mathbb{R}} \geq \langle z_2, z_2 \rangle_{\mathbb{R}} - \sqrt{\langle z_2, z_2 \rangle_{\mathbb{R}} \langle z_1, z_1 \rangle_{\mathbb{R}}} \geq \langle z_2, z_2 \rangle_{\mathbb{R}} - \sqrt{\langle z_2, z_2 \rangle_{\mathbb{R}} \langle z_2, z_2 \rangle_{\mathbb{R}}} = 0$.

Observation 4.6 (Local maximum modulus principle). Let M be an FOD digraph and let $\Lambda \in \mathbb{R}_{\geq 0}^{V(M)}$. Suppose $f \in \text{Ker } \mathcal{S}_{M,\Lambda}$. If $|f(v)| \geq |f(w)|$ for every out-neighbor w of v in M , then $f(v) = f(w)$ for $w \in N_M^+(v)$ and $\Lambda(v)f(v) = 0$. In particular, if $|f(v)| \geq |f(w)|$ for every $w \in v \uparrow_M$, then $f(v) = f(w)$ for every $w \in v \uparrow_M$.

Proof. By our remarks preceding Observation 4.6, $\langle f(v), f(v) - f(w) \rangle_{\mathbb{R}} \geq 0$ for all $w \in N_M^+(v)$, with equality if and only if $f(v) = f(w)$. Observe that

$$\begin{aligned}
 0 &= \langle f(v), 0 \rangle_{\mathbb{R}} = \left\langle f(v), \Lambda(v)f(v) + \sum_{w \in N_M^+(v)} M(v, w)(f(v) - f(w)) \right\rangle_{\mathbb{R}} \\
 &= \Lambda(v)|f(v)|^2 + \sum_{w \in N_M^+(v)} M(v, w) \langle f(v), (f(v) - f(w)) \rangle_{\mathbb{R}} \\
 &\geq \Lambda(v)|f(v)|^2.
 \end{aligned}$$

It then follows that $\Lambda(v) = 0$ and $f(v) = f(w)$ for all $w \in N_M^+(v)$, as claimed. $\square$

Lemma 4.7. Let M be an FOD digraph. Let $\Lambda \in \mathbb{R}_{\geq 0}^{V(M)}$ and let S be a finite sink component of M . If $x \in \text{Ker } \mathcal{S}_{M,\Lambda}$ and $S \cap \text{supp}(\Lambda) \neq \emptyset$, then $\text{supp}(x) \cap S = \emptyset$.

Proof. By Observation 4.6, x takes a constant value B inside S . Take $w \in S \cap \text{supp}(\Lambda)$. Then $0 = \Lambda(w)x(w) + \sum_{v \in N_M^+(w)} M(w, v)(x(w) - x(v)) = \Lambda(w)B$, showing that $B = 0$. This gives $\text{supp}(x) \cap S = \emptyset$. $\square$

The next result characterizes when a basic open set of a countable digraph M can be the support of a nonnegative element from the right null space of a Schrödinger operator on it.

Theorem 4.8. Let M be a countable FOD digraph and take $\Lambda \in \mathbb{R}_{\geq 0}^{V(M)}$. Suppose that S is a finite sink component of M . Then we can find $h_S \in \ell^\infty(V(M)) \cap \text{Ker } \mathcal{S}_{M,\Lambda}$ such that $h_S \geq 0$ and $\text{supp}(h_S) = S \downarrow_M$ if and only if $S \cap \text{supp}(\Lambda) = \emptyset$.

Proof. In view of Lemma 4.7, we only need to prove the backward direction.

Fix an element in S , say p_0 . Let $Q := \{h \in \mathbb{R}_{\geq 0}^{V(M)} : h(v) \leq h(p_0), \forall v \in V(M)\} \subseteq \ell^\infty(V(M))$. Suppose that $S \cap \text{supp}(\Lambda)$ is nonempty and we want to find $h \in Q$ such that $\mathcal{S}_{M,\Lambda}(h) = \mathbf{0}$ and $\text{supp}(h) = S \downarrow_M$.

We enumerate the vertices in $V(M) \setminus S$ as $p_1, p_2, \dots$. If $V(M) \setminus S$ is a finite set of c elements, we use the convention that $p_0 = p_{c+1} = p_{c+2} = \dots$. We put $X_0 := S$ and $X_{n+1} := X_n \cup \{p_{n+1}\}$ for all $n \in \mathbb{N}_0$.

