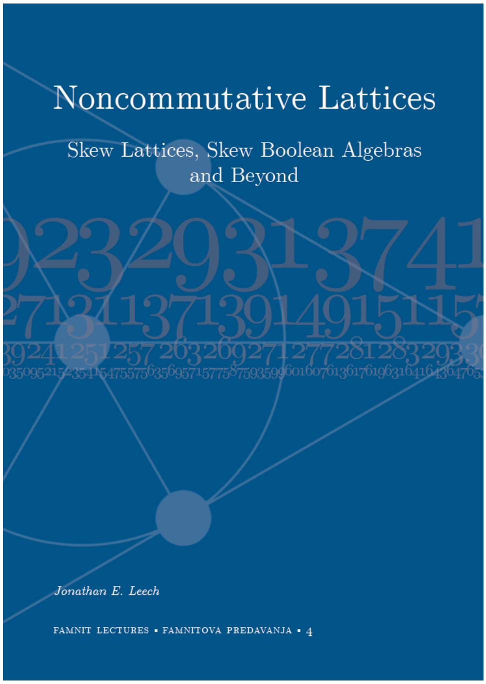
Noncommutative Lattices
by Jonathan E. Leech



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Weakenings of lattices, where the meet and join operations may fail to be commutative, attracted from time to time the attention of various communities of scholars, including ordered algebra theorists (for their connection with preordered sets) and semigroup theorists (who viewed them as structurally enriched bands possessing a dual multiplication). Recently, noncommutative generalisations of lattices and related structures have seen a growth in interest, with new ideas and applications emerging. The adjective "noncommutative" is used here in the inclusive sense of "not-necessarily-commutative". Much of this recent activity derives in some way from the initiation, thirty years ago, by Jonathan Leech, of a research program into structures based on Pascual Jordan's notion of a noncommutative lattice. Indeed, the research began by studying multiplicative bands of idempotents in rings, and realising that under certain conditions such bands would also be closed under an "upward multiplication". In particular, for multiplicative bands that were left regular, any maximal such band in a ring was also closed under the circle operation (or quadratic join) $x \circ y = x + y - xy$. And any band closed under both operations satisfied certain absorption identities, e.g., $e(e \circ f) = e = e \circ (ef)$. These observations indicated the presence of structurally strengthened bands with a roughly lattice-like structure. These algebras are called *skew lattices* and are defined as algebras $(S; \wedge, \vee)$ of type $(2, 2)$, where both operations $\wedge$ and $\vee$ are associative and satisfy the four absorption identities $x \wedge (x \vee y) = x = (y \vee x) \wedge x$ and their dual. Absorption causes both operations to be idempotent. In the case of maximal left regular bands in rings, $\wedge$ and $\vee$ are given as $e \wedge f = ef$ and $e \vee f = e + f - ef$.

Parallel to this was an expanding role of results related to structures that were weakened or modified forms of (generalised) Boolean algebras. This was especially important in the study of a second class of motivating examples, algebras of partial functions between pairs of sets, A and B. Here, the skew operations are defined as follows:

$$f \wedge g = g_{|G \cap F}\,; \quad f \vee g = f \cup g_{|G-F}\,,$$

where $F, G \subseteq A$ denote the actual functional domains of the partial functions f and g respectively. A relative complement, defined by $f \setminus g = f_{|F-G}$, can be added to the algebraic structure, making definable the complement in the Boolean interval $\{g : g \subseteq f\}$ of all functions approximating a given f. These algebras of partial functions provided examples of so-called *skew Boolean algebras* and related structures, much as subsets of a given set led to basic examples of Boolean algebras and distributive lattices. The pioneering papers by Leech on skew lattices and skew Boolean algebras have attracted the attention of mathematicians from around the world, and in the last thirty years many interesting papers have been published on the subject. As a result of these developments, skew lattices have grown into a theory worth studying for its own sake. The 2020/21 monograph *Noncommutative Lattices: Skew Lattices, Skew Boolean Algebras and Beyond* by Jonathan E. Leech provides an excellent, organised and comprehensive account of much that has been published on the subject up through 2017. The book is mainly written for algebraists and mathematicians, but readers interested in applications to logic and computer science may also find it useful. The core of this monograph is the first four chapters. More specialised topics are studied in the last three chapters. The content of the monograph will be explained in more detail in the remaining part of this review.

In the first chapter of the book the author recalls various basic facts about bands (idempotent semigroups) that are pertinent to the rest of the monograph. In particular, he emphasises that a knowledge of regular bands, and their left and right-sided cases, is crucial to understanding much that will be said about skew lattices. The author of this review has particularly appreciated Section 1.3, where a noncommutative lattice is defined as a double band satisfying a specified set among eight possible absorption identities. The comparison of these absorption laws naturally brings the reader into the definitions of quasilattices, paralattices, antilattices and skew lattices.

