ISSN 2590-9770
The Art of Discrete and Applied Mathematics
https://doi.org/10.26493/2590-9770.1414.f67
(Also available at http://adam-journal.eu)

Circulant almost cross intersecting families

Michal Parnas*
The Academic College of Tel-Aviv-Yaffo, Tel-Aviv, Israel

Received 16 July 2020, accepted 29 April 2021

Abstract

Let $\mathcal{F}$ and $\mathcal{G}$ be two t-uniform families of subsets over $[k] = \{1, 2, ..., k\}$, where $|\mathcal{F}| = |\mathcal{G}|$, and let C be the adjacency matrix of the bipartite graph whose vertices are the subsets in $\mathcal{F}$ and $\mathcal{G}$, where there is an edge between $A \in \mathcal{F}$ and $B \in \mathcal{G}$ if and only if $A \cap B \neq \emptyset$. The pair $(\mathcal{F}, \mathcal{G})$ is q-almost cross intersecting if every row and column of C has exactly q zeros.

We further restrict our attention to q-almost cross intersecting pairs that have a circulant intersection matrix $C_{p,q}$, determined by a column vector with $p > 0$ ones followed by $q > 0$ zeros. This family of matrices includes the identity matrix in one extreme, and the adjacency matrix of the bipartite crown graph in the other extreme.

We give constructions of pairs $(\mathcal{F}, \mathcal{G})$ whose intersection matrix is $C_{p,q}$, for a wide range of values of the parameters p and q, and in some cases also prove matching upper bounds. Specifically, we prove results for the following values of the parameters: (1) $1 \leq p \leq 2t - 1$ and $1 \leq q \leq k - 2t + 1$. (2) $2t \leq p \leq t^2$ and any $q > 0$, where $k \geq p + q$. (3) p that is exponential in t, for large enough k.

Using the first result we show that if $k \geq 4t - 3$ then $C_{2t-1,k-2t+1}$ is a maximal isolation submatrix of size $k \times k$ in the $0,1$-matrix $A_{k,t}$, whose rows and columns are labeled by all subsets of size t of $[k]$, and there is a one in the entry on row x and column y if and only if subsets x, y intersect.

Keywords: Circulant matrix, intersecting sets, Boolean rank, isolation set.

Math. Subj. Class.: 05D05, 15B34

1 Introduction

One of the fundamental results of extremal set theory is the theorem of Erdős, Ko and Rado [8], which shows that the size of an intersecting t-uniform family of subsets over

*The author would like to thank the referee for the useful comments that helped improve the presentation of the paper.

E-mail address: michalp@mta.ac.il (Michal Parnas)

$[k] = \{1, 2, ..., k\}$ is bounded above by $\binom{k-1}{t-1}$. Numerous variations of the original problem have been suggested and studied over the years. Among them is the problem of cross intersecting families of subsets (e.g. [4, 10, 14, 17, 21]). Specifically, if $\mathcal{F}$ and $\mathcal{G}$ are two t-uniform families of subsets over $[k]$, then the pair $(\mathcal{F}, \mathcal{G})$ is cross intersecting if every subset in $\mathcal{F}$ intersects with every subset in $\mathcal{G}$ and vice versa. Pyber [21] proved that in this case $|\mathcal{F}| \cdot |\mathcal{G}| \leq \binom{k-1}{t-1}^2$.

Many of the extremal combinatorial problems considered so far can be inferred as results about maximal submatrices of the $0,1$-matrix $A_{k,t}$ of size $\binom{k}{t} \times \binom{k}{t}$, whose rows and columns are labeled by all subsets of size t of $[k]$, and there is a one in the entry on row x and column y if and only if subsets x, y intersect. Hence, in this setting, the result of Erdős, Ko and Rado can be inferred as stating the size of the largest all-one square principal submatrix of $A_{k,t}$, and the result of Pyber states the size of the largest all-one submatrix of $A_{k,t}$. We note that considering the classic results of extremal combinatorics as maximal submatrices of $A_{k,t}$, allows us to employ tools from algebra in addition to the combinatorial techniques.

Another variation of the problem of cross intersecting families was introduced by Gerbner et al. [11], which defined the notion of a q-almost cross intersecting pair $(\mathcal{F}, \mathcal{G})$. Here every subset in $\mathcal{F}$ does not intersect with exactly q subsets in $\mathcal{G}$ and vice versa. If $\mathcal{F} = \mathcal{G}$ then $\mathcal{F}$ is called a q-almost intersecting family of subsets. Hence, if C is the submatrix of $A_{k,t}$ whose rows are indexed by the subsets of $\mathcal{F}$ and columns by the subsets of $\mathcal{G}$, then every row and column of C has exactly q zeros. Using a classic theorem of Bollobás [3], it is possible to prove that the largest square submatrix C of $A_{k,t}$, representing a 1-almost cross intersecting pair, is of size $\binom{2t}{t} \times \binom{2t}{t}$. A theorem proved in [11] shows that if C is a submatrix of size $n \times n$ of $A_{k,t}$, with exactly q zeros in each row and column, then $n \leq (2q-1)\binom{2t}{t}$.

In this paper we consider the problem of finding maximal *circulant* submatrices of $A_{k,t}$, representing an almost cross intersecting pair, for a range of parameters. A circulant matrix is a matrix in which each row is shifted one position to the right compared to the preceding row (or alternatively, each column is shifted one position compared to the preceding column). Therefore, such a matrix C is determined completely by its first row or first column. Circulant matrices were studied extensively in the context of the multiplicative commutative semi-group of circulant Boolean matrices and also when discussing Cayley graphs of cyclic groups (see e.g. [1, 5, 6, 7, 22]). However, they were not studied in the context of extremal combinatorics, besides some special cases that will be discussed shortly.

$$C_{4,4} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Figure 1: The circulant matrix $C_{p,q}$, where $p = q = 4$.

Our focus will be on circulant matrices that are determined by a column vector with

p ones followed by q zeros. Such a matrix will be denoted by $C_{p,q}$. See Figure 1 for an example. Thus, in one extreme, if $p = 1$ and $q > 0$, then $C_{p,q}$ is the identity matrix. The other extreme is $q = 1$ and $p > 0$, and then $C_{p,q}$ is the adjacency matrix of a crown graph (where a crown graph is a complete bipartite graph from which the edges of a perfect matching have been removed). Hence, the structure of the circulant matrix $C_{p,q}$ forms a bridge which connects these two extreme cases, and it is interesting to find a unifying theorem which determines the maximal size of the matrix $C_{p,q}$ as a function of p, q, k and t.

We note that two trivial examples of circulant submatrices of $A_{k,t}$ include the case of $q = 0$, where we get an all-one submatrix of $A_{k,t}$ of maximal size $\binom{k-1}{t-1} \times \binom{k-1}{t-1}$, and the case of $p = 0$, where we get an all-zero submatrix of $A_{k,t}$ of maximal size $\binom{k/2}{t} \times \binom{k/2}{t}$. Hence, the problem of studying the size of circulant submatrices $C_{p,q}$ of $A_{k,t}$ is interesting only if both $p, q > 0$. Furthermore, we must require that $k \geq 2t$, as otherwise, $A_{k,t}$ is the all-one matrix itself.

