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On 12-regular nut graphs*

Nino Bašić †

FAMNIT & IAM, University of Primorska, 6000 Koper, Slovenia, and
Institute of Mathematics, Physics and Mechanics, 1000 Ljubljana, Slovenia

Martin Knor ‡

Department of Mathematics, Faculty of Civil Engineering, Slovak University of Technology in Bratislava, Radlinského 11, 810 05, Bratislava, Slovakia

Riste Škrekovski §

Faculty of Information Studies, 8000 Novo mesto, Slovenia, and
FMF, University of Ljubljana, 1000 Ljubljana, Slovenia, and
FAMNIT, University of Primorska, 6000 Koper, Slovenia

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Abstract

A nut graph is a simple graph whose adjacency matrix is singular with 1-dimensional kernel such that the corresponding eigenvector has no zero entries. In 2020, Fowler *et al.* characterised for each $d \in \{3, 4, \dots, 11\}$ all values n such that there exists a d-regular nut graph of order n. In the present paper, we resolve the first open case $d = 12$, i.e. we show that there exists a 12-regular nut graph of order n if and only if $n \geq 16$. We also present a result by which there are infinitely many circulant nut graphs of degree $d \equiv 0 \pmod 4$ and no circulant nut graphs of degree $d \equiv 2 \pmod 4$. The former result partially resolves a question by Fowler *et al.* on existence of vertex-transitive nut graphs of order n and degree d. We conclude the paper with problems, conjectures and ideas for further work.

Keywords: Nut graph, adjacency matrix, singular matrix, core graph, Fowler construction, regular graph.

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1 Introduction

Let G be a simple graph with the vertex set $V(G) = \{0, 1, \ldots, n-1\}$. Its adjacency matrix $\mathbb{A}$ is a symmetric $n \times n$ matrix with entries $a_{i,j}$, where $0 \le i, j \le n-1$, such that $a_{i,j} = 1$ if $\{i, j\}$ is an edge of G and $a_{i,j} = 0$ otherwise. A graph G is a *nut graph* if $\mathbb{A}$ has eigenvalue 0 and no eigenvector has zero entries. As a consequence, the eigenspace corresponding to the eigenvalue 0 is 1-dimensional. Observe that if the eigenspace corresponding to 0 has dimension greater than one, then there exists an eigenvector containing entry 0 that is different from $\mathbf{0} = (0, 0, \ldots, 0)^T$. For an introductory treatment of spectral graph theory, which links graphs to linear algebra, see e.g. [3, 4, 7].

Nut graphs have been studied in [6, 9, 11, 12, 16, 17, 18, 19, 20, 22], see also the webpage `https://hog.grinvin.org/Nuts` within the House of Graphs [2, 5]. Recently, this concept was extended to signed graphs [1]. Nut graphs have chemical applications, see e.g. [9, 8, 21]. However, in the present paper we consider 12-regular graphs, so our motivation is purely mathematical.

In [22], Gutman and Sciriha showed that the smallest non-trivial nut graph has order 7. In [10], Fowler *et al.* determined all nut graphs on up to 10 vertices and all chemical nut graphs on up to 16 vertices. The smallest order for which a regular nut graph exists is 8; see also [9]. In [9], Fowler *et al.* presented the following question.

If there exists a d-regular graph of order n, then we say that the order n is *admissible* regarding the degree d. Obviously, if d is even then every $n \ge d + 1$ is admissible. If d is odd then every even $n \ge d + 1$ is admissible.

Question 1.1. Is it true that for each degree $d \ge 3$ there are only finitely many admissible orders n such that there does not exist a d-regular nut graph of order n?

In the attempt to answer Question 1.1, the 'Fowler Construction' played an important role; see also [11]. This construction implies the following theorem.

Theorem 1.2. *Let G be a nut graph on n vertices and let u be a vertex of G of degree d. Then there exists a nut graph of order $n + 2d$ that is obtained from G by adding $2d$ new vertices and rearranging the edges in a certain way. In the newly obtained nut graph the degrees of the new vertices are d and the degrees of the original vertices are not changed.*

Obviously, if G is a d-regular graph of order n, then the new graph is d-regular of order $n + 2d$. Hence, to answer Question 1.1 positively for specific degree d, it suffices to find d-regular graphs for $2d$ consecutive orders. In [11] ($d = 3, 4$) and [9] ($5 \le d \le 11$), the authors found all pairs (d, n), such that $d \le 11$ and there exists a d-regular nut graph of order n. In the present paper, we extend this result to $d = 12$. We prove the following statement.

Theorem 1.3. *There exists a 12-regular nut graph of order n if and only if $n \ge 16$.*

To prove the 'positive part' of Theorem 1.3, it suffices to find 12-regular nut graphs of orders $n \in \{16, 17, \ldots, 39\}$. We present these graphs in the following section. For odd orders there is not much to say; we did a computer search and thus we provide a list of graphs that we found. However, for even orders we can say more.

E-mail addresses: nino.basic@famnit.upr.si (Nino Bašić), knor@math.sk (Martin Knor), skrekovski@gmail.com (Riste Škrekovski)

A graph G is called *vertex-transitive* if all vertices are equivalent under the action of the automorphism group $\mathrm{Aut}(G)$. In other words, for each pair of vertices $u, v \in V(G)$ there exist an automorphism $\alpha \in \mathrm{Aut}(G)$ such that $\alpha(u) = v$. In [9], the following necessary condition for a vertex-transitive nut graph was given.

Theorem 1.4. *Let G be a vertex-transitive nut graph of degree d on n vertices. Then n and d satisfy the following conditions. Either*

(1) $d \equiv 0 \pmod 4$, $n \equiv 0 \pmod 2$ *and* $n \geq d + 4$, *or*

(2) $d \equiv 0 \pmod 2$, $n \equiv 0 \pmod 4$ *and* $n \geq d + 6$.

The existence of vertex-transitive nut graphs is interesting in its own right, see [9, Question 9]. For our research it is important that, by Theorem 1.4, there may exist vertex-transitive 12-regular graphs of even orders $n \geq 16$. We found such graphs among circulant graphs.

2 Results

We start with the 'negative part' of Theorem 1.3. There is only one 12-regular graph of order 13, namely the complete graph K_{13}, and it is not a nut graph. The unique 12-regular graph of order 14 is obtained by removing a matching from K_{14}, and again, this graph is not a nut graph. Finally, there are 17 graphs of order 15 which are 12-regular. They are obtained by removing a 2-factor from K_{15}. Using the SageMath software [23], we analysed all such graphs and concluded that none of them is a nut graph.

Now we turn our attention to the 'positive part' of Theorem 1.3. We start with more general results for even orders. The following lemma is in fact implied in the text preceding Proposition 1 in [11]. We present it here in a slightly more general setting together with its short proof.

Lemma 2.1. *Let G be a d-regular graph on n vertices such that its adjacency matrix $\mathbb{A}$ is singular. Then for every eigenvector $\mathbf{c} = (c_0, c_1, \ldots, c_{n-1})^T$ corresponding to eigenvalue 0, we have*

$$\sum_{i=0}^{n-1} c_i = 0.$$

Proof. In every d-regular graph, the eigenvector $\mathbf{1} = (1, 1, \ldots, 1)$ corresponds to the eigenvalue d. Since eigenspaces are mutually orthogonal, we have $\mathbf{c} \cdot \mathbf{1} = 0$. □

Let $V = \{0, 1, \ldots, n-1\}$ and let $1 \leq a_1 < a_2 < \cdots < a_t \leq \frac{n}{2}$. By $C(n, \{a_1, a_2, \ldots, a_t\})$ we denote a graph on the vertex set V in which two vertices $i, j \in V$ are adjacent if and only if $|i - j| = a_k$, where $1 \leq k \leq t$. The graph $C(n, \{a_1, a_2, \ldots, a_t\})$ is called a *circulant graph* and it is regular. Its degree is $2t - 1$ if $a_t = \frac{n}{2}$ and $2t$ otherwise. In fact, circulant graphs are vertex-transitive since $\varphi\colon i \to i + 1$ is an automorphism of $C(n, \{a_1, a_2, \ldots, a_t\})$ (the addition is modulo n).