For each $n \in \mathbb{N}_0$, define

$$K_n := \{\mathbf{1}_{X_n} h : (\mathcal{S}_{M,\Lambda} h)|_{X_n} = \mathbf{0}, h \in Q, \text{supp}(h) \subseteq S \downarrow_M\}.$$

Observations 3.6, 4.5 and 4.6 tell us that $K_n \supsetneq \{\mathbf{0}\}$. Let $\hat{K}_n := \bigcap_{m=n}^\infty K_m$ be the intersection of K_m for all $m \geq n$. We claim that $\hat{K}_n$ is not the trivial cone with a single point. Let S_n denote the unit sphere in the finite-dimensional inner product space $\mathbb{K}^{V(M), X_n}$. Since S_n is compact, the descending chain of nonempty closed sets

$$K_n \cap S_n \supseteq u_{n+1,n}(K_{n+1}) \cap S_n \supseteq u_{n+2,n}(K_{n+2}) \cap S_n \supseteq \dots$$

has a nonempty intersection $\hat{S}_n := \bigcap_{m=n}^\infty u_{m,n}(K_m) \cap S_n = \hat{K}_n \cap S_n$. Therefore $\hat{K}_n$ is not a trivial cone. Note that $\hat{K}_n \subseteq u_{n+1,n}(\hat{K}_{n+1})$ for all $n \in \mathbb{N}_0$. This allows us to find $h_n \in \hat{K}_n$ for all nonnegative integers n so that $h_0 = \mathbf{1}_S$ and $h_n = u_{n+1,n}(h_{n+1})$. We claim that the required function h can now be taken as the one satisfying $h|_{X_n} = h_n$ for all $n \in \mathbb{N}_0$.

What remains to be verified is $h(v) > 0$ for all $v \in S \downarrow_M$. Indeed, we can find m such that there is a path P_v from v to a vertex in S such that $P_v \subseteq X_m$. By the definition of K_m , there exists $\tilde{h} \in Q$ such that $\mathcal{S}_{M,\Lambda} \tilde{h}|_{X_m} = \mathbf{0}$ and $h_m = \mathbf{1}_{X_m} \tilde{h}$. Note that $S \subseteq \text{supp}(\tilde{h}) \cap X_m = \text{supp}(h_m)$. Accordingly, Lemma 4.3 tells us that $v \in \text{supp}(\tilde{h}) \cap X_m = \text{supp}(h_m)$, and hence $h(v) = h_m(v) > 0$, as desired. $\square$

Example 4.9. Let M be an infinite directed path with a single sink vertex, namely $V(M) = \mathbb{N}_0$, $M(i+1, i) = 1$ and $M(j, i) = 0$ if $j \neq i+1$ for every $i, j \in \mathbb{N}_0$. Then M is an FOD digraph and it has a single finite sink component. If $h \in \text{Ker } \mathcal{L}_M$ and $h(0) = c$, then $h(i) = c$ for every $i \in \mathbb{N}_0$. This gives $\ell^1(V(M)) \cap \text{Ker } \mathcal{L}_M = \{\mathbf{0}\}$. We thus see that the analogue of Theorem 4.8 with FOD replaced by BOD does not hold.

The next result is a generalization of Theorem 3.8 (1).

Theorem 4.10. Let M be a countable FOD digraph and take $\Lambda \in \mathbb{R}_{\geq 0}^{V(M)}$. Let $\mathfrak{S} = \{S : S \text{ is a sink component of } M, S \cap \text{supp}(\Lambda) = \emptyset\}$ be the collection of sink components without intersection with $\text{supp}(\Lambda)$. Suppose $|v \uparrow_M| < \infty$ for every $v \in V(M)$. Then, for each $S \in \mathfrak{S}$, the function h_S as claimed in Theorem 4.8 is uniquely determined. Furthermore, the harmonic functions h_S , $S \in \mathfrak{S}$, are linearly independent, and they form a basis of $\text{Ker } \mathcal{S}_{M,\Lambda}$ when $|\mathfrak{S}| < \infty$.

Proof. For each $S \in \mathfrak{S}$, we fix one function h_S as claimed in Theorem 4.8. To finish the proof, it suffices to show that every function $h \in \text{Ker } \mathcal{S}_{M,\Lambda}$ can be expressed as a unique linear combination of these h_S for $S \in \mathfrak{S}$.

Since $|v \uparrow_M| < \infty$ for every $v \in V(M)$, each sink component of M is finite. What is more, M is actually an FON digraph. For every sink component S such that $S \cap \text{supp}(\Lambda) \neq$

$\emptyset$, Lemma 4.7 has shown that h vanishes on S . In view of Observation 3.4, we now find that, for every sink component $S \in \mathfrak{S}$, the harmonic function h must take a constant value on S , say c_S . Since the harmonic functions h_S , where $S \in \mathfrak{S}$, are linearly independent, we will complete the proof by verifying that $h' = h - \sum_{S \in \mathfrak{S}} c_S h_S$ is the zero function.