As we discuss shortly, one of our results also provides a simple construction of maximal isolation submatrices of $A_{k,t}$, thus providing simple small witnesses to the Boolean rank of $A_{k,t}$. The *Boolean rank* of a $0,1$-matrix A of size $n \times m$ is equal to the smallest integer r, such that A can be factorized as a product of two $0,1$-matrices, $X \cdot Y = A$, where X is a matrix of size $n \times r$ and Y is a matrix of size $r \times m$, and all additions and multiplications are Boolean (that is, $1+1 = 1, 1+0 = 0+1 = 1, 1 \cdot 1 = 1, 1 \cdot 0 = 0 \cdot 1 = 0$). A $0,1$-matrix B of size $s \times s$ is called an *isolation matrix*, if we can select s ones in B, so that no two of the selected ones are in the same row or column of B, and no two of the selected ones are contained in a 2×2 all-one submatrix of B. It is well known that if B is an isolation submatrix of size $s \times s$ in a given $0,1$-matrix A, then s bounds below the Boolean rank of A (see [2, 16]). However, computing the Boolean rank or finding a maximal isolation submatrix in general is an NP-hard problem (see [13, 18, 24]). Hence, it is interesting to find and characterize families of maximal isolation sets for specific given matrices.

1.1 Our results

Our main goal is to determine the range of parameters, p and q, for which $C_{p,q}$ is a submatrix of $A_{k,t}$. The constructions and upper bounds we present differ in their structure and proof methods according to the size of p, q compared to t, k.

We first consider the range of values of relatively small p, that is $1 \leq p \leq 2t-1$, and prove in Section 2 the following positive result.

Theorem 1.1. *Let $k \geq 2t$, $1 \leq p \leq 2t-1$ and $1 \leq q \leq k-2t+1$. Then $C_{p,q}$ is a submatrix of $A_{k,t}$.*

In the extreme case of $p = 1$ and $q = k-2t+1$, this construction gives the identity submatrix of size $(k-2t+2) \times (k-2t+2)$. Recently, [20] proved that this is the maximal size of an identity submatrix in $A_{k,t}$.

The other extreme is $p = 2t-1$ and $q = k-2t+1$, in which case we get a circulant submatrix of size $k \times k$. As we show in Section 2, if $k \geq 4t-3$ then $C_{2t-1,k-2t+1}$ is a maximal isolation submatrix of size $k \times k$ in $A_{k,t}$. Since the Boolean rank of $A_{k,t}$ is k for $k \geq 2t$ (see [19]), then the size of a maximal isolation submatrix of $A_{k,t}$ is upper bounded by $k \times k$, and thus, our result is optimal in this case.

Furthermore, for $k = 2t+p-2$ and $p \geq 2$, the construction described in Theorem 1.1 provides an isolation submatrix of size $(2p-1) \times (2p-1)$. We note that [20] gave constructions of isolation submatrices in $A_{k,t}$, of the same size as achieved here. However, the constructions described in [20] are quite complex, and thus, the result described in Theorem 1.1 provides an alternative simpler construction of a maximal isolation submatrix in $A_{k,t}$, for large enough k.

We then prove the following upper bound that matches the size of the construction given in Theorem 1.1, for the range of values of $1 \leq p \leq 2t-1$ and $q \geq p-1$. The proof of this result characterizes the structure of the Boolean decompositions of $C_{p,q}$ for this range of parameters.

Theorem 1.2. *Let $C_{p,q}$ be a submatrix of $A_{k,t}$, where $k \geq 2t$, $1 \leq p \leq 2t-1$ and $q > 0$. If $q \geq p-1$ then $q \leq k-2t+1$.*

In Section 3 we address the range of slightly larger values of p, that is, $2t \leq p \leq t^2$, and provide a different construction of circulant submatrices of $A_{k,t}$ of the form $C_{p,q}$. As we show, for this range of values of p, there is no upper bound on the size of q, as we had in Theorem 1.1 and Theorem 1.2, as long as $k \geq p+q$.

Furthermore, the proof for this range of parameters provides a decomposition of $C_{p,q}$ into a product of two Boolean circulant matrices X, Y, where X has t ones in each row and Y has t ones in each column. If we view the rows of X and the columns of Y as the characteristic vectors of subsets of size t, then X and Y each represents a *circulant t-uniform family*. Thus, the construction used in the proof of the next theorem, uses a pair $\mathcal{F}, \mathcal{G}$ of circulant families to construct $C_{p,q}$.

Theorem 1.3. *Let $2t \leq p \leq t^2$ and $q > 0$. Then $C_{p,q}$ is a submatrix of $A_{k,t}$ for $k \geq q+p$.*

Finally, in Section 4, we consider the range of large p. Using the result of [11] stated above, we know that if $C_{p,q}$ is a submatrix of $A_{k,t}$ of size $n \times n$, then $n \leq (2q-1)\binom{2t}{t}$, and [23] proved a conjecture of [12] and showed that for large enough q and t, the size of a q-almost intersecting family $\mathcal{F}$ is bounded by $(q+1)\binom{2t-2}{t-1}$. Note that this last result refers to q-almost cross intersecting pairs $(\mathcal{F}, \mathcal{G})$ in which $\mathcal{F} = \mathcal{G}$. Furthermore, the constructions presented in [23], which achieve this bound, do not have a circulant intersection matrix.

Indeed, we can get a better upper bound for circulant submatrices of the form $C_{p,q}$. Using a theorem of Frankl [9] and Kalai [15] about skew matrices, it is possible to show that $p \leq \binom{2t}{t} - 1$. Hence, if $C_{p,q}$ is a submatrix of size $n \times n$ of $A_{k,t}$ then $n \leq \binom{2t}{t} + q - 1$.

In the extreme case of $p = \binom{2t}{t} - 1$ and $q = 1$, the simple construction that takes all subsets of size t of $[2t]$ as row and column indices, results in a submatrix $C_{p,q}$ of size $\binom{2t}{t} \times \binom{2t}{t}$. This is optimal, as it matches the upper bound of $\binom{2t}{t} + q - 1$.

For larger q, we give a simple construction of $C_{p,q}$ for $p = q \cdot (\binom{2t/q}{t/q} - 1)$, when $t \bmod q = 0$ and k is large enough. Note that there is a relatively large gap between the size of $C_{p,q}$ stated here, and the upper bound of $\binom{2t}{t} + q - 1$. As we prove, this gap can be slightly narrowed for $q = 2$:

Theorem 1.4. *Let $q = 2$ and $p = 2^t + 2^{t-2} - 2$, where $t > 2$. Then $C_{p,q}$ is a submatrix of $A_{k,t}$ for large enough k.*

We conclude by considering the case of $t = 2$ and $p = \binom{2t}{t} - 1 = 5$ and fully characterize it. As we show, in this case, $C_{p,q}$ is a submatrix of $A_{k,t}$, for $q = 1$ and $k \geq 5$, or for $q = 3$ and $k \geq 6$. Thus, for $t = 2, p = 5$ and $q = 1, 3$, we get a result which matches the upper bound of $\binom{2t}{t} + q - 1$. However, as we prove, for $t = 2, p = 5$ and

$q > 0$, $q \neq 1, 3$, there is no k for which $C_{p,q}$ is a submatrix of $A_{k,t}$. This implies that the upper bound of $\binom{2t}{t} + q - 1$ is not tight in general. It remains an open problem to determine for what values of $q > 1$ is $C_{p,q}$ a submatrix of $A_{k,t}$, given that $p = \binom{2t}{t} - 1$ and $t > 2$.

2 The range of $1 \leq p \leq 2t - 1$

In this section we prove Theorems 1.1 and 1.2, which address the range of small p, that is, $1 \leq p \leq 2t - 1$. As stated above, this range of values includes the identity matrix, as well as allows us to provide a simple construction of maximal isolation sets for large enough k.