Circulant graphs are easy to describe and easy to handle. Therefore, it would be nice if there were many nut graphs among them. We prove one positive and one negative result about circulant graphs. We start with the following lemma.

Lemma 2.2. *Let $G = C(n, \{a_1, a_2, \ldots, a_t\})$ be a circulant nut graph, and let $\mathbb{A}$ be its adjacency matrix. Then $(1, -1, 1, -1, \ldots)^T$ is an eigenvector corresponding to eigenvalue 0.*

Proof. We use the well-known fact that if $\mathbf{b}$ and $\mathbf{c}$ are eigenvectors corresponding to eigenvalue λ, then $\mathbf{b}+\mathbf{c}$ is also an eigenvector corresponding to eigenvalue λ.

Let $\mathbf{b} = (b_0, b_1, \dots, b_{n-1})^T$ be an eigenvector corresponding to 0. Denote $b_0 = p$ and $b_1 = q$. Since $\varphi\colon i \to 2-i$ is an automorphism of G (the addition being modulo n), there is an eigenvector $\mathbf{c} = (c_0, c_1, \dots, c_{n-1})^T$ such that $c_{2-i} = -b_i$, $0 \le i \le n-1$. Then $c_1 = -b_1 = -q$ and $c_2 = -b_0 = -p$. Since $b_1 + c_1 = 0$ and $\mathbf{b}+\mathbf{c}$ is an eigenvector, we must have $\mathbf{b}+\mathbf{c} = \mathbf{0}$ because G is a nut graph. Hence, $b_2 + c_2 = 0$ and therefore $b_2 = p$. Now repeating the process we get $\mathbf{b} = (p, q, p, q, \dots)$. Observe that n is even by Theorem 1.4. Thus, by Lemma 2.1, we have $q = -p$ and so $(1, -1, 1, -1, \dots)$ is an eigenvector corresponding to eigenvalue 0. □

Our negative result covers all circulant graphs of degree $d \equiv 2 \pmod 4$.

Theorem 2.3. *There is no circulant nut graph of degree d if $d \equiv 2 \pmod 4$.*

Proof. Let $d \equiv 2 \pmod 4$. Denote $t = \frac{d}{2}$. Observe that t is an odd number. By way of contradiction, assume that $G = C(n, \{a_1, a_2, \dots, a_t\})$ is a circulant nut graph. Then n is even by Theorem 1.4. Let $\mathbb{A} = (\mathbf{a}_0, \mathbf{a}_1, \dots, \mathbf{a}_{n-1})^T$ be the adjacency matrix of G. By Lemma 2.2, $\mathbf{c} = (1, -1, 1, -1, \dots)^T$ is an eigenvector corresponding to eigenvalue 0, so that $\mathbb{A}\mathbf{c} = \mathbf{0}$, and in particular $\mathbf{a}_0\mathbf{c} = 0$. However,

$$\mathbf{a}_0\mathbf{c} = c_{a_1} + c_{a_2} + \dots + c_{a_t} + c_{n-a_1} + c_{n-a_2} + \dots + c_{n-a_t}.$$

Since $c_{a_i} = c_{n-a_i}$ for every i, $1 \le i \le t$ (observe that the difference between indices a_i and $n - a_i$ is even), we have $\mathbf{a}_0\mathbf{c} = 2(c_{a_1} + c_{a_2} + \dots + c_{a_t})$, which implies that $c_{a_1} + c_{a_2} + \dots + c_{a_t} = 0$. However, the sum of an odd number of odd numbers is odd, a contradiction. □

Now we prove the positive result. Notice that this result also partially resolves Question 9 from [9] about the existence of vertex-transitive nut graphs of order n and degree d.

Theorem 2.4. *Let $d \equiv 0 \pmod 4$ and let n be even. Then $C(n, \{1, 2, \dots, \frac{d}{2}\})$ is a nut graph if and only if $\frac{d}{2}+1$ is coprime to n and $\frac{d}{4}$ is coprime to $\frac{n}{2}$.*

Proof. Let $t = \frac{d}{2}$. Then t is even and the graph is $G = C(n, \{1, 2, \dots, t\})$.

Let $\mathbb{A}$ be the adjacency matrix of G. By Lemma 2.2, $\mathbf{b} = (1, -1, 1, -1, \dots)^T$ is an eigenvector of $\mathbb{A}$ corresponding to eigenvalue 0. Thus $\mathbb{A}\mathbf{b} = \mathbf{0}$. Our aim is to show that if $t+1$ is coprime to n and $\frac{t}{2}$ is coprime to $\frac{n}{2}$, then $\mathbb{A}\mathbf{c} = \mathbf{0}$ if and only if $\mathbf{c}$ is a multiple of $\mathbf{b}$.

So let $\mathbb{A}\mathbf{c} = \mathbf{0}$, where $\mathbf{c} = (c_0, c_1, \dots, c_{n-1})^T$. Let $\mathbb{A} = (\mathbf{a}_0, \mathbf{a}_1, \dots, \mathbf{a}_{n-1})^T$. Then

$$\mathbf{a}_t\mathbf{c} = c_0 + c_1 + \dots + c_{t-1} + c_{t+1} + c_{t+2} + \dots + c_{2t} = 0,$$
$$\mathbf{a}_{t+1}\mathbf{c} = c_1 + c_2 + \dots + c_t + c_{t+2} + c_{t+3} + \dots + c_{2t+1} = 0.$$

Subtracting the two equations we get

$$\mathbf{a}_t\mathbf{c} - \mathbf{a}_{t+1}\mathbf{c} = c_0 - c_t + c_{t+1} - c_{2t+1} = 0,$$

and analogously

$$\mathbf{a}_{2t+1}\mathbf{c} - \mathbf{a}_{2t+2}\mathbf{c} = c_{t+1} - c_{2t+1} + c_{2t+2} - c_{3t+2} = 0.$$

This gives

$$c_0 - c_t = c_{2t+2} - c_{3t+2},$$

and analogously

$$c_{2t+2} - c_{3t+2} = c_{4t+4} - c_{5t+4},$$
$$c_{4t+4} - c_{5t+4} = c_{6t+6} - c_{7t+5}, \qquad \text{etc.}$$

So if the odd number $t+1$ is coprime to the even number n, we get

$$c_0 - c_t = c_{2(t+1)} - c_{t+2(t+1)} = \cdots = c_2 - c_{t+2},$$

which gives

$$c_2 - c_0 = c_{t+2} - c_t,$$

and analogously we get

$$c_{t+2} - c_t = c_{2t+2} - c_{2t},$$
$$c_{2t+2} - c_{2t} = c_{3t+2} - c_{3t}, \qquad \text{etc.}$$

Here, t and n are both even. But if $\frac{t}{2}$ is coprime to $\frac{n}{2}$ then

$$c_2 - c_0 = c_{t+2} - c_t = \cdots = c_4 - c_2.$$

Hence,

$$c_2 - c_0 = c_4 - c_2 = c_6 - c_4 = \cdots$$

Now, if $c_2 > c_0$ then $c_0 < c_2 < c_4 < \cdots < c_0$, a contradiction. Analogously, if $c_2 < c_0$ then $c_0 > c_2 > c_4 > \cdots > c_0$, a contradiction. So $c_0 = c_2 = \cdots = c_{n-2}$ and analogously $c_1 = c_3 = \cdots = c_{n-1}$. Hence if $c_0 = p$, then $\mathbf{c} = (p, -p, p, -p, \dots)$ by Lemma 2.1, and the eigenspace corresponding to eigenvalue 0 is 1-dimensional.