Take $v \in V(M)$. Consider the induced finite digraph $M' = M(v \uparrow_M, v \uparrow_M)$. Note that $h'|_{V(M')}$ vanishes on all sink components of M' and that $\mathcal{S}_{M', \Lambda'} h'|_{V(M')} = \mathbf{0}$. By Theorem 3.8 and Observation 4.5, we get $h'(v) = 0$ and therefore we are done. $\square$

Example 4.11. In Theorem 4.10 we cannot replace the assumption $|v \uparrow_M| < \infty$ by the condition that all sink components of M are finite. One may consider modifying Example 4.2 by adding a single sink vertex as an out-neighbour of v_0 in Figure 4 and associating with the new arc an arbitrary positive weight. The modified digraph is a countable FOD digraph. It has a single sink component, but it bears at least two linearly independent harmonic functions.

Lemma 4.12. Let M be a digraph and let $\Lambda \in \mathbb{R}_{\geq 0}^{V(M)}$. If M has a finite sink component S such that $\text{supp}(\Lambda) \cap S = \emptyset$, then $\mathcal{S}_{M, \Lambda}$ is not surjective (to the corresponding codomain).

Proof. Note that the Laplacian $\mathcal{L}_{M(S, S)}$ of the finite digraph $M(S, S)$ is not surjective. But, considering that S is a sink component, for every $g \in \mathbb{R}^{V(M)}$ it holds $(\mathcal{S}_{M, \Lambda} g)|_S = \mathcal{L}_{M(S, S)} g|_S$. This is the proof. $\square$

It is known that a linear operator is surjective provided it has finite hopping range and satisfies the pointwise maximum principle [15, 33]. Our Theorem 4.13 is in the same vein.

Theorem 4.13. Let M be a countable FON digraph and let $\Lambda \in \mathbb{R}_{\geq 0}^{V(M)}$. Then $\mathcal{S}_{M, \Lambda}$ is a surjective map from $\mathbb{K}^{V(M)}$ to itself if and only if it holds $S \cap \text{supp}(\Lambda) \neq \emptyset$ for every finite sink component S of M .

Proof. In view of Lemma 4.12, we only need to prove the backward direction.

Suppose that $\text{supp}(\Lambda) \cap S$ is nonempty for every finite sink component S of M . Since M is countable, we may enumerate the vertices of M as $v_0, v_1, v_2, \dots$. For each non-negative integer n , let X_n denote the set $\{v_0, v_1, \dots, v_n\}$ when $n + 1 \leq |V(M)|$ and let $X_n = V(M)$ otherwise, let $\Gamma_n = \mathbb{K}^{V(M), X_n}$ and let $\mathcal{S}_n$ be the linear map from Γ_n to itself so that $\mathcal{S}_n(h) = \mathbf{1}_{X_n} \mathcal{S}_{M, \Lambda}(h)$ for all $h \in \Gamma_n$. For any $n \in \mathbb{N}_0$, let us show that $\mathcal{S}_n$ is a surjective map on Γ_n . As $\dim \Gamma_n = n + 1 < \infty$, our task is to verify that $\mathcal{S}_n$ is injective. Take $h \in \text{Ker } \mathcal{S}_n$. Suppose $\max_{v \in V(M)} |h(v)| = |h(v_i)| = B$ for some $i \in \{0, \dots, n\}$. We intend to show that $B = 0$. By Lemma 4.6, we have $h(w) = h(v_i)$ for every $w \in v_i \uparrow_M$. If $v_i \uparrow_M$ contains a vertex x outside of X_n , then we have $B = h(x) = 0$. We thus turn to the case that $v_i \uparrow_M \subseteq X_n$. Since X_n is finite, we find that $v_i \uparrow_M$ contains a finite sink component of M , say S . By assumption, there exists $w \in \text{supp}(\Lambda) \cap S$. It follows $0 = \Lambda(w)h(w) + \sum_{v \in N_M^+(w)} M(w, v)(h(w) - h(v)) = \Lambda(w)B$, yielding that $B = 0$, as wanted.