It will be useful to identify subsets of $[k]$ with their characteristic vectors. Thus, a subset of size t of $[k]$ will be represented by a $0, 1$-vector of length k with exactly t ones. Furthermore, in order to show that some matrix C of size $n \times m$ is a submatrix of $A_{k,t}$, it will be enough to show that there exists a Boolean decomposition $C = X \cdot Y$, where X is a Boolean matrix of size $n \times k$ with exactly t ones in each row, and Y is a Boolean matrix of size $k \times m$ with exactly t ones in each column, and all operations are Boolean.

2.1 A construction of $C_{p,q}$ for $1 \leq p \leq 2t - 1$

The following lemma will be useful in proving Theorem 1.1. It shows that it is possible to decompose a matrix of the form $C_{p,q}$ into a product of two circulant matrices of the same type, for a wide range of parameters.

Lemma 2.1. *Let i, j, z be three integers, such that $i, j \geq 1$ and $i + j - 1 \leq z$. Then*

$$C_{i,z-i} \cdot C_{j,z-j} = C_{i+j-1,z-i-j+1}.$$

Proof. It is well known that the product of two circulant Boolean matrices is a circulant Boolean matrix (where all operations are Boolean). Thus, it is enough to determine the first column $c = (c_1, c_2, ..., c_z)$ of the product matrix $C_{i,z-i} \cdot C_{j,z-j}$, and to show that it has $i + j - 1$ ones, followed by $z - i - j + 1$ zeros. The proof follows directly from the definition of matrix multiplication using the Boolean operations.

Specifically, it is clear that $c_s = 1$ for $1 \leq s \leq i$, since the first element in each of the first i rows of $C_{i,z-i}$ is a 1, and the first element of the first column of $C_{j,z-j}$ is also a 1 (since $i, j \geq 1$). Next consider element c_{i+s} for $1 \leq s \leq j - 1$. Note that row $i + s$ of $C_{i,z-i}$ begins with s zeros and then has i ones, and the first j elements of the first column of $C_{j,z-j}$ are ones. Since $s \leq j - 1$, then the result of multiplying row $i + s$ of $C_{i,z-i}$ with the first column of $C_{j,z-j}$, is a one.

It remains to show that the remaining elements of c are all zeros. But the last $z-i-j+1$ rows of $C_{i,z-i}$ begin with at least j zeros. Therefore, multiplying any of these rows with the first column of $C_{j,z-j}$, results in a zero. □

Using Lemma 2.1, we can now prove Theorem 1.1.

Proof of Theorem 1.1. Let $1 \leq i, j \leq t$ such that $i + j - 1 = p$. Let $J_{n,m}$ be the all-one matrix of size $n \times m$, and $O_{n,m}$ the all-zero matrix of size $n \times m$. Define, two matrices X and Y as follows:

$$X = [C_{i,p+q-i} O_{p+q,t-j} J_{p+q,t-i}], \qquad Y = \begin{bmatrix} C_{j,p+q-j} \\ J_{t-j,p+q} \\ O_{t-i,p+q} \end{bmatrix}.$$

Using Lemma 2.1, where $z = p + q$, we have that

$$X \cdot Y = C_{i,p+q-i} \cdot C_{j,p+q-j} = C_{i+j-1,p+q-i-j+1} = C_{p,q}.$$

Furthermore, each row of X and each column of Y is a vector with exactly t ones, whose length is:

$$(p+q) + (t-j) + (t-i) = p+q+2t-i-j = p+q+2t-(p+1) = q+2t-1.$$

Therefore, if $k \geq q + 2t - 1$, then we can view the rows of X and columns of Y as the characteristic vectors of subsets in $\binom{[k]}{t}$. Thus, $X \cdot Y = C_{p,q}$ is a submatrix of $A_{k,t}$ as claimed. □

As we show next, if $k \geq 4t - 3$ then the construction described in the proof of Theorem 1.1, provides a maximal isolation submatrix of size $k \times k$ in $A_{k,t}$. This result is optimal since the Boolean rank of $A_{k,t}$ is k for $k \geq 2t$ (see [19]).

Corollary 2.2. *Let $2 \leq p \leq 2t - 1$ and let $k = 2t + p - 2$. Then $C_{p,p-1}$ is an isolation submatrix of size $(2p-1) \times (2p-1)$ in $A_{k,t}$. Furthermore, if $k \geq 4t-3$ then $C_{2t-1,k-2t+1}$ is an isolation submatrix of size $k \times k$ in $A_{k,t}$.*

Proof. Let $k = 2t+p-2$. If we set $q = k-2t+1 = (2t+p-2)-2t+1 = p-1$, then by Theorem 1.1, $C_{p,q}$ is a submatrix of $A_{k,t}$ of size $(2p-1) \times (2p-1)$ since $p+q = 2p-1$. It is easy to verify that in this case, since $q = p - 1$, the ones on the main diagonal of $C_{p,q}$ form an isolation set of size $p + q$.

In the extreme case of $p = 2t-1$, and if $k \geq 4t-3$, then $q = k-2t+1 \geq 2t-2 \geq p-1$, and we get an isolation matrix $C_{p,q}$ of size $k \times k$, since $p + k - 2t + 1 = k$. □

2.2 Upper bounds on the size of $C_{p,q}$ for $1 \leq p \leq 2t - 1$

We now turn to prove Theorem 1.2, which provides a matching upper bound to the size of the construction given in Theorem 1.1, for $1 \leq p \leq 2t - 1$ and $q \geq p - 1$. We note that if $q \geq p - 1$ then $p + q \leq k$ (for any value of p), since in this case $C_{p,q}$ is an isolation submatrix of $A_{k,t}$. Thus, its Boolean rank, which is $p + q$, is bounded above by k, which is the Boolean rank of $A_{k,t}$. However, the proof of Theorem 1.2, which provides a tight upper bound on $p + q$, will require a more elaborate proof.

The following simple claim is easy to verify, and will be needed for the proof of Theorem 1.2.

Claim 2.3. *Let B be an all-one submatrix of size $i \times j$ of $C_{p,q}$, where $p, q > 0$. Then, $1 \leq i, j \leq p$ and $i + j \leq p + 1$.*

The next lemma is a generalization of a claim proved in [19], which characterizes the Boolean decompositions of the identity matrix. Here we characterize the Boolean decompositions of circulant isolation matrices of the form $C_{p,q}$.

Denote by $|x|$ the number of ones in a vector x, and let $x \otimes y$ denote the outer product of a column vector x and a row vector y, where both x, y are of length n. That is, $x \otimes y$ is a matrix of size $n \times n$.

Lemma 2.4. *Let $p, q > 0$ and $n = p + q$. Let $X \cdot Y = C_{p,q}$ be a Boolean decomposition of $C_{p,q}$, where X is an $n \times r$ Boolean matrix and Y is an $r \times n$ Boolean matrix. Denote by $x_1, \ldots, x_r$ the columns of X, and by $y_1, \ldots, y_r$ the rows of Y. Then:*

1. *For each $i \in [r]$, if x_i has more than p ones then y_i is the all-zero vector, and if y_i has more than p ones then x_i is the all-zero vector.*
2. *For each $i \in [r]$, if $|x_i|, |y_i| > 0$, then $|x_i| + |y_i| \leq p + 1$.*
3. *If $q \geq p - 1$, then there exist n indices $i_1, ..., i_n$, such that $|x_{i_j}|, |y_{i_j}| > 0$ for every $j \in [n]$.*

Proof. For simplicity, denote $C = C_{p,q}$. If we write the decomposition $X \cdot Y = C$ with outer products, then $C = \sum_{i=1}^{r} x_i \otimes y_i$.