Now suppose that $t+1$ is not coprime to n. Set $\mathbf{b} = \mathbf{0}$. We will change some entries of $\mathbf{b}$. Since $t+1$ is odd, there is an even k such that $(t+1)k \equiv 0 \pmod{n}$ and $1 \le k < n$. Set

$$b_0 = 1, \quad b_{t+1} = -1, \quad b_{2(t+1)} = 1, \quad b_{3(t+1)} = -1, \quad \dots,$$

where the indices are modulo n. We have changed k entries of $\mathbf{b}$ and since k is even, the last changed entry has value -1. Thus some entries of $\mathbf{b}$ remained 0's and nevertheless $\mathbb{A}\mathbf{b} = \mathbf{0}$, since if j-th entry of $\mathbf{a}_i$ is 1, then either $(j + (t+1))$-th or $(j - (t+1))$-th (modulo n) entry of $\mathbf{a}_i$ is also 1 (while the other is 0). Hence, G is not a nut graph in this case.

Finally, suppose that $\frac{t}{2}$ is not coprime to $\frac{n}{2}$. Then there exists a number k such that $k \mid \frac{t}{2}$, $k \mid \frac{n}{2}$ and $k > 1$. Again, set $\mathbf{b} = \mathbf{0}$. We will change some entries of $\mathbf{b}$. Set

$$b_0 = b_2 = b_4 = \cdots = b_{2(k-2)} = 1 \qquad \text{and} \qquad b_{2(k-1)} = -(k-1),$$

and repeat this pattern for all even indices of $\mathbf{b}$. Since $k \mid \frac{n}{2}$, this pattern is repeated exactly $\frac{n}{2k}$ times. And since every $\mathbf{a}_i$ contains two disjoint sets of t consecutive 1's, we have $\mathbb{A}\mathbf{b} = \mathbf{0}$. But half of the entries of $\mathbf{b}$ are 0's and therefore G is not a nut graph. □

Observe that the only requirement for n in Theorem 2.4 is that n is even and $n > d$. However, if $n = d + 2$ then $\frac{d}{2} + 1$ is not coprime to n, and so $n \geq d + 4$. Hence, by Theorem 2.4, for $d = 12$ the following circulant graphs are nut graphs:

$C(16, \{1, 2, 3, 4, 5, 6\})$, $C(20, \{1, 2, 3, 4, 5, 6\})$, $C(22, \{1, 2, 3, 4, 5, 6\})$, $C(26, \{1, 2, 3, 4, 5, 6\})$, $C(32, \{1, 2, 3, 4, 5, 6\})$, $C(34, \{1, 2, 3, 4, 5, 6\})$, and $C(38, \{1, 2, 3, 4, 5, 6\})$.

Using the computer [23] we found the following graphs that are nut graphs:

$C(18, \{1, 2, 3, 4, 5, 8\})$, $C(24, \{1, 2, 3, 4, 5, 8\})$, $C(28, \{1, 2, 3, 4, 5, 10\})$, $C(30, \{1, 2, 3, 4, 5, 8\})$, and $C(36, \{1, 2, 3, 4, 5, 8\})$.

3 Concluding remarks and further work

From the very beginning of our work on this paper, the nut circulant graphs were continuously present, which fact motivates us explicitly to pose here the following problem.

Problem 3.1. Find which circulant graphs are nut graphs.

By the arguments in this paper, any circulant nut graph must satisfy the conditions of Theorem 1.4(1), i.e. the order n is even, the degree d is divisible by 4, and $n \geq d + 4$. We believe that for any such pairs n and d, there exists a circulant nut graph.

Conjecture 3.2. *For every d, where $d \equiv 0 \pmod 4$, and for every even n, $n \geq d + 4$, there exists a circulant nut graph $C(n, \{a_1, a_2, \ldots, a_{d/2}\})$ of degree d.*

And, as a very particular case of the above conjecture, by restricting to 12-regular graphs, we also propose.

Conjecture 3.3. *For every even n, $n \geq 16$, there exists a circulant nut graph $C(n, \{a_1, a_2, \ldots, a_6\})$ of degree 12.*

By Theorem 1.4, if n is odd then there is no vertex-transitive nut graph of order n and degree 12. In this case all graphs were found by a computer search. If G is a regular graph that contains edges u_1v_1 and u_2v_2 but does not contain edges u_1v_2, u_2v_1, then *rewiring* (i.e. removing edges u_1v_1, u_2v_2 and adding edges u_1v_2, u_2v_1; it is also known as a *Ryser switch* [15]) yields another regular graph. Our approach was to start with a 'random' 12-regular graph of odd order and perform a sequence of rewirings. In this way all graphs in the Appendix were obtained. For instance, the graph on 21 vertices, whose kernel eigenvector contains only values 1 and -2, was obtained from $C(21, \{1, 2, 3, 4, 5, 6\})$ by removing the edges $(0, 16)$ and $(2, 7)$ and adding the edges $(0, 7)$ and $(2, 16)$.

Note that kernel eigenvectors of all graphs in the Appendix on $n = 3k$ vertices (for $k = 7, 9, 11, 13$) contain only values 1 and -2. All those graphs have a special structure. Let $V = V_1 \cup V_{-2}$ be the partition of vertices with respect to the kernel eigenvector entry. In each case, the graph induced by V_{-2} is isomorphic to a graph that can be obtained from $C(k, \{1, 2\})$ by at most one rewiring, while the graph induced by V_1 is isomorphic to a graph that can be obtained from $C(2k, \{1, 2, 3, 4\})$ by at most one rewiring. Moreover, let $\mathrm{BiC}(n, S)$ be the graph with the vertex set $V = \{v_0, \ldots, v_{n-1}, u_0, \ldots, u_{n-1}\}$ and the edge set $E = \{v_i u_{(i+s) \bmod n} : 0 \leq i < n, s \in S\}$. This graph is a special kind of bicirculant (see [13, 14] and references cited therein). The set V_1 can be partitioned into two subsets $V_1 = V_1' \cup V_1''$, $|V_1'| = |V_1''|$, such that the graph induced by edges from V_{-2} to V_1' is isomorphic to a graph that can be obtained from $\mathrm{BiC}(k, \{0, 1, 2, 3\})$ by at most one rewiring. Similarly, the graph induced by edges from V_{-2} to V_1'' is also isomorphic to a graph that can be obtained from $\mathrm{BiC}(k, \{0, 1, 2, 3\})$ by at most one rewiring.

ORCID iDs

Nino Bašić https://orcid.org/0000-0002-6555-8668
Martin Knor https://orcid.org/0000-0003-3555-3994
Riste Škrekovski https://orcid.org/0000-0001-6851-3214

References

[1] N. Bašić, P. W. Fowler, T. Pisanski and I. Sciriha, On singular signed graphs with nullspace spanned by a full vector: Signed nut graphs, 2020, arXiv:2009.09018 [math.CO].

[2] G. Brinkmann, K. Coolsaet, J. Goedgebeur and H. Mélot, House of Graphs: A database of interesting graphs, *Discrete Appl. Math.* **161** (2013), 311–314, doi:10.1016/j.dam.2012.07.018.

