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Sierpiński products of r-uniform hypergraphs

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Abstract

If H_1 and H_2 are r-uniform hypergraphs and f is a function from the set of all $(r-1)$-element subsets of $V(H_1)$ into $V(H_2)$, then the Sierpiński product $H_1 \otimes_f H_2$ is defined to have vertex set $V(H_1) \times V(H_2)$ and hyperedges falling into two classes:

$$(g, h_1)(g, h_2) \cdots (g, h_r), \quad \text{such that } g \in V(H_1) \text{ and } h_1 h_2 \cdots h_r \in E(H_2),$$

and

$$(g_1, f(\{g_2, g_3, \dots, g_r\}))(g_2, f(\{g_1, g_3, \dots, g_r\})) \cdots (g_r, f(\{g_1, g_2, \dots, g_{r-1}\})),$$

such that $g_1 g_2 \cdots g_r \in E(H_1)$. We develop the basic structure possessed by this product and offer proofs of numerous extremal properties involving connectivity, clique numbers, and strong chromatic numbers.

Keywords: Hypergraph products, cliques, chromatic numbers.

Math. Subj. Class.: 05C65, 05C15, 05C40

1 Introduction

Sierpiński graphs were first introduced in 1997 by Klavžar and Milutinović [8] stemming from their work on the Tower of Hanoi problem. Since then, numerous properties and generalizations of Sierpiński graphs have been extensively studied (e.g., see [6, 7, 9, 10, 12, 13], and [14]). Recently, Kovič, Pisanski, Zemljič, and Žitnik [11] have used Sierpiński graphs

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as a motivation for a graph product structure, which they referred to as a Sierpiński product. Their introductory work on this product included proofs of the product's basic properties involving connectivity, planarity, automorphism groups, and a consideration of the product with multiple factors. The present paper seeks to generalize the Sierpiński product to the setting of r-uniform hypergraphs and to describe its structure, with an emphasis on extremal properties.

We begin with the construction of a Sierpiński product in the setting of graphs. Given graphs G_1 and G_2, and a function $f : V(G_1) \longrightarrow V(G_2)$, the Sierpiński product $G_1 \otimes_f G_2$ is defined to have vertex set $V(G_1) \times V(G_2)$ and edge set consisting of edges that fall into two classes:

$$\begin{aligned} (g,h)(g,h'), &\quad \text{such that } g \in V(G_1) \text{ and } hh' \in E(G_2), \\ (g, f(g'))(g', f(g)), &\quad \text{such that } gg' \in E(G_1). \end{aligned}$$

Edges in these classes are referred to as inner and connecting edges, respectively. Observe that regardless of the choice of function f, the graph $G_1 \otimes_f G_2$ is a subgraph of the lexicographic product $G_1[G_2]$, defined to have vertex set $V(G_1) \times V(H_2)$ and edge set

$$E(G_1[G_2]) = \{(g,h)(g',h') \mid (g = g' \text{ and } hh' \in E(G_2)) \text{ or } gg' \in E(G_1)\}.$$

Like the lexicographic product, the Sierpiński product is not commutative in general.

For each vertex $g \in V(G_1)$, the subgraph induced by the set

$$gG_2 = \{(g,h) \mid h \in V(G_2)\}$$

is isomorphic to G_2. It follows that when $|V(G_1)| = 1$, the Sierpiński product $G_1 \otimes_f G_2$ is isomorphic to G_2, regardless of the choice of f. It is also easily confirmed that when $|V(G_2)| = 1$, the function f must be constant and the Sierpiński product $G_1 \otimes_f G_2$ is isomorphic to G_1. Among these properties, it was proven in [11] that $G_1 \otimes_f G_2$ is connected if and only if G_1 and G_2 are both connected.

In Section 2, we consider a generalization of the Sierpiński product to r-uniform hypergraphs and prove several properties regarding connectivity. In Section 3, we turn our attention to clique numbers and the strong chromatic number. We note that in the case of the strong chromatic number, Theorems 3.2 and 3.4 and Corollary 3.5 are stated to include the case $r = 2$, offering new results involving the chromatic number of Sierpiński products of graphs. In Section 4, we conclude by offering some directions for future research and an alternate generalization of the Sierpiński product of r-uniform hypergraphs.