First note that if we have an i such that x_i has more than p ones, and y_i is not the all-zero vector, then $x_i \otimes y_i$ has a column with more than p ones. Since the addition is the Boolean addition, then $\sum_{i=1}^{r} x_i \otimes y_i \neq C$. A similar argument shows that if y_i has more than p ones then x_i is the all-zero vector. Thus, item (1) follows.

Assume now, by contradiction, that item (2) does not hold. Thus, there exists an i, such that $|x_i|, |y_i| > 0$ and $|x_i|+|y_i| > p+1$. Let $|x_i| = s$ and $|y_i| = \ell$, where by our assumption $\ell \geq p - s + 2$. Thus, the matrix $x_i \otimes y_i$ has an all-one submatrix B of size $s \times \ell$. Since the addition is Boolean, $C_{p,q}$, also has an all-one submatrix of size $s \times \ell \geq s \times (p - s + 2)$, in contradiction to Claim 2.3.

It remains to prove item (3). Since $q \geq p - 1$, then C is an isolation matrix. Therefore, its Boolean rank is $n = p + q$. Assume by contradiction that there are strictly less than n pairs x_i, y_i such that $|x_i|, |y_i| > 0$. Note that if x_i or y_i is the all-zero vector then $x_i \otimes y_i$ is the all-zero matrix. Thus, we can remove from X any column x_i which is the all-zero vector, and remove the corresponding row y_i from Y, and similarly, remove from Y any row y_j which is the all-zero vector, and remove the corresponding column x_j from X. We get two new matrices X', Y', such that $X' \cdot Y' = C$, where the size of X' is $n \times \ell$, the size of Y' is $\ell \times n$, and by our assumption $\ell < n$. Therefore, the Boolean rank of C is strictly less than n, and we get a contradiction. □

Lemma 2.5. *Let $p, q > 0$ and $q \geq p-1$, and let $n = p+q$. Let $X \cdot Y = C_{p,q}$ be a Boolean decomposition of $C_{p,q}$, where X is an $n \times r$ Boolean matrix and Y is an $r \times n$ Boolean matrix. Then the sum of the number of ones in X and the number of ones in Y is at most $(p + 1)n + (r - n)n$.*

Proof. Let $x_1, \ldots, x_r$ be the columns of X, and $y_1, \ldots, y_r$ the rows of Y. By Lemma 2.4, there exist n indices $i_1, ..., i_n$, such that $|x_{i_j}|, |y_{i_j}| > 0$ for every $j \in [n]$. Furthermore, for these indices it holds that $|x_{i_j}| + |y_{i_j}| \leq p + 1$. Assume, without loss of generality, that these are indices $1, ..., n$.

As for the remaining pairs, x_i, y_i, for $n < i \leq r$: by Lemma 2.4, if $|x_i|, |y_i| > 0$ then $|x_i| + |y_i| \leq p + 1$, and if $|x_i| \geq p + 1$ then y_i is the all-zero vector, and similarly if $|y_i| \geq p + 1$ then x_i is the all-zero vector. Thus, $|x_i| + |y_i|$ is maximized when x_i or y_i is the all-zero vector and the other is the all-one vector, since in this case $|x_i| + |y_i| = n = p + q \geq p + 1$.

Hence, the sum of the number of ones in X and the number of ones in Y is at most $(p + 1)n + (r - n)n$. □

Proof of Theorem 1.2. Consider the Boolean decomposition $X \cdot Y = A_{k,t}$, where X is a matrix of size $\binom{k}{t} \times k$ and Y is a matrix of size $k \times \binom{k}{t}$, and X and Y each contain all characteristic vectors of subsets in $\binom{[k]}{t}$.

Since $C_{p,q}$ is a submatrix of $A_{k,t}$ then there exist two matrices $X' \subseteq X, Y' \subseteq Y$, such that $X' \cdot Y' = C_{p,q}$. Notice that X' is an $n \times k$ matrix and Y' is an $k \times n$ matrix, where $n = p + q$, and the sum of the number of ones in X and the number of ones in Y is exactly $2nt$. But, by Lemma 2.5, the total number of 1's in both X' and Y' is at most $(p+1)n+(k-n)n$. Thus, $2nt \leq (p+1)n+(k-n)n$. Hence, $p+q = n \leq k-2t+p+1$, as claimed. □

3 The range of $2t \leq p \leq t^2$

The circulant decomposition given in Lemma 2.1 is not suitable for $p \geq 2t$, since if we take the decomposition $C_{i+j-1,z-i-j+1} = C_{i,z-i} \cdot C_{j,z-j}$, and let $p = i + j - 1$ and $p \geq 2t$, then $i + j \geq 2t + 1$. Thus, either i or j are strictly larger than t, and therefore, the rows of $C_{i,z-i}$ or the columns of $C_{j,z-j}$ cannot represent subsets of size t of $[k]$.

However, Theorem 1.3 stated in Section 1.1 and proved next, shows that when $2t \leq p \leq t^2$, there exists a different circulant decomposition $C_{p,q} = X \cdot Y$, in which each row of X and each column of Y has exactly t ones as required. See Figure 2 for an illustration, and note also that since $2t \leq p \leq t^2$ then $t \geq 2$.

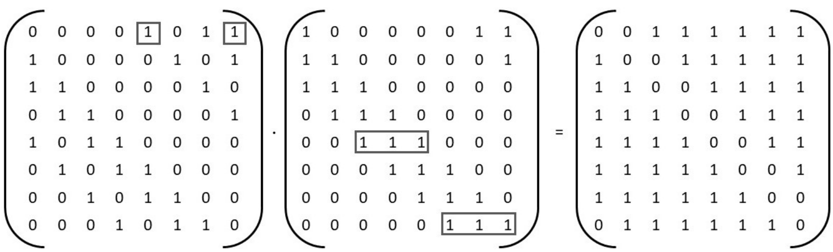

Figure 2: The construction described in Theorem 1.3 for $t = 3, p = 6, q = 2$. The matrices presented are $X \cdot Y = W$, where W is achieved from $C_{p,q}$ by moving the last row of $C_{p,q}$ as defined in the introduction to be first. The first row x_1 of X begins with $q + t - 1$ zeros, and then every t positions there is a one outlined with a rectangle. The remaining positions of x_1 contain zeros and ones for a total of t ones in x_1. The matrix $Y = C_{t,p+q-t}$. The ones in the columns of Y that intersect the outlined ones in x_1 are also outlined with rectangles.

Proof of Theorem 1.3. Let W be the matrix achieved from $C_{p,q}$ by moving the last row of $C_{p,q}$ as defined in the introduction to be first. Hence, the first row of W has q zeros followed by p ones. It is enough to show that there exist two square matrices X, Y, of size $p + q$, such that each row of X and each column of Y has exactly t ones, and $X \cdot Y = W$.

The matrix Y is just the matrix $C_{t,p+q-t}$ as defined in the introduction, and thus each column has t ones as required. The matrix X is also circulant and is defined as follows. The first row x_1 of X begins with $q+t-1$ zeros. The remaining $p-t+1 \geq 2t-t+1 = t+1$ coordinates of x_1 start with a one and then there is a one every t positions, and a one in the last position of x_1. The remaining positions have ones and zeros in an arbitrary order, for a total of t ones in x_1. Note that there are at most $t - 1$ zeros between every two consecutive ones in x_1, and in the extreme case of $p = t^2$, there are exactly $t - 1$ zeros between every two consecutive ones.

Finally, we must show that $X \cdot Y = W$. Since both X and Y are circulant, the resulting product matrix $X \cdot Y$ is circulant. Thus, it is sufficient to prove that the first row of $X \cdot Y$ is equal to the first row $(0, 0, ..., 0, 1, 1, ..., 1)$ of W. Let $y_1, ..., y_{p+q}$ be the columns of Y.