[3] A. E. Brouwer and W. H. Haemers, *Spectra of Graphs*, Universitext, Springer, New York, 2012, doi:10.1007/978-1-4614-1939-6.

[4] F. R. K. Chung, *Spectral Graph Theory*, volume 92 of *CBMS Regional Conference Series in Mathematics*, American Mathematical Society, Providence, RI, 1997.

[5] K. Coolsaet, P. W. Fowler and J. Goedgebeur, Nut graphs, homepage of Nutgen, http://caagt.ugent.be/nutgen/.

[6] K. Coolsaet, P. W. Fowler and J. Goedgebeur, Generation and properties of nut graphs, *MATCH Commun. Math. Comput. Chem.* **80** (2018), 423–444.

[7] D. M. Cvetković, M. Doob and H. Sachs, *Spectra of Graphs: Theory and Applications*, Johann Ambrosius Barth, Heidelberg, 3rd edition, 1995.

[8] P. W. Fowler, N. Bašić and T. Pisanski, Charting the space of chemical nut graphs, *MATCH Commun. Math. Comput. Chem.* **86** (2021), 519–538.

[9] P. W. Fowler, J. B. Gauci, J. Goedgebeur, T. Pisanski and I. Sciriha, Existence of regular nut graphs for degree at most 11, *Discuss. Math. Graph Theory* **40** (2020), 533–557, doi:10.7151/dmgt.2283.

[10] P. W. Fowler, B. T. Pickup, T. Z. Todorova, M. Borg and I. Sciriha, Omni-conducting and omni-insulating molecules, *J. Chem. Phys.* **140** (2014), Article No. 054115, doi:10.1063/1.4863559.

[11] J. B. Gauci, T. Pisanski and I. Sciriha, Existence of regular nut graphs and the Fowler construction, *Appl. Anal. Discrete Math.* (2021), doi:10.2298/aadm190517028g.

[12] I. Gutman and I. Sciriha, Graphs with maximum singularity, *Graph Theory Notes N. Y.* **30** (1996), 17–20.

[13] I. Kovács, B. Kuzman, A. Malnič and S. Wilson, Characterization of edge-transitive 4-valent bicirculants, *J. Graph Theory* **69** (2012), 441–463, doi:10.1002/jgt.20594.

[14] A. Malnič, D. Marušič, P. Šparl and B. Frelih, Symmetry structure of bicirculants, *Discrete Math.* **307** (2007), 409–414, doi:10.1016/j.disc.2005.09.033.

[15] H. J. Ryser, Combinatorial properties of matrices of zeros and ones, *Canadian J. Math.* **9** (1957), 371–377, doi:10.4153/cjm-1957-044-3.

[16] I. Sciriha, On the construction of graphs of nullity one, *Discrete Math.* **181** (1998), 193–211, doi:10.1016/s0012-365x(97)00036-8.

[17] I. Sciriha, A characterization of singular graphs, *Electron. J. Linear Algebra* **16** (2007), 451–462, doi:10.13001/1081-3810.1215.

[18] I. Sciriha, Coalesced and embedded nut graphs in singular graphs, *Ars Math. Contemp.* **1** (2008), 20–31, doi:10.26493/1855-3974.20.7cc.

[19] I. Sciriha, Graphs with a common eigenvalue deck, *Linear Algebra Appl.* **430** (2009), 78–85, doi:10.1016/j.laa.2008.06.033.

[20] I. Sciriha, Maximal core size in singular graphs, *Ars Math. Contemp.* **2** (2009), 217–229, doi:10.26493/1855-3974.115.891.

[21] I. Sciriha and P. W. Fowler, Nonbonding orbitals in fullerenes: Nuts and cores in singular polyhedral graphs, *J. Chem. Inf. Model.* **47** (2007), 1763–1775, doi:10.1021/ci700097j.

[22] I. Sciriha and I. Gutman, Nut graphs: Maximally extending cores, *Util. Math.* **54** (1998), 257–272.

[23] The Sage Developers, SageMath, the Sage Mathematics Software System (Version 9.2), 2020, http://www.sagemath.org/.

Appendix A 12-regular nut graphs of odd orders

Here, we list one 12-regular nut graph of odd order n for each $n \in \{17, 19, \ldots, 39\}$. Each graph is given in the adjacency-lists (of neighbours of each vertex) representation, formatted as a Python dictionary. We also give the corresponding kernel eigenvector **c** as a list of integer entries.

Order $n = 17$.
{0: [1, 2, 3, 4, 5, 8, 9, 10, 11, 12, 15, 16], 1: [0, 2, 3, 4, 6, 7, 8, 9, 10, 11, 15, 16], 2: [0, 1, 4, 5, 6, 7, 8, 9, 10, 11, 13, 15], 3: [0, 1, 4, 6, 7, 8, 9, 11, 12, 14, 15, 16], 4: [0, 1, 2, 3, 5, 6, 8, 9, 10, 11, 13, 16], 5: [0, 2, 4, 6, 7, 8, 9, 10, 12, 13, 14, 15], 6: [1, 2, 3, 4, 5, 7, 8, 9, 12, 13, 14, 15], 7: [1, 2, 3, 5, 6, 8, 10, 11, 12, 13, 14, 16], 8: [0, 1, 2, 3, 4, 5, 6, 7, 10, 11, 13, 14], 9: [0, 1, 2, 3, 4, 5, 6, 10, 12, 13, 14, 16], 10: [0, 1, 2, 4, 5, 7, 8, 9, 12, 14, 15, 16], 11: [0, 1, 2, 3, 4, 7, 8, 12, 13, 14, 15, 16], 12: [0, 3, 5, 6, 7, 9, 10, 11, 13, 14, 15, 16], 13: [2, 4, 5, 6, 7, 8, 9, 11, 12, 14, 15, 16], 14: [3, 5, 6, 7, 8, 9, 10, 11, 12, 13, 15, 16], 15: [0, 1, 2, 3, 5, 6, 10, 11, 12, 13, 14, 16], 16: [0, 1, 3, 4, 7, 9, 10, 11, 12, 13, 14, 15]}

$\mathbf{c} = [3, -3, -2, 2, 1, 2, -1, -2, 3, -1, -1, 1, 1, -1, 1, -1, -2]$

Order $n = 19$.
{0: [1, 2, 5, 7, 9, 10, 11, 12, 13, 14, 16, 18], 1: [0, 3, 5, 6, 7, 10, 12, 13, 14, 15, 17, 18], 2: [0, 4, 6, 7, 8, 9, 10, 11, 12, 16, 17, 18], 3: [1, 6, 7, 8, 10, 11, 12, 13, 14, 16, 17, 18], 4: [2, 5, 6, 7, 8, 11, 12, 13, 14, 15, 17, 18], 5: [0, 1, 4, 7, 8, 9, 11, 12, 13, 14, 15, 17], 6: [1, 2, 3, 4, 7, 8, 9, 10, 14, 15, 16, 17], 7: [0, 1, 2, 3, 4, 5, 6, 8, 9, 11, 15, 16], 8: [2, 3, 4, 5, 6, 7, 9, 11, 14, 15, 17, 18], 9: [0, 2, 5, 6, 7, 8, 10, 11, 12, 13, 16, 17], 10: [0, 1, 2, 3, 6, 9, 11, 12, 13, 14, 16, 18], 11: [0, 2, 3, 4, 5, 7, 8, 9, 10, 16, 17, 18], 12: [0, 1, 2, 3, 4, 5, 9, 10, 13, 14, 15, 16], 13: [0, 1, 3, 4, 5, 9, 10, 12, 14, 15, 16, 17], 14: [0, 1, 3, 4, 5, 6, 8, 10, 12, 13, 15, 18], 15: [1, 4, 5, 6, 7, 8, 12, 13, 14, 16, 17, 18], 16: [0, 2, 3, 6, 7, 9, 10, 11, 12, 13, 15, 18], 17: [1, 2, 3, 4, 5, 6, 8, 9, 11, 13, 15, 18], 18: [0, 1, 2, 3, 4, 8, 10, 11, 14, 15, 16, 17]}