The basic theory for skew lattices is developed in Chapter 2. Of particular importance in Section 2.1 are the two core structural results for skew lattices, analogues of the Clifford-McLean Theorem and the Kimura Factorization Theorem, given originally for bands and regular bands, respectively. There are two basic subvarieties of skew lattices: lattices (full commutation) and anti-lattices (no non trivial commutation). The Clifford-McLean Theorem for skew lattices thus states that every skew lattice is a lattice of anti-lattices. More precisely, Green's relation $\mathcal{D}$ on a skew lattice S, defined by $x \,\mathcal{D}\, y$ iff $x \wedge y \wedge x = x$ and $y \wedge x \wedge y = y$, is a congruence making $S/\mathcal{D}$ the maximal lattice image of S, and all congruence classes of $\mathcal{D}$ are maximal anti-lattices in S (Theorem 2.1.3). The Kimura Factorisation Theorem for skew lattices interestingly states that every skew lattice S factorises as the fibered product of its maximal right-handed image and its maximal left-handed image (Theorem 2.1.5).

An element that join commutes with all elements in a skew lattice also meet commutes with all elements (and conversely). In general two elements commuting under one operation need not commute under the other operation. A skew lattice S is called *symmetric* if this does not happen. All skew lattices in rings (using multiplication and the circle operation) are symmetric. In Section 2.2 many results on symmetric skew lattices are presented. We mention here that, if S is a symmetric skew lattice for which $S/\mathcal{D}$ is countable, then S has a lattice section (Theorem 2.2.7), i.e., a sublattice T of S having nonempty intersection with each $\mathcal{D}$-class of S, in which case, $T \cong S/\mathcal{D}$. A characterisation of having a left-handed section and a right-handed section is also given in Theorem 2.2.8.

In Section 2.3 *normal* skew lattices are studied. In a normal skew lattice the lower set $\{x : x \leq e\}$ is a lattice, for every e. Of special interest are distributive, symmetric, normal skew lattices characterised in Theorem 2.3.4 by the identities $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ and $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$. This strengthened form of distributivity is called *strong distributivity*. Thanks to Theorem 2.3.6, every normal skew lattice of idempotents in a ring is strongly distributive. In this case the operations are given by $e \wedge f = ef$ again, but $e \vee f = (e + f - ef)^2 = e + f + fe - efe - fef$, the cubic join. Of course when $e + f - ef$ is idempotent, both outcomes agree. Strongly distributive skew lattices are also of interest due to their connections to skew Boolean algebras, the subject of Chapter 4. A skew lattice can be embedded into (the skew lattice reduct of) a skew Boolean algebra precisely when it is strongly distributive.

In Section 2.4 and 2.5 a detailed study of the natural partial order $\leq$ on a skew lattice is provided. This study is based on the behaviour of *primitive* skew lattices, consisting of exactly two $\mathcal{D}$-classes. Primitive skew lattices have a simple description given in terms of cosets (Theorem 2.4.1). In Section 2.6 the decompositions of (mostly symmetric) normal skew lattices are studied. For instance, the Reduction Theorem 2.6.9 implies that every symmetric normal skew lattice can be embedded in the product of its maximal lattice image and its maximal distributive image. In Theorem 2.6.12 and its corollaries the variety of strongly distributive skew lattices is shown to be generated by a special primitive skew lattice $\mathbf{5}$, a noncommutative 5-element variant of the lattice $\mathbf{2}$ for which the latter is its maximal lattice image. A similar result holds for the variety of symmetric, normal skew lattices.

Chapter 3 is devoted to the study of quasilattices, paralattices, and especially refined quasilattices. The variety of refined quasilattices contains the variety of skew lattices and it is defined as intersection of the variety of quasilattices and of the variety of paralattices. Particular attention is given to their congruence lattices and to related topics such as Green's equivalences and simple algebras. Since all skew lattices are refined quasilattices, the study in this chapter has implications for skew lattices.

Skew Boolean algebras are studied in Chapters 4 and 7. In Section 4.1 skew Boolean algebras are formally defined as structural enhancements of strongly distributive skew lattices. Skew Boolean algebras are shown to be subdirect products of primitive skew Boolean algebras; moreover, every skew Boolean algebra can be embedded into a power of $\mathbf{5}$, a 5-element primitive algebra (Corollaries 4.1.6 and 4.1.7). In Section 4.2, special attention is given to classifying finite algebras, and in particular, to classifying finitely generated (and thus finite) free skew Boolean algebras. The main results are Theorems 4.2.2 and 4.2.6, with the latter describing the structure of finitely generated free algebras. In Section 4.4 skew Boolean algebras with finite intersections are introduced, that is, algebras for which the natural partial order has meets that are called intersections. All skew Boolean algebras S for which $S/\mathcal{D}$ is finite have intersections as do, more generally, all complete skew Boolean algebras. In Chapter 7 the role of skew Boolean algebras in universal algebra is examined, in particular in the study of what might be termed "generalised Boolean phenomena", a topic of interest in universal algebra, with connections to discriminator varieties, iBCK-algebras and more recently, Church algebras. The reviewer thinks that the characterisation of one-pointed discriminator varieties in terms of right-handed skew Boolean algebras with intersections is one of the most beautiful results of the theory. Chapter 6 (Skew Lattices in Rings) is also devoted to the skew Boolean algebras of idempotents in rings, and in particular, the case where the idempotents in a ring are closed under multiplication and thus naturally form a skew Boolean algebra.

We conclude the review of this excellent monograph with the belief that it will be the main reference on the subject of noncommutative lattices for many years.

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