- Columns $y_1, ..., y_q$ have ones only in positions at most $q + t - 1$, whereas x_1 has zeros in the first $q + t - 1$ positions. Hence, $x_1 \cdot y_i = 0$, for $1 \leq i \leq q$, as required.
- Now consider the last p columns of Y. Note that for each such column y_j, at least one of the t consecutive ones of y_j avoids the $q + t - 1$ consecutive zeros of x_1. Furthermore, every t consecutive columns of Y have a one in a common row. It is easy to verify that columns y_j, for $q + 1 \leq j \leq q + t$, intersect with the first one in x_1, as they all have a one in position $q + t$. The next t columns of Y all have a one in position $q + 2t$, and since x_1 has a one every t coordinates, these columns also intersect with x_1, and so on. The last t columns of Y intersect with the last one of x_1.

Hence, $X \cdot Y = W$ as claimed, and the theorem follows. □

4 The range of large p

Bollobás [3] proved that for any m pairs of subsets (A_i, B_i), such that $|A_i| = a, |B_i| = b$ for $1 \leq i \leq m$, and $A_i \cap B_j = \emptyset$ if and only if $i = j$, it holds that $m \leq \binom{a+b}{a}$. An immediate corollary of this theorem is that the largest circulant submatrix $C_{p,q}$ of $A_{k,t}$, for $q = 1$, is of size $\binom{2t}{t} \times \binom{2t}{t}$. This result is tight.

This theorem has several generalizations. Among them is a result of Frankl [9] and Kalai [15] that considered the skew version of the problem, and showed that the same bound holds even under the following relaxed assumptions: Let (A_i, B_i) be pairs of sets, such that $|A_i| = a, |B_i| = b$ for $1 \leq i \leq m$, $A_i \cap B_i = \emptyset$ for every $1 \leq i \leq m$, and $A_i \cap B_j \neq \emptyset$ if $i > j$. Then $m \leq \binom{a+b}{a}$.

We immediately get the following corollary, where here and throughout this section we assume that the first row of $C_{p,q}$ has q zeros followed by p ones.

Corollary 4.1. *Let $C_{p,q}$ be a submatrix of $A_{k,t}$ of size $n \times n$, for a given fixed q. Then, $n \leq \binom{2t}{t} + q - 1$, that is, $p \leq \binom{2t}{t} - 1$.*

Proof. Consider the submatrix B of $C_{p,q}$ that is defined by the first $p+1$ rows and columns of $C_{p,q}$. The matrix B maintains the conditions of the Theorem of Frankl and Kalai, and, thus, its size is at most $\binom{2t}{t} \times \binom{2t}{t}$. Hence, $n \leq \binom{2t}{t} + q - 1$ as claimed, and $p \leq \binom{2t}{t} - 1$. □

The following lemma presents a simple construction of a large circulant submatrix $C_{p,q}$ of $A_{k,t}$ for a given fixed q. See also Figure 3 for an illustration.

Lemma 4.2. *Let $q > 0$, $t \geq q$, where $t \bmod q = 0$. Then $C_{p,q}$ is a submatrix of $A_{k,t}$, for $p = q \cdot (\binom{2t/q}{t/q} - 1)$ and $k \geq 3t - t/q$.*

Proof. Let $n = \binom{2t/q}{t/q}$. The matrix $C_{p,q}$, where $p + q = q \cdot n$, can be partitioned into q disjoint submatrices of size $n \times (p+q)$, as follows. The ith submatrix, $1 \leq i \leq q$, contains rows $i + j \cdot q$, $0 \leq j \leq n - 1$, of $C_{p,q}$. Each such submatrix is a blowup of $C_{n-1,1}$, since we can partition each row of these q submatrices into blocks of q consecutive entries, where the blocks of the ith submatrix are shifted by one position compared to the blocks of the

$C_{10,2}$ =

	1 2 a b	1 2 a c	1 3 a c	1 3 a d	1 4 a d	1 4 b c	2 3 b c	2 3 b d	2 4 b d	2 4 c d	3 4 c d	3 4 a b
3, 4	0	0	1	1	1	1	1	1	1	1	1	1
b, d	1	0	0	1	1	1	1	1	1	1	1	1
2, 4	1	1	0	0	1	1	1	1	1	1	1	1
b, c	1	1	1	0	0	1	1	1	1	1	1	1
2, 3	1	1	1	1	0	0	1	1	1	1	1	1
a, d	1	1	1	1	1	0	0	1	1	1	1	1
1, 4	1	1	1	1	1	1	0	0	1	1	1	1
a, c	1	1	1	1	1	1	1	0	0	1	1	1
1, 3	1	1	1	1	1	1	1	1	0	0	1	1
a, b	1	1	1	1	1	1	1	1	1	0	0	1
1, 2	1	1	1	1	1	1	1	1	1	1	0	0
c, d	0	1	1	1	1	1	1	1	1	1	1	0

Figure 3: The construction described in Lemma 4.2, for $t = 4, p = 10, q = 2$. The matrix $C_{10,2}$ is composed of two submatrices, one containing the odd rows and one the even rows. Each row of these submatrices is partitioned into $n = 6$ blocks of size $q = 2$, as outlined by rectangles. The first submatrix is the intersection matrix of all subsets of size 2 of $[4]$, and the second submatrix is the intersection matrix of all subsets of size 2 of $\{a, b, c, d\}$, where in both cases each of the subsets assigned to the columns appears twice (in columns belonging to the same block). Since the subsets assigned to the first submatrix are disjoint from the subsets assigned to the second submatrix, and the blocks in the two submatrices are shifted, this defines an assignment of different subsets of $\{1, 2, 3, 4, a, b, c, d\}$, each of size at most 4, to the columns and rows of $C_{10,2}$.

previous submatrix (in a circulant way). Thus, the entries in each block are identical (either all ones or all zeros). For example, the first row of the first submatrix starts with a block of q zeros, followed by $n - 1$ blocks of q ones.

Hence, we can view each of these q submatrices as the intersection matrix of the two families of all subsets of size t/q of $[2t/q]$ (since each subset intersects with all subsets but one), where columns belonging to the same block in a submatrix are assigned the same subset. Now, if we take disjoint copies of the subsets assigned to each submatrix, and label each column in $C_{p,q}$ with the subset that is the union of all subsets of size t/q assigned to this column, then we get $q \cdot n$ different subsets, each of size t, assigned to the columns (the subsets are different because the blocks in each submatrix are shifted compared to the other submatrices). As to the rows, each row is assigned a different subset of size t/q, and therefore, we can define $t - t/q$ additional new elements that do not belong to any of the subsets, and add them to each row, so that the rows are also assigned subsets of size t. Finally, by taking the union of all subsets, we get that $k \geq 2t + t - t/q = 3t - t/q$. □

The size of the construction given in Lemma 4.2 is quite far from the upper bound given in Corollary 4.1. As we show in the next subsection, there exists a slightly larger construction for $q = 2$. Finally, in Subsection 4.2, we show that the upper bound of Corollary 4.1 is tight for $t = 2, p = \binom{2t}{t} - 1 = 5$ and $q = 1, 3$, but there is no k for which $C_{5,2}$ is a submatrix of $A_{k,2}$ when $q \neq 1, 3$.