$\mathbf{c} = [5, 10, 6, -10, -3, -1, 4, -1, -5, 1, 1, -5, -4, -3, -4, 2, -4, 7, 4]$

Order $n = 21$.
{0: [1, 2, 3, 4, 5, 6, 7, 15, 17, 18, 19, 20], 1: [0, 2, 3, 4, 5, 6, 7, 16, 17, 18, 19, 20], 2: [0, 1, 3, 4, 5, 6, 8, 16, 17, 18, 19, 20], 3: [0, 1, 2, 4, 5, 6, 7, 8, 9, 18, 19, 20], 4: [0, 1, 2, 3, 5, 6, 7, 8, 9, 10, 19, 20], 5: [0, 1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 20], 6: [0, 1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 12], 7: [0, 1, 3, 4, 5, 6, 8, 9, 10, 11, 12, 13], 8: [2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 13, 14], 9: [3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15], 10: [4, 5, 6, 7, 8, 9, 11, 12, 13, 14, 15, 16], 11: [5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17], 12: [6, 7, 8, 9, 10, 11, 13, 14, 15, 16, 17, 18], 13: [7, 8, 9, 10, 11, 12, 14, 15, 16, 17, 18, 19], 14: [8, 9, 10, 11, 12, 13, 15, 16, 17, 18, 19, 20], 15: [0, 9, 10, 11, 12, 13, 14, 16, 17, 18, 19, 20], 16: [1, 2, 10, 11, 12, 13, 14, 15, 17, 18, 19, 20], 17: [0, 1, 2, 11, 12, 13, 14, 15, 16, 18, 19, 20], 18: [0, 1, 2, 3, 12, 13, 14, 15, 16, 17, 19, 20], 19: [0, 1, 2, 3, 4, 13, 14, 15, 16, 17, 18, 20], 20: [0, 1, 2, 3, 4, 5, 14, 15, 16, 17, 18, 19]}

$\mathbf{c} = [1, -2, 1, 1, -2, 1, 1, -2, 1, 1, -2, 1, 1, -2, 1, 1, -2, 1, 1, -2, 1]$

Order $n = 23$.
{0: [1, 2, 4, 6, 7, 8, 10, 11, 13, 19, 20, 21], 1: [0, 4, 5, 6, 7, 9, 11, 13, 16, 17, 20, 22], 2: [0, 3, 4, 6, 8, 11, 12, 13, 16, 19, 20, 21], 3: [2, 4, 5, 8, 9, 10, 12, 13, 14, 16, 17, 18], 4: [0, 1, 2, 3, 6, 7, 8, 14, 15, 16, 21, 22], 5: [1, 3, 7, 10, 11, 12, 14, 15, 17, 18, 19, 20], 6: [0, 1, 2, 4, 11, 12, 14, 17, 18, 19, 20, 22], 7: [0, 1, 4, 5, 10, 11, 12, 16, 18, 19, 21, 22], 8: [0, 2, 3, 4, 9, 10, 12, 13, 15, 16, 21, 22], 9: [1, 3,

8, 10, 11, 13, 14, 15, 18, 19, 21, 22], 10: [0, 3, 5, 7, 8, 9, 11, 13, 17, 18, 19, 21], 11: [0, 1, 2, 5, 6, 7, 9, 10, 13, 14, 15, 20], 12: [2, 3, 5, 6, 7, 8, 13, 14, 17, 19, 20, 22], 13: [0, 1, 2, 3, 8, 9, 10, 11, 12, 14, 15, 19], 14: [3, 4, 5, 6, 9, 11, 12, 13, 15, 16, 17, 20], 15: [4, 5, 8, 9, 11, 13, 14, 17, 18, 20, 21, 22], 16: [1, 2, 3, 4, 7, 8, 14, 17, 18, 20, 21, 22], 17: [1, 3, 5, 6, 10, 12, 14, 15, 16, 19, 20, 21], 18: [3, 5, 6, 7, 9, 10, 15, 16, 19, 20, 21, 22], 19: [0, 2, 5, 6, 7, 9, 10, 12, 13, 17, 18, 22], 20: [0, 1, 2, 5, 6, 11, 12, 14, 15, 16, 17, 18], 21: [0, 2, 4, 7, 8, 9, 10, 15, 16, 17, 18, 22], 22: [1, 4, 6, 7, 8, 9, 12, 15, 16, 18, 19, 21]}

$\mathbf{c} = [6, -24, -7, 13, 39, 1, 27, 4, -18, -4, 10, 3, -14, -14, 28, 1, -22, -2, 3, 6, -28, 2, -10]$

Order $n = 25$.

{0: [3, 4, 5, 7, 9, 10, 12, 13, 17, 19, 22, 23], 1: [2, 3, 5, 11, 12, 15, 16, 18, 19, 20, 21, 23], 2: [1, 3, 4, 5, 10, 13, 14, 17, 20, 21, 23, 24], 3: [0, 1, 2, 5, 8, 10, 14, 16, 20, 21, 23, 24], 4: [0, 2, 6, 8, 9, 10, 11, 13, 18, 21, 23, 24], 5: [0, 1, 2, 3, 10, 13, 14, 17, 18, 19, 20, 24], 6: [4, 8, 9, 10, 11, 12, 14, 17, 19, 20, 21, 22], 7: [0, 8, 9, 11, 12, 15, 16, 18, 19, 22, 23, 24], 8: [3, 4, 6, 7, 9, 10, 11, 13, 17, 18, 22, 23], 9: [0, 4, 6, 7, 8, 10, 11, 12, 14, 15, 18, 21], 10: [0, 2, 3, 4, 5, 6, 8, 9, 15, 16, 17, 18], 11: [1, 4, 6, 7, 8, 9, 12, 13, 14, 17, 19, 20], 12: [0, 1, 6, 7, 9, 11, 13, 14, 15, 18, 21, 22], 13: [0, 2, 4, 5, 8, 11, 12, 16, 20, 21, 22, 23], 14: [2, 3, 5, 6, 9, 11, 12, 15, 16, 17, 19, 22], 15: [1, 7, 9, 10, 12, 14, 16, 17, 19, 20, 22, 24], 16: [1, 3, 7, 10, 13, 14, 15, 17, 18, 19, 20, 24], 17: [0, 2, 5, 6, 8, 10, 11, 14, 15, 16, 21, 23], 18: [1, 4, 5, 7, 8, 9, 10, 12, 16, 21, 22, 24], 19: [0, 1, 5, 6, 7, 11, 14, 15, 16, 21, 22, 24], 20: [1, 2, 3, 5, 6, 11, 13, 15, 16, 22, 23, 24], 21: [1, 2, 3, 4, 6, 9, 12, 13, 17, 18, 19, 23], 22: [0, 6, 7, 8, 12, 13, 14, 15, 18, 19, 20, 24], 23: [0, 1, 2, 3, 4, 7, 8, 13, 17, 20, 21, 24], 24: [2, 3, 4, 5, 7, 15, 16, 18, 19, 20, 22, 23]}

$\mathbf{c} = [29, 20, -31, 7, 5, -13, 32, -19, -12, 1, 31, -12, -8, -6, -49, 17, 3, -17, -21, 20, 33, 7, 1, -2, -16]$

Order $n = 27$.