For every $g \in \mathbb{K}^{V(M)}$, we want to find $f \in \mathbb{K}^{V(M)}$ such that $g = \mathcal{S}_{M, \Lambda} f$. For each $n \in \mathbb{N}_0$, we set $H_n = \{\mathbf{1}_{X_n} h : (\mathcal{S}_{M, \Lambda} h)|_{X_n} = g|_{X_n}, h \in \mathbb{K}^{V(M)}\}$. It is nonempty since $\mathcal{S}_n^{-1}(\mathbf{1}_{X_n} g) \in H_n$. Note that for every $n \geq m \geq 0$ there is an affine map $u_{n, m}$ from the affine space H_n to H_m given by $u_{n, m}(h) = \mathbf{1}_{X_m} h$ for all $h \in H_n$. Observe that we have a descending chain of finite-dimensional affine subspaces of $\mathbb{K}^{V(M)}$:

$$H_m = \text{Im } u_{m, m} \supseteq \text{Im } u_{m+1, m} \supseteq \text{Im } u_{m+2, m} \supseteq \dots$$

This sequence must stabilize after finitely many steps, namely there exists $m' \geq m$ such that $\text{Im } u_{m',m} = \text{Im } u_{m'',m}$ for all $m'' \geq m'$. We write $\hat{H}_m = \text{Im } u_{m',m}$ for this nonempty affine subspace of H_m . We can verify that $u_{n+1,n}$ maps $\hat{H}_{n+1}$ onto $\hat{H}_n$ for all $n \in \mathbb{N}_0$. It then follows that we can take $(h_n)_{n \in \mathbb{N}_0} \in \prod_{n \in \mathbb{N}_0} \hat{H}_n$ such that $h_n = u_{n+1,n}(h_{n+1})$. The required function f can now be taken as the one satisfying $f|_{X_n} = h_n$ for all $n \in \mathbb{N}_0$. $\square$

Remark 4.14. Theorem 4.13 is not a special case of [33, Theorem 1] as the latter requires the digraph to be both out-finite and in-finite. For example, Figure 4 shows a strongly connected FON digraph M for which $M^\top$ is not FON. For this digraph, Theorem 4.13 is applicable but [33, Theorem 1] is not.

Remark 4.15. Let M be the infinite directed path described in Example 4.1. Then M is a countable weakly connected digraph without finite sink component, and it is BOD and hence FOD. For any $g_1 \in \mathbb{R}^{V(M)}$ with $|\text{supp}(g_1)| = 1$, there is no $h \in \ell^1(V(M))$ such that $\mathcal{L}_M h = g_1$. For $g_2 := \mathbf{1}_{V(M)} \in \mathbb{R}^{V(M)}$, there is no $h \in \ell^\infty(V(M))$ such that $\mathcal{L}_M h = g_2$. These show that the analogues of Theorem 4.13 with FON replaced by FOD or BOD do not hold. It also tells us that we cannot change FON to be either FOD or BOD in Theorem 4.4.

For a linear map, or for a map and its adjoint map, the relationship between injectivity and surjectivity is of lots of interest [14, 34]. Theorem 4.13 is about surjectivity. Let us also include a simple observation on injectivity. For each set V , let $\mathbb{K}[V] = \mathbb{K}_{\text{fin}}^V$ denote the linear space spanned by V , namely the set of functions on V with a finite support. Note that $\dim \mathbb{K}^V > \dim \mathbb{K}[V]$ if and only if V is infinite.

Theorem 4.16. Let M be an infinite FOD digraph. Let $\Lambda \in \mathbb{R}_{\geq 0}^{V(M)}$. If $S \cap \text{supp}(\Lambda) \neq \emptyset$ for every finite sink component S of M , then $\text{Ker } \mathcal{S}_{M,\Lambda} \cap \ell^p(V(M)) = \{\mathbf{0}\}$ for every $0 < p < \infty$ and $\text{Ker } \mathcal{S}_{M,\Lambda} \cap \mathbb{K}_{\text{fin}}^{V(M)} = \{\mathbf{0}\}$.

Proof. Let $\mathbf{0} \neq g \in \text{Ker } \mathcal{S}_{M,\Lambda}$. There are two cases to consider.

Suppose the maximum of $|g|$ over $V(M)$ can be achieved at some vertex v . Observation 4.6 then ensures $g(w) = g(v) \neq 0$ for every $w \in v \uparrow_M$. By Lemma 4.7, the harmonic function g must vanish on each finite sink component of M . Therefore, $v \uparrow_M$ contains infinitely many vertices and $|\text{supp}(g)| = \infty$. It also says that the summation $\sum_{w \in v \uparrow} |g(w)|^p$ diverges for every $p > 0$.