4.1 The values $q = 2$ and $p = 2^t + 2^{t-2} - 2$

We now prove Theorem 1.4 and show a construction of $C_{p,q}$ for $q = 2$ and p that is exponential in t. The construction we present is recursive in nature, and exploits the fact that $C_{p,2}$ has two blocks on the main diagonal, such that each one of these blocks is half the size of $C_{p,2}$, and the structure of each block is almost identical to that of $C_{p,q}$, where the only difference is that there is a 1 in the first position of the last row instead of a zero in $C_{p,q}$. This small difference complicates the recursive argument. The details of the proof follow. See Figure 4 for an illustration of the proof of Theorem 1.4.

$C_{8,2}$ =

1,5,8	1,2,8	3,2,8	3,5,8	3,7,8	1,6,9	1,2,9	4,2,9	4,6,9	4,7,9	
0	0	1	1	1	1	1	1	1	1	7, 3, 9
1	0	0	1	1	1	1	1	1	1	7, 5, 9
1	1	0	0	1	1	1	1	1	1	7, 1, 9
1	1	1	0	0	1	1	1	1	1	2, 1, 9
1	1	1	1	0	0	1	1	1	1	2, 5, 4
1	1	1	1	1	0	0	1	1	1	7, 4, 8
1	1	1	1	1	1	0	0	1	1	7, 6, 8
1	1	1	1	1	1	1	0	0	1	7, 1, 8
1	1	1	1	1	1	1	1	0	0	2, 1, 8
0	1	1	1	1	1	1	1	1	0	2, 6, 3

Figure 4: The construction of $C_{p,q}$ described in Theorem 1.4, for $q = 2$ and $p = 2^t + 2^{t-2} - 2$, where $t = 3$. The column indices are written above the matrix $C_{8,2}$ and the row indices to the right of the matrix.

Proof of Theorem 1.4. Let $h = (p + q)/2$. We prove by induction on t that $C_{p,q}$ is the intersection matrix of two families of t-subsets

$$\mathcal{F}_{a,b} = \{F_1, ..., F_{p+q}\}, \qquad \mathcal{G}_{a,b} = \{G_1, ..., G_{p+q}\},$$

where the subsets in $\mathcal{F}_{a,b}$ are the row indices and the subsets in $\mathcal{G}_{a,b}$ are the column indices, and a, b are two integers with the following properties:

- $a \in G_1, ..., G_h$, and $a \in F_{h+1}, ..., F_{p+q-1}$.
- $b \in G_{h+1}, ..., G_{p+q}$, and $b \in F_1, ..., F_{h-1}$.
- a, b appear only in the subsets specified above. In particular, $a, b \notin F_{p+q}$.

Let $\tilde{C}_{p,q}$ be the matrix that is achieved from $C_{p,q}$, by modifying to 1 the first position of the last row of $C_{p,q}$, and let $\tilde{\mathcal{F}}_{a,b}$ be a family that is identical to $\mathcal{F}_{a,b}$ with one difference: the subset F_{p+q} also contains the element a. It is not hard to verify that if $C_{p,q}$ is the intersection matrix of $\mathcal{F}_{a,b}$ and $\mathcal{G}_{a,b}$, then $\tilde{C}_{p,q}$ is the intersection matrix of $\tilde{\mathcal{F}}_{a,b}$ and $\mathcal{G}_{a,b}$.

The base of the induction is $t = 3$, and the construction of $C_{8,2}$ is given in Figure 4, where in this case $a = 8, b = 9$. Note that if we modify the last row index $\{2, 6, 3\}$ to be $\{2, 6, 3, 8\}$, then we get a construction of $\tilde{C}_{8,2}$ as claimed.

Assume now that $t > 3$, let $p_t = 2^t + 2^{t-2} - 2$ and $p_{t-1} = 2^{t-1} + 2^{t-3} - 2$, and consider $C_{p_t,q}$. Then it has the following structure: there are two matrices of the form $\tilde{C}_{p_{t-1},q}$ on the main diagonal, and two blocks of size $(p_t + q)/2$ that are all one, but the leftmost entry on the bottom row of each of these blocks that is a 0.

By the induction hypothesis there exist, as specified above, two families of $(t-1)$-subsets

$$\mathcal{F}_{a,b} = \{F_1, ..., F_{p_{t-1}+q}\}, \qquad \mathcal{G}_{a,b} = \{G_1, ..., G_{p_{t-1}+q}\},$$

whose intersection matrix is $C_{p_{t-1},q}$.

Let $\mathcal{F'}_{b,a} = \{F'_1, ..., F'_{p_{t-1}+q}\}$ be a family of subsets that is identical to $\mathcal{F}_{a,b}$, but a, b are interchanged in all subsets. That is, for $1 \leq i \leq p_{t-1} + q$:

$$F'_i = \begin{cases} F_i \setminus \{a\} \cup \{b\}, & \text{if } a \in F_i, \\ F_i \setminus \{b\} \cup \{a\}, & \text{if } b \in F_i, \\ F_i, & \text{if } a, b \notin F_i. \end{cases}$$

Similarly define $\mathcal{G'}_{b,a} = \{G'_1, ..., G'_{p_{t-1}+q}\}$, which is identical to $\mathcal{G}_{a,b}$, but a, b are interchanged in all subsets. Note that since a, b appear only in subsets as specified above, then it also holds that $C_{p_{t-1},q}$ is the intersection matrix of the two families $\mathcal{F'}_{b,a}$ and $\mathcal{G'}_{b,a}$.

Now let c, d be two new elements that do not appear in any of the above families, and define the following families:

$$\begin{aligned} \mathcal{F}_d &= \{F_1 \cup \{d\}, F_2 \cup \{d\}, ..., F_{p_{t-1}+q-1} \cup \{d\}, F_{p_{t-1}+q} \cup \{a\}\}, \\ \mathcal{F}_c &= \{F'_1 \cup \{c\}, F'_2 \cup \{c\}, ..., F'_{p_{t-1}+q-1} \cup \{c\}, F'_{p_{t-1}+q} \cup \{b\}\}, \\ \mathcal{G}_c &= \{G_1 \cup \{c\}, G_2 \cup \{c\}, ..., G_{p_{t-1}+q} \cup \{c\}\}, \\ \mathcal{G}_d &= \{G'_1 \cup \{d\}, G'_2 \cup \{d\}, ..., G'_{p_{t-1}+q} \cup \{d\}\}. \end{aligned}$$

Finally, define the families $\mathcal{F}_{c,d}, \mathcal{G}_{c,d}$ as follows:

$$\mathcal{F}_{c,d} = \mathcal{F}_d \cup \mathcal{F}_c, \qquad \mathcal{G}_{c,d} = \mathcal{G}_c \cup \mathcal{G}_d.$$

It is clear that $\mathcal{F}_{c,d}, \mathcal{G}_{c,d}$ are two families of t-sets, each of size $p_t + q$, and their structure is as claimed above, where c and d are in the role of a and b, respectively. It remains to prove that $C_{p_t,q}$ is the intersection matrix of $\mathcal{F}_{c,d}, \mathcal{G}_{c,d}$. First note that by the induction hypothesis, and using the structure of the subsets we defined, $\tilde{C}_{p_{t-1},q}$ is the intersection matrix of $\mathcal{F}_d, \mathcal{G}_c$, as well as the intersection of $\mathcal{F}_c, \mathcal{G}_d$.