{0: [2, 3, 4, 5, 6, 7, 21, 22, 23, 24, 25, 26], 1: [2, 3, 4, 5, 6, 7, 8, 22, 23, 24, 25, 26], 2: [0, 1, 3, 4, 5, 6, 7, 8, 23, 24, 25, 26], 3: [0, 1, 2, 4, 5, 6, 7, 8, 9, 24, 25, 26], 4: [0, 1, 2, 3, 5, 6, 7, 8, 9, 10, 25, 26], 5: [0, 1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 26], 6: [0, 1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 12], 7: [0, 1, 2, 3, 4, 5, 6, 9, 10, 11, 12, 13], 8: [1, 2, 3, 4, 5, 6, 9, 10, 11, 12, 13, 14], 9: [3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15], 10: [4, 5, 6, 7, 8, 9, 11, 12, 13, 14, 15, 16], 11: [5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17], 12: [6, 7, 8, 9, 10, 11, 13, 14, 15, 16, 17, 18], 13: [7, 8, 9, 10, 11, 12, 14, 15, 16, 17, 18, 19], 14: [8, 9, 10, 11, 12, 13, 15, 16, 17, 18, 19, 20], 15: [9, 10, 11, 12, 13, 14, 16, 17, 18, 19, 20, 21], 16: [10, 11, 12, 13, 14, 15, 17, 18, 19, 20, 21, 22], 17: [11, 12, 13, 14, 15, 16, 18, 19, 20, 21, 22, 23], 18: [12, 13, 14, 15, 16, 17, 19, 20, 21, 22, 23, 24], 19: [13, 14, 15, 16, 17, 18, 20, 21, 22, 23, 24, 25], 20: [14, 15, 16, 17, 18, 19, 21, 22, 23, 24, 25, 26], 21: [0, 15, 16, 17, 18, 19, 20, 22, 23, 24, 25, 26], 22: [0, 1, 16, 17, 18, 19, 20, 21, 23, 24, 25, 26], 23: [0, 1, 2, 17, 18, 19, 20, 21, 22, 24, 25, 26], 24: [0, 1, 2, 3, 18, 19, 20, 21, 22, 23, 25, 26], 25: [0, 1, 2, 3, 4, 19, 20, 21, 22, 23, 24, 26], 26: [0, 1, 2, 3, 4, 5, 20, 21, 22, 23, 24, 25]}

$\mathbf{c} = [1, -2, 1, 1, -2, 1, 1, -2, 1, 1, -2, 1, 1, -2, 1, 1, -2, 1, 1, -2, 1, 1, -2, 1, 1, -2, 1]$

Order $n = 29$.

{0: [1, 3, 5, 6, 9, 10, 11, 13, 14, 19, 26, 28], 1: [0, 2, 4, 5, 11, 16, 17, 18, 19, 21, 26, 27], 2: [1, 3, 8, 9, 10, 11, 13, 24, 25, 26, 27, 28], 3: [0, 2, 12, 13, 17, 20, 21, 23, 24, 25, 26, 27], 4: [1, 5, 6, 9, 11, 15, 16, 17, 20, 22, 23, 28], 5: [0, 1, 4, 7, 12, 15, 16, 19, 20, 22, 24, 25], 6: [0, 4, 7, 8, 9, 11, 15, 17, 18, 19, 21, 22], 7: [5, 6, 8, 11, 12, 13, 15, 16, 18, 20, 22, 24], 8: [2, 6, 7, 10, 12, 15, 19, 20, 21, 24, 26, 27], 9: [0, 2, 4, 6, 12, 14, 15, 20, 22, 23, 24, 27], 10: [0, 2, 8, 13, 16, 17, 18, 20, 21, 23, 25, 26], 11: [0, 1, 2, 4, 6, 7, 12, 16, 17, 19, 20, 23], 12: [3, 5, 7, 8, 9, 11, 14, 15, 18, 19, 21, 25], 13: [0, 2, 3, 7, 10, 14, 15, 21, 23, 25, 27, 28], 14: [0, 9, 12, 13, 15, 18, 22, 23, 24, 26, 27, 28], 15: [4, 5, 6, 7, 8, 9, 12, 13, 14, 18, 22, 27], 16: [1, 4, 5, 7, 10, 11, 18, 20, 21, 25, 27, 28], 17: [1, 3, 4, 6, 10, 11, 18, 19, 22, 24, 27, 28], 18: [1, 6, 7, 10, 12, 14, 15, 16, 17, 19, 23, 24], 19: [0, 1, 5, 6, 8, 11, 12, 17, 18, 23, 26, 27], 20: [3, 4, 5, 7, 8, 9, 10, 11, 16, 25, 26, 28], 21: [1, 3, 6, 8, 10, 12, 13, 16, 22, 23, 25, 26], 22: [4, 5, 6, 7, 9, 14, 15, 17, 21, 24, 25, 27], 23: [3, 4, 9, 10, 11, 13, 14, 18, 19, 21, 24, 28], 24: [2, 3, 5, 7, 8, 9, 14, 17, 18, 22, 23, 28], 25: [2, 3, 5, 10, 12, 13, 16, 20, 21, 22, 26, 28], 26: [0, 1, 2, 3, 8, 10, 14, 19, 20, 21, 25, 28], 27: [1, 2, 3, 8, 9, 13, 14, 15, 16, 17, 19, 22], 28: [0, 2, 4, 13, 14, 16, 17, 20, 23, 24, 25, 26]}

$\mathbf{c} = [1, 1, 37, -13, -20, -42, 21, -5, -36, 25, 5, 30, 41, -25, 21, -6, 6, 17, 34, -34, -14, -13, 7, -51, -16, 39, 5, -21, 6]$

Order $n = 31$.

{0: [5, 10, 12, 13, 17, 18, 21, 22, 24, 26, 27, 29], 1: [3, 6, 7, 8, 10, 14, 17, 20, 23, 25, 27, 30], 2: [4, 7, 9, 10, 18, 21, 22, 23, 24, 25, 27, 28], 3: [1, 4, 5, 11, 13, 16, 17, 18, 19, 24, 25, 29], 4: [2, 3, 5, 11, 12, 13, 18, 21, 25, 26, 28, 29], 5: [0, 3, 4, 6, 7, 9, 11, 14, 17, 25, 27, 29], 6: [1, 5, 8, 9, 11, 13, 18, 20, 22, 26, 29, 30], 7: [1, 2, 5, 9, 10, 12, 20, 24, 25, 26, 27, 30], 8: [1, 6, 9, 14, 15, 17, 18, 20, 21, 22, 23, 30], 9: [2, 5, 6, 7, 8, 12, 14, 15, 19, 24, 27, 28], 10: [0, 1, 2, 7, 12, 13, 15, 18, 19, 21, 24, 28], 11: [3, 4, 5, 6, 12, 15, 17, 20, 22, 23, 29, 30], 12: [0, 4, 7, 9, 10, 11, 14, 16, 18, 21, 27, 30], 13: [0, 3, 4, 6, 10, 16, 20, 23, 24, 25, 26, 27], 14: [1, 5, 8, 9, 12, 15, 17, 18, 19, 20, 22, 23], 15: [8, 9, 10, 11, 14, 17, 19, 20, 21, 27, 28, 30], 16: [3, 12, 13, 18, 19, 21, 22, 23, 24, 26, 28, 29], 17: [0, 1, 3, 5, 8, 11, 14, 15, 20, 22, 23, 29], 18: [0, 2, 3, 4, 6, 8, 10, 12, 14, 16, 24, 25], 19: [3, 9, 10, 14, 15, 16, 20, 21, 22, 23, 26, 28], 20: [1, 6, 7, 8, 11, 13, 14, 15, 17, 19, 24, 25], 21: [0, 2, 4, 8, 10, 12, 15, 16, 19, 25, 27, 29], 22: [0, 2, 6, 8, 11, 14, 16, 17, 19, 23, 28, 30], 23: [1, 2, 8, 11, 13, 14, 16, 17, 19, 22, 26, 28], 24: [0, 2, 3, 7, 9, 10, 13, 16, 18, 20, 28, 30], 25: [1, 2, 3, 4, 5, 7, 13, 18, 20, 21, 26, 29], 26: [0, 4, 6, 7, 13, 16, 19, 23, 25, 27, 29, 30], 27: [0, 1, 2, 5, 7, 9, 12, 13, 15, 21, 26, 30], 28: [2, 4, 9, 10, 15, 16, 19, 22, 23, 24, 29, 30], 29: [0, 3, 4, 5, 6, 11, 16, 17, 21, 25, 26, 28], 30: [1, 6, 7, 8, 11, 12, 15, 22, 24, 26, 27, 28]}