Suppose the maximum of $|g|$ over $V(M)$ cannot be achieved anywhere. This means that there exists a sequence of vertices $v_1, v_2, \dots$ such that $|g(v_1)| < |g(v_2)| < \dots$. Henceforth, $|\text{supp}(g)| = \infty$, and the summation $\sum_{i=1}^\infty |g(v_i)|^p$ diverges for every $p > 0$.

We now conclude that $\text{Ker } \mathcal{S}_{M,\Lambda} \cap \ell^p(V(M)) = \{\mathbf{0}\}$ and $\text{Ker } \mathcal{S}_{M,\Lambda} \cap \mathbb{K}_{\text{fin}}^{V(M)} = \{\mathbf{0}\}$, as wanted. $\square$

Proof of Theorem 4.4. Combine Theorems 4.8 and 4.13. $\square$

Question 4.17. There is a countability assumption in the statements of Theorems 4.4, 4.8, 4.10 and 4.13. Can this be relaxed?

5 Final remarks

Starting from the conjectures of Ostrander, we have tried to link the Alexandrov topology of a countable digraph and the support of nonnegative harmonic functions of a Schrödinger operator on that digraph. It may be interesting to go forward to the measurable framework [5, 6, 7, 19, 38], like the existing theory on Dirichlet forms, graphons, Borel equivalence relations, etc. To get stronger results, one may need to focus on some natural and important situations, say models built on Bratteli diagrams. In some sense our work in this paper is based on the traditional homogeneous Markov chain theory. There is a beautiful decomposition-separation Theorem for finite nonhomogeneous Markov chains [49]. It looks interesting if we can also adapt that research to the study of countable weighted digraphs. In the more algebraic direction, one can consider harmonic functions with values in a field of positive characteristic [60]. We conclude the paper with two simple examples about harmonic functions of finite Laplacians and invite the readers to see if it is possible to find a generalization to some infinite settings.

It is well-known that Laplacian is related with energy minimization [11, 19]. The following result of Harper provides one such example.

Example 5.1 (Minimum mean-square error [23, 26]). The n -cube H_n has vertex set $\{0, 1\}^n$ and two vertices are adjacent if and only if their Hamming distance is one, i.e., they differ exactly in one coordinate. How to design a bijection f from $V(M)$ to $[2^n]$ so that $\sum_{uv \in E(H_n)} (g(u) - g(v))^2$ attains the minimum value? For each $i \in [n]$, denote by F_i the function with $F_i(a_1 \cdots a_n) = a_i$ for all $a_1 \cdots a_n \in \{0, 1\}^n = V(H_n)$. We consider the canonical bijection $g = f + 1$ where the function f is given by $f = \sum_{t=1}^n 2^{t-1} F_t$. To see that this function g really has minimum energy, we note that $\sum_{uv \in E(H_n)} (g(u) - g(v))^2 = \sum_{uv \in E(H_n)} (f(u) - f(v))^2 = \langle f, \mathcal{L}_{H_n} f \rangle$. The minimum eigenvalue of $\mathcal{L}_{H_n}$ is 0 and the second smallest eigenvalue of $\mathcal{L}_{H_n}$ is 2. The eigenspace of 0 is the space of constant functions and the eigenspace of 2 is spanned by those functions F_i for $i \in [n]$.

In Section 3, we let a matrix act both from right and from left on some vectors, namely consider an operator and its adjoint. This trick sometimes connects deterministic side and stochastic side, as the next example will show. Note that if we take a digraph M to be the probability transition matrix of a Markov chain, its Laplacian is simply $I - M$ and so the invariant measure lies in the left null space of $\mathcal{L}_M$.

Example 5.2. Let M be an $n \times n$ row stochastic matrix. For each $i \in [n]$, let M_i be the matrix obtained from the identity matrix of order n by replacing its i -th row with the i -th row of M . Assume that the set of sink vertices of M is S and that every vertex in M has a path leading into S . Note that M_i is the identity matrix if and only if $i \in S$. Let $f \in [n]^{\mathbb{N}_0}$ such that $f^{-1}(i)$ is an infinite set for all $i \in [n]$. When multiplying these matrices M_i on column vectors, we are modelling the process of opinion dynamics. If we apply $M_{f(t)}$ at time t , the limiting opinion profile is determined in [58, Theorem 1.1]. Basically, this is equivalent to determining the limit of finite products $M_{f(0)} M_{f(1)} \cdots M_{f(t)}$. However, if we apply $M_{f(0)} M_{f(1)} \cdots M_{f(t)}$ on the right of row vectors for all t , we see that it is simply iteratively applying the matrix M on row spaces and so we know its limit [58, Theorem 3.3] immediately from the theory of absorbing Markov chains.

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