Consider now the matrix C which is the intersection matrix of $\mathcal{F}_d, \mathcal{G}_d$. It is clear that the first $p_{t-1} + q - 1$ rows of C are all ones, since the first $p_{t-1} + q - 1$ families of $\mathcal{F}_d, \mathcal{G}_d$ all contain d. We next show that the last row of C is of the form $(0, 1, 1, ..., 1)$. By the induction hypothesis, the intersection of $F_{p_{t-1}+q}$ with all subsets of $\mathcal{G}_{a,b}$ gives a vector of the form $(0, 1, 1, ...1, 0)$. Thus, since $a, b \notin F_{p_{t-1}+q}$ and $\mathcal{G}'_{b,a}$ is identical to $\mathcal{G}_{a,b}$, but a, b are interchanged in all subsets, then the intersection of of $F_{p_{t-1}+q}$ with all subsets of $\mathcal{G}'_{b,a}$ results also with the vector $(0, 1, 1, ...1, 0)$. Since the last subset of $\mathcal{F}_d$ is defined as $F_{p_{t-1}+q} \cup \{a\}$ and the last subset of $\mathcal{G}_d$ is $G'_{p_{t-1}+q} \cup \{d\}$, and $a \in G'_{p_{t-1}+q}$, then we get that $F_{p_{t-1}+q} \cup \{a\}$ and $G'_{p_{t-1}+q} \cup \{d\}$ also intersect as required.

A similar argument shows that the intersection matrix of $\mathcal{F}_d, \mathcal{G}_d$ is also a matrix that is all one, but the first element on the last row of this matrix, which is a zero. This completes the proof of the theorem. □

4.2 The values $t = 2, p = \binom{2t}{t} - 1, q > 0$

Finally, we address the range of values of $t = 2$ and $p = \binom{2t}{t} - 1 = 5$. We first show that $C_{p,q}$ is a submatrix of $A_{k,t}$ for these values of p and t, and for $q = 1, 3$.

Lemma 4.3. *Let $t = 2$ and $p = \binom{2t}{t} - 1 = 5$. Then $C_{p,q}$ is a submatrix of $A_{k,t}$ for $q = 1$ and $k \geq 5$, or for $q = 3$ and $k \geq 6$.*

Proof. If $t = 2, p = 5, q = 1$, then $C_{5,1}$ is a submatrix of $A_{5,2}$. Simply take as row/column indices all subsets of size 2 of $[4]$. As to the case of $t = 2, p = 5, q = 3$, Figure 5 shows that $C_{5,3}$ is a submatrix of $A_{k,2}$, for any $k \geq 6$. □

$$\begin{pmatrix} 1&0&0&1&0&0\\ 1&0&0&0&1&0\\ 1&1&0&0&0&0\\ 0&1&0&0&0&1\\ 0&1&1&0&0&0\\ 0&0&1&0&1&0\\ 0&0&1&1&0&0\\ 0&0&0&1&0&1 \end{pmatrix} \cdot \begin{pmatrix} 0&0&0&0&0&1&1&1\\ 1&1&0&0&0&0&0&1\\ 0&1&1&1&0&0&0&0\\ 0&0&0&1&1&1&0&0\\ 1&0&0&0&1&0&0&0\\ 0&0&1&0&0&0&1&0 \end{pmatrix} = \begin{pmatrix} 0&0&0&1&1&1&1&1\\ 1&0&0&0&1&1&1&1\\ 1&1&0&0&0&1&1&1\\ 1&1&1&0&0&0&1&1\\ 1&1&1&1&0&0&0&1\\ 1&1&1&1&1&0&0&0\\ 0&1&1&1&1&1&0&0\\ 0&0&1&1&1&1&1&0 \end{pmatrix}$$

Figure 5: A construction of $C_{p,q}$ for $t = 2, p = 5, q = 3$.

We conclude by proving that $C_{p,q}$ is not a submatrix of $A_{k,t}$ for $t = 2, p = 5$ and $q \neq 1, 3$. Unfortunately, this proof cannot be generalized to the case of $p = \binom{2t}{t} - 1$ and

$t > 2$. Thus, it remains an open problem to determine for what values of $q > 1$ is $C_{p,q}$ a submatrix of $A_{k,t}$, when $p = \binom{2t}{t} - 1$ and $t > 2$.

Lemma 4.4. *Let $t = 2$, $p = \binom{2t}{t} - 1 = 5, q \neq 1, 3$, $q > 0$. Then $C_{p,q}$ is not a submatrix of $A_{k,t}$ for any k.*

Proof. Assume by contradiction that $C_{5,q}$ is a submatrix of $A_{k,2}$ for some k, where the first row of $C_{5,q}$ starts with $q \neq 1, 3$ zeros, followed by p ones. Let $n = p + q = q + 5 \geq 7$ be the size of $C_{5,q}$, and let A_i, B_i, $0 \leq i \leq n-1$, be the 2-uniform subsets defining the row and column indices, respectively, of $C_{5,q}$.

Assume first that there exists some i such that $B_i \cap B_{(i+1) \bmod n} = \emptyset$, that is, two consecutive column indices are disjoint. Since $C_{5,q}$ is circulant, then we can assume that $i = 0$, that is, $B_0 \cap B_1 = \emptyset$. Since B_0 and B_1 both intersect with A_3, A_4, A_5, then each of these three subsets contains one element from each of B_0, B_1. Furthermore, as all subsets are different and of size 2, then each element of B_0, B_1 is contained in at most two of these three subsets.

Next consider B_2. It also intersects with A_3, A_4, A_5, and since there is no common element of B_0, B_1 in these three subsets, then B_2 also includes two elements from $B_0 \cup B_1$ (although here B_2 can contain two elements from the same subset B_0 or B_1).

Now, consider $A_{7 \bmod n}$, where if $q = 2$ then $A_{7 \bmod n} = A_0$ and otherwise, $A_{7 \bmod n} = A_7$. In both cases, since $p = 5$, the row labeled by $A_{7 \bmod n}$ starts with two zeros followed by a one. Thus, since $A_{7 \bmod n} \cap B_2 \neq \emptyset$, then $A_{7 \bmod n}$ contains an element from $B_0 \cup B_1$, in contradiction to the fact that $A_{7 \bmod n} \cap B_0 = A_{7 \bmod n} \cap B_1 = \emptyset$.

Hence, we can assume from now on that $B_i \cap B_{(i+1) \bmod n} \neq \emptyset$, and similarly that $A_i \cap A_{(i+1) \bmod n} \neq \emptyset$, for $0 \leq i \leq n-1$. There are two cases:

- There exists an i such that $B_i \cap B_{(i+1) \bmod n} \cap B_{(i+2) \bmod n} \neq \emptyset$. Since $C_{p,q}$ is circulant, then assume that $i = 0$, and let $b \in B_0, B_1, B_2$. Thus, $B_0 = \{b_0, b\}, B_1 = \{b_1, b\}, B_2 = \{b_2, b\}$. From this and the structure of $C_{5,q}$, we can deduce the following:

  1. $b_0 \in A_1$, $A_2 = \{b_0, b_1\}$, and $b \in A_3, A_4, A_5$.
  2. The row labeled by A_6 starts with a zero followed by 5 ones, and so $b \notin A_6$. But $A_6 \cap B_1 \neq \emptyset, A_6 \cap B_2 \neq \emptyset$. Thus, $A_6 = \{b_1, b_2\}$.
  3. $b \notin B_3$ as $B_3 \cap A_3 = \emptyset$. But $B_3 \cap B_2 \neq \emptyset$, and therefore, $b_2 \in B_3$.
  4. $b, b_0, b_1 \notin B_3$ as also $B_3 \cap A_2 = \emptyset$. But $B_3 \cap A_5 \neq \emptyset$ and $A_5 \cap A_6 \neq \emptyset$. Therefore, $b_2 \in A_5$.
  5. Since $b, b_0, b_1 \notin B_3$ and $B_3 \cap A_4 \neq \emptyset$, then there exists a new element $b_3 \in B_3 \cap A_4$.
  6. $A_4 \cap B_4 = \emptyset$ and so $b \notin B_4$. Hence, $b_2 \in B_4$ since $B_4 \cap A_5 \neq \emptyset$. In a similar way, $b_1 \in B_5$.