$\mathbf{c} = [1, 91, -39, 14, 39, 33, 75, -48, -37, 2, 146, -14, -13, 23, 20, 6, -84, -32, 27, 38, -93, -66, -43, 21, -79, -43, 18, -15, 59, 1, -8]$

Order $n = 33$.

{0: [1, 2, 3, 4, 5, 6, 27, 28, 29, 30, 31, 32], 1: [0, 2, 3, 4, 5, 6, 7, 11, 28, 29, 31, 32], 2: [0, 1, 3, 4, 5, 6, 7, 8, 29, 30, 31, 32], 3: [0, 1, 2, 4, 5, 6, 7, 8, 9, 30, 31, 32], 4: [0, 1, 2, 3, 5, 6, 7, 8, 9, 10, 31, 32], 5: [0, 1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 32], 6: [0, 1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 12], 7: [1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 13], 8: [2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 13, 14], 9: [3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15], 10: [4, 5, 6, 7, 8, 9, 12, 13, 14, 15, 16, 30], 11: [1, 5, 6, 7, 8, 9, 12, 13, 14, 15, 16, 17], 12: [6, 7, 8, 9, 10, 11, 13, 14, 15, 16, 17, 18], 13: [7, 8, 9, 10, 11, 12, 14, 15, 16, 17, 18, 19], 14: [8, 9, 10, 11, 12, 13, 15, 16, 17, 18, 19, 20], 15: [9, 10, 11, 12, 13, 14, 16, 17, 18, 19, 20, 21], 16: [10, 11, 12, 13, 14, 15, 17, 18, 19, 20, 21, 22], 17: [11, 12, 13, 14, 15, 16, 18, 19, 20, 21, 22, 23], 18: [12, 13, 14, 15, 16, 17, 19, 20, 21, 22, 23, 24], 19: [13, 14, 15, 16, 17, 18, 20, 21, 22, 23, 24, 25], 20: [14, 15, 16, 17, 18, 19, 21, 22, 23, 24, 25, 26], 21: [15, 16, 17, 18, 19, 20, 22, 23, 24, 25, 26, 27], 22: [16, 17, 18, 19, 20, 21, 23, 24, 25, 26, 27, 28], 23: [17, 18, 19, 20, 21, 22, 24, 25, 26, 27, 28, 29], 24: [18, 19, 20, 21, 22, 23, 25, 26, 27, 28, 29, 30], 25: [19, 20, 21, 22, 23, 24, 26, 27, 28, 29, 30, 31], 26: [20, 21, 22, 23, 24, 25, 27, 28, 29, 30, 31, 32], 27: [0, 21, 22, 23, 24, 25, 26, 28, 29, 30, 31, 32], 28: [0, 1, 22, 23, 24, 25, 26, 27, 29, 30, 31, 32], 29: [0, 1, 2, 23, 24, 25, 26, 27, 28, 30, 31, 32], 30: [0, 2, 3, 10, 24, 25, 26, 27, 28, 29, 31, 32], 31: [0, 1, 2, 3, 4, 25, 26, 27, 28, 29, 30, 32], 32: [0, 1, 2, 3, 4, 5, 26, 27, 28, 29, 30, 31]}

$\mathbf{c} = [1, -2, 1, 1, -2, 1, 1, -2, 1, 1, -2, 1, 1, -2, 1, 1, -2, 1, 1, -2, 1, 1, -2, 1, 1, -2, 1, 1, -2, 1, 1, -2, 1]$

Order $n = 35$.

{0: [1, 2, 3, 4, 5, 6, 29, 30, 31, 32, 33, 34], 1: [0, 2, 3, 4, 5, 6, 7, 30, 31, 32, 33, 34], 2: [0, 1, 3, 4, 5, 6, 7, 8, 15, 31, 32, 33], 3: [0, 1, 2, 4, 5, 6, 8, 9, 15, 32, 33, 34], 4: [0, 1, 2, 3, 5, 6, 7, 8, 9, 31, 33, 34], 5: [0, 1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 34], 6: [0, 1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 12], 7: [1, 2, 4, 5, 6, 8, 9, 10, 11, 12, 13, 21], 8: [2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 13, 14], 9: [3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15], 10: [5, 6, 7, 8, 9, 11, 12, 13, 14, 15, 16, 25], 11: [5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17], 12: [6, 7, 8, 9, 10, 11, 13, 14, 15, 16, 17, 18], 13: [7, 8, 9, 10, 11, 12, 14, 16, 17, 18, 19, 34], 14: [8, 9, 10, 11, 12, 13, 15, 16, 17, 18, 19, 20], 15: [2, 3, 9, 10, 11, 12, 14, 16, 17, 18, 19, 20], 16: [10, 11, 12, 13, 14, 15, 17, 18, 19, 20, 21, 22], 17: [11, 12, 13, 14, 15, 16, 18, 19, 20, 21, 22, 23], 18: [12, 13, 14, 15, 16, 17, 19, 20, 21, 22, 23, 24], 19: [13, 14, 15, 16, 17, 18, 20, 21, 22, 23, 24, 25], 20: [14, 15, 16, 17, 18, 19, 21, 22, 23, 24, 25, 26], 21: [7, 16, 17, 18, 19, 20, 22, 23, 24, 25, 26, 27], 22: [16, 17, 18, 19, 20, 21, 23, 24, 25, 26, 27, 28], 23: [17, 18, 19, 20, 21, 22, 24, 25, 26, 27, 28, 29], 24: [18, 19, 20, 21, 22, 23, 25, 26, 27, 28, 29, 30], 25: [10, 19, 20, 21, 22, 23, 24, 26, 27, 28, 29, 30], 26: [20, 21, 22, 23, 24, 25, 27, 28, 29, 30, 31, 32], 27: [21, 22, 23, 24, 25, 26, 28, 29, 30, 31, 32, 33], 28: [22, 23, 24, 25, 26, 27, 29, 30, 31, 32, 33, 34], 29: [0, 23, 24, 25, 26, 27, 28, 30, 31, 32, 33, 34], 30: [0, 1, 24, 25, 26, 27, 28, 29, 31, 32, 33, 34], 31: [0, 1, 2, 4, 26, 27, 28, 29, 30, 32, 33, 34], 32: [0, 1, 2, 3, 26, 27, 28, 29, 30, 31, 33, 34], 33: [0, 1, 2, 3, 4, 27, 28, 29, 30, 31, 32, 34], 34: [0, 1, 3, 4, 5, 13, 28, 29, 30, 31, 32, 33]}

$\mathbf{c} = [1, -1, -1, -3, 3, 2, -1, -1, 1, 1, -2, 2, -2, -1, 3, -1, -1, 2, -2, -2, 5, -1, -1, 1, -2, -2, 6, -3, -1, 1, -1, 5, -1, -4, 1]$

Order $n = 37$.

{0: [1, 2, 3, 4, 5, 6, 31, 32, 33, 34, 35, 36], 1: [0, 2, 3, 4, 5, 6, 7, 18, 22, 32, 33, 35], 2: [0, 1, 3, 4, 5, 6, 7, 8, 33, 34, 35, 36], 3: [0, 1, 2, 4, 5, 6, 7, 8, 9, 34, 35, 36], 4: [0, 1, 2, 3, 5, 6, 7, 8, 9, 10, 35, 36], 5: [0, 1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 36], 6: [0, 1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 12], 7: [1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 13], 8: [2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 13, 14], 9: [3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 15, 32], 10: [4, 5, 6, 7, 8, 9, 11, 12, 13, 14, 15, 16], 11: [5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17], 12: [6, 7, 8, 9, 10, 11, 13, 14, 15, 16, 17, 18], 13: [7, 8, 9, 10, 11, 12, 14, 15, 16, 17, 19, 36], 14: [8, 10, 11, 12, 13, 15, 16, 17, 18, 19, 20, 35], 15: [9, 10, 11, 12, 13, 14, 16, 17, 18, 19, 20, 21], 16: [10, 11, 12, 13, 14, 15, 17, 18, 19, 20, 21, 22], 17: [11, 12, 13, 14, 15, 16, 18, 19, 20, 21, 22, 23], 18: [1, 12, 14, 15, 16, 17, 19, 20, 21, 22, 23, 24], 19: [13, 14, 15, 16, 17, 18, 20, 21, 22, 23, 24, 25], 20: [14, 15, 16, 17, 18, 19, 21, 22, 23, 24, 25, 26], 21: [15, 16, 17, 18, 19, 20, 22, 23, 24, 25, 26, 27], 22: [1, 16, 17, 18, 19, 20, 21, 23, 24, 26, 27, 28], 23: [17, 18, 19, 20, 21, 22, 24, 25, 26, 27, 28, 29], 24: [18, 19, 20, 21, 22, 23, 25, 26, 27, 28, 29, 30], 25: [19, 20, 21, 23, 24, 26, 27, 28, 29, 30, 31, 34], 26: [20, 21, 22, 23, 24, 25, 27, 28, 29, 30, 31, 32], 27: [21, 22, 23, 24, 25, 26, 28, 29, 30, 31, 32, 33], 28: [22, 23, 24, 25, 26, 27, 29, 30, 31, 32, 33, 34], 29: [23, 24, 25, 26, 27, 28, 30, 31, 32, 33, 34, 35], 30: [24, 25, 26, 27, 28, 29, 31, 32, 33, 34, 35, 36], 31: [0, 25, 26, 27, 28, 29, 30, 32, 33, 34, 35, 36], 32: [0, 1, 9, 26, 27, 28, 29, 30, 31, 33, 34, 36], 33: [0, 1, 2, 27, 28, 29, 30, 31, 32, 34, 35, 36], 34: [0, 2, 3, 25, 28, 29, 30, 31, 32, 33, 35, 36], 35: [0, 1, 2, 3, 4, 14, 29, 30, 31, 33, 34, 36], 36: [0, 2, 3, 4, 5, 13, 30, 31, 32, 33, 34, 35]}

$\mathbf{c} = [2, -3, -4, 5, 1, -1, -1, -4, 5, 2, -5, 1, 1, -1, 6, -5, -4, 7, -1, -5, 4, -5, 3, 6, -5, -5, 8, -3, 1, 1, -4, 3, 4, -7, -1, 3, 1]$

Order $n = 39$.

{0: [1, 2, 3, 4, 5, 6, 15, 33, 34, 36, 37, 38], 1: [0, 2, 3, 4, 5, 6, 7, 34, 35, 36, 37, 38], 2: [0, 1, 3, 4, 5, 6, 7, 8, 35, 36, 37, 38], 3: [0, 1, 2, 4, 5, 6, 7, 8, 9, 36, 37, 38], 4: [0, 1, 2, 3, 5, 6, 7, 8, 9, 10, 37, 38], 5: [0, 1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 38], 6: [0, 1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 12], 7: [1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 13], 8: [2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 13, 14], 9: [3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15], 10: [4, 5, 6, 7, 8, 9, 11, 12, 13, 14, 15, 16], 11: [5, 6, 7, 8, 9, 10, 12, 13, 14, 16, 17, 35], 12: [6, 7, 8, 9, 10, 11, 13, 14, 15, 16, 17, 18], 13: [7, 8, 9, 10, 11, 12, 14, 15, 16, 17, 18, 19], 14: [8, 9, 10, 11, 12, 13, 15, 16, 17, 18, 19, 20], 15: [0, 9, 10, 12, 13, 14, 16, 17, 18, 19, 20, 21], 16: [10, 11, 12, 13, 14, 15, 17, 18, 19, 20, 21, 22], 17: [11, 12, 13, 14, 15, 16, 18, 19, 20, 21, 22, 23], 18: [12, 13, 14, 15, 16, 17, 19, 20, 21, 22, 23, 24], 19: [13, 14, 15, 16, 17, 18, 20, 21, 22, 23, 24, 25], 20: [14, 15, 16, 17, 18, 19, 21, 22, 23, 24, 25, 26], 21: [15, 16, 17, 18, 19, 20, 22, 23, 24, 25, 26, 27], 22: [16, 17, 18, 19, 20, 21, 23, 24, 25, 26, 27, 28], 23: [17, 18, 19, 20, 21, 22, 24, 25, 26, 27, 28, 29], 24: [18, 19, 20, 21, 22, 23, 25, 26, 27, 28, 29, 30], 25: [19, 20, 21, 22, 23, 24, 26, 27, 28, 29, 30, 31], 26: [20, 21, 22, 23, 24, 25, 27, 28, 29, 30, 31, 32], 27: [21, 22, 23, 24, 25, 26, 28, 29, 30, 31, 32, 33], 28: [22, 23, 24, 25, 26, 27, 29, 30, 31, 32, 33, 34], 29: [23, 24, 25, 26, 27, 28, 30, 31, 32, 33, 34, 35], 30: [24, 25, 26, 27, 28, 29, 31, 32, 33, 34, 35, 36], 31: [25, 26, 27, 28, 29, 30, 32, 33, 34, 35, 36, 37], 32: [26, 27, 28, 29, 30, 31, 33, 34, 35, 36, 37, 38], 33: [0, 27, 28, 29, 30, 31, 32, 34, 35, 36, 37, 38], 34: [0, 1, 28, 29, 30, 31, 32, 33, 35, 36, 37, 38], 35: [1, 2, 11, 29, 30, 31, 32, 33, 34, 36, 37, 38], 36: [0, 1, 2, 3, 30, 31, 32, 33, 34, 35, 37, 38], 37: [0, 1, 2, 3, 4, 31, 32, 33, 34, 35, 36, 38], 38: [0, 1, 2, 3, 4, 5, 32, 33, 34, 35, 36, 37]}

$\mathbf{c} = [1, -2, 1, 1, -2, 1, 1, -2, 1, 1, -2, 1, 1, -2, 1, 1, -2, 1, 1, -2, 1, 1, -2, 1, 1, -2, 1, 1, -2, 1, 1, -2, 1, 1, -2, 1, 1, -2, 1]$