  Hence, the subsets defining the first seven rows and columns of $C_{5,q}$ have the following structure so far, where they are written to the left and above the submatrix:

	b_0 b	b_1 b	b_2 b	b_2 b_3	b_2	b_1	
	0	0					
b_0	1	0	0				
b_0, b_1	1	1	0	0			
b	1	1	1	0	0		
b, b_3	1	1	1	1	0	0	
b, b_2	1	1	1	1	1	0	0
b_1, b_2	0	1	1	1	1	1	0

Now, if $q \geq 4$, we already get a contradiction, since in $C_{5,q}$ it holds that $A_2 \cap B_5 = \emptyset$, whereas here $b_1 \in A_2 \cap B_5$.

Therefore, assume that $q = 2$, and so all remaining entries in the submatrix above are ones. From the structure of the submatrix and the information we have so far, we can deduce that $A_1 = \{b_0, b_3\}$ and hence $B_5 = \{b_1, b_0\}$ (since $B_5 \cap A_1 \neq \emptyset$ and $B_5 \cap A_4 = \emptyset$ and so $b_3 \notin B_5$). But then since $b_0, b_1 \notin A_0$, we get a contradiction since $A_0 \cap B_5 \neq \emptyset$.

- $B_i \cap B_{(i+1) \bmod n} \cap B_{(i+2) \bmod n} = \emptyset$, but $B_i \cap B_{(i+1) \bmod n} \neq \emptyset$, for $0 \leq i \leq n-1$. Thus, $B_i = \{b_i, b_{(i+1) \bmod n}\}$, where some of the b_i's may be identical.

  If all b_i's in the subsets $B_q, B_{q+1}, B_{q+2}, B_{q+3}, B_{q+4}$ are different, then A_0 cannot intersect with these subsets, since $|A_0| = 2$. Hence, there exist $0 \leq i \neq j \leq 4$ such that $b_{q+i} = b_{q+j}$. Assume, without loss of generality, that $i = 0$ (as the matrix is circulant). Since the intersection of every three consecutive subsets is empty, and each subset contains two different elements, then $j \neq 1, 2$. If $j = 3$ then $b_q = b_{q+3}$, and since A_2 does not intersect with B_q, B_{q+1} then $b_q, b_{q+1}, b_{q+2} \notin A_2$. But A_2 intersects with $B_{q+2} = \{b_{q+2}, b_{q+3} = b_q\}$, and we get a contradiction. A similar contradiction is achieved if $j = 4$ when considering A_5.

Thus, in all cases we get a contradiction and the lemma follows. □

References

[1] B. Alspach, Isomorphism and cayley graphs on abelian groups, in: *Graph Symmetry*, Springer, volume 497 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pp. 1–22, 1997, doi:10.1007/978-94-015-8937-6_1.

[2] L. B. Beasley, Isolation number versus Boolean rank, *Linear Algebra Appl.* **436** (2012), 3469–3474, doi:10.1016/j.laa.2011.12.013.

[3] B. Bollobás, On generalized graphs, *Acta Math. Acad. Sci. Hungar.* **16** (1965), 447–452, doi:10.1007/bf01904851.

[4] P. Borg, The maximum sum and the maximum product of sizes of cross-intersecting families, *European J. Combin.* **35** (2014), 117–130, doi:10.1016/j.ejc.2013.06.029.

[5] K. Butler and J. R. Krabill, Circulant Boolean relation matrices, *Czechoslovak Math. J.* **24** (1974), 247–251.

[6] K. Butler and Š. Schwarz, The semigroup of circulant Boolean matrices, *Czechoslovak Math. J.* **26** (1976), 632–635.

[7] H. Daode, On circulant Boolean matrices, *Linear Algebra Appl.* **136** (1990), 107–117, doi:10.1016/0024-3795(90)90022-5.

[8] P. Erdős, C. Ko and R. Rado, Intersection theorems for systems of finite sets, *Q. J. Math.* **12** (1961), 313–320, doi:10.1093/qmath/12.1.313.

[9] P. Frankl, An extremal problem for two families of sets, *European J. Combin.* **3** (1982), 125–127, doi:10.1016/s0195-6698(82)80025-5.

[10] P. Frankl and N. Tokushige, Some best possible inequalities concerning cross-intersecting families, *J. Combin. Theory Ser. A* **61** (1992), 87–97, doi:10.1016/0097-3165(92)90054-x.

[11] D. Gerbner, N. Lemons, C. Palmer, D. Pálvölgyi, B. Patkós and V. Szécsi, Almost cross-intersecting and almost cross-sperner pairs of families of sets, *Graphs Combin.* **29** (2013), 489–498, doi:10.1007/s00373-012-1138-2.

[12] D. Gerbner, N. Lemons, C. Palmer, B. Patkós and V. Szécsi, Almost intersecting families of sets, *SIAM J. Discrete Math.* **26** (2012), 1657–1669, doi:10.1137/120878744.

[13] H. Gruber and M. Holzer, Inapproximability of nondeterministic state and transition complexity assuming $\mathrm{P} \neq \mathrm{NP}$, in: *Developments in language theory*, Springer, Berlin, volume 4588 of *Lecture Notes in Comput. Sci.*, pp. 205–216, 2007, doi:10.1007/978-3-540-73208-2_21.

[14] A. J. Hilton and E. C. Milner, Some intersection theorems for systems of finite sets, *Quart. J. Math. Oxford Ser. (2)* **18** (1967), 369–384, doi:10.1093/qmath/18.1.369.

[15] G. Kalai, Intersection patterns of convex sets, *Israel J. Math.* **48** (1984), 161–174, doi:10.1007/bf02761162.

[16] E. Kushilevitz and N. Nisan, *Communication Complexity*, Cambridge University Press, Cambridge, 1996, doi:10.1017/cbo9780511574948.

[17] M. Matsumoto and N. Tokushige, The exact bound in the erdös-ko-rado theorem for cross-intersecting families, *J. Combin. Theory Ser. A* **52** (1989), 90–97, doi:10.1016/0097-3165(89)90065-4.

[18] J. Orlin, Contentment in graph theory: covering graphs with cliques, *Indag. Math.* **80** (1977), 406–424, doi:10.1016/1385-7258(77)90055-5.

[19] M. Parnas, D. Ron and A. Shraibman, The Boolean rank of the uniform intersection matrix and a family of its submatrices, *Linear Algebra Appl.* **574** (2019), 67–83, doi:10.1016/j.laa.2019.03.027.

[20] M. Parnas and A. Shraibman, On maximal isolation sets in the uniform intersection matrix, *Australas. J. Combin.* **77** (2020), 285–300, https://ajc.maths.uq.edu.au/?page=get_volumes&volume=77.

[21] L. Pyber, A new generalization of the Erdős-Ko-Rado theorem, *J. Combin. Theory Ser. A* **43** (1986), 85–90, doi:10.1016/0097-3165(86)90025-7.

[22] Š. Schwarz, Circulant Boolean relation matrices, *Czechoslovak Math. J.* **24** (1974), 252–253.

[23] A. Scott and E. Wilmer, Hypergraphs of bounded disjointness, *SIAM J. Discrete Math.* **28** (2014), 372–384, doi:10.1137/130925670.

[24] Y. Shitov, On the complexity of Boolean matrix ranks, *Linear Algebra Appl.* **439** (2013), 2500–2502, doi:10.1016/j.laa.2013.06.033.