2 The Sierpiński product of r-uniform hypergraphs

Recall that an r-uniform hypergraph H consists of a nonempty vertex set $V(H)$ and a hyperedge set $E(H)$, consisting of unordered r-tuples of distinct elements from $V(H)$. For our purposes, we assume that all r-uniform hypergraphs are simple (i.e., we do not allow multi-hyperedges). When $r = 2$, this definition coincides with that of simple graphs.

If H_1 and H_2 are r-uniform hypergraphs, then denote by $\binom{V(H_1)}{r-1}$ the set of all unordered $(r-1)$-tuples of elements in $V(H_1)$. For a function

$$f : \binom{V(H_1)}{r-1} \longrightarrow V(H_2),$$

the Sierpiński product $H_1 \otimes_f H_2$ has vertex set $V(H_1) \times V(H_2)$. The hyperedges in $E(H_1 \otimes_f H_2)$ have the following forms:

$$(g, h_1)(g, h_2) \cdots (g, h_r), \quad \text{such that } g \in V(H_1) \text{ and } h_1 h_2 \cdots h_r \in E(H_2),$$

and

$$(g_1, f(\{g_2, g_3, \ldots, g_r\}))(g_2, f(\{g_1, g_3, \ldots, g_r\})) \cdots (g_r, f(\{g_1, g_2, \ldots, g_{r-1}\})),$$

such that $g_1 g_2 \cdots g_r \in E(H_1)$. Hyperedges in the first class are called inner hyperedges, while those in the second class are called connecting hyperedges. This product agrees with the definition in Section 1 in the case where $r = 2$.

For each $g \in V(H_1)$, the subhypergraph of $H_1 \otimes_f H_2$ induced by

$$gH_2 = \{(g, h) \mid h \in V(H_2)\}$$

is isomorphic to H_2. Among any r distinct $g_1H_2, g_2H_2, \ldots, g_rH_2$ chosen, there exists at most a single connecting hyperedge. In total, we find that $H_1 \otimes_f H_2$ contains $|V(H_1)| \cdot |E(H_2)|$ inner hyperedges and $|E(H_1)|$ connecting hyperedges.

Before considering examples and properties involving connectivity, we must recall some definitions. Recall that a Berge path of length ℓ is a sequence of $\ell + 1$ distinct vertices $v_1, v_2, \ldots, v_{\ell+1}$ and distinct hyperedges $e_1, e_2, \ldots, e_\ell$ such that $v_i, v_{i+1} \in e_i$ for all $i \in \{1, 2, \ldots, \ell\}$. We write such a path as

$$P = v_1 e_1 v_2 e_2 \cdots e_\ell v_{\ell+1}$$

and observe that although the hyperedges are distinct, each pair of hyperedges may have numerous vertices in common. A Berge path $P = v_1 e_1 v_2 e_2 \cdots e_\ell v_{\ell+1}$ forms a loose path if all vertices in P other than $v_2, v_3, \ldots, v_\ell$ have degree 1. In this case, all vertices are necessarily distinct and P has order $r + (r-1)(\ell-1)$. While we have defined Berge paths and loose paths as "stand alone" hypergraphs, we also refer to subhypergraphs isomorphic to these hypergraph constructions by the same names.

An r-uniform hypergraph H is called connected if for any distinct pair of vertices, there exists a Berge path that contains them both. An r-uniform hypergraph that is not connected is called disconnected. When an r-uniform hypergraph is connected, but the removal of any hyperedge (while retaining all vertices) disconnects it, then it is called minimally connected (e.g., see [2]). Given a Berge path $P = v_1 e_1 v_2 e_2 \cdots e_\ell v_{\ell+1}$, if there exists a hyperedge $e_{\ell+1}$ distinct from $e_1, e_2, \ldots, e_\ell$ such that $v_1, v_{\ell+1} \in e_{\ell+1}$, then we say that

$$C = v_1 e_1 v_2 e_2 \cdots e_\ell v_{\ell+1} e_{\ell+1} v_1$$

is a Berge cycle. An r-uniform hypergraph is an r-uniform tree if it is connected and does not contain any Berge cycles. Other equivalent definitions for an r-uniform tree are given in [2] and [3]. In particular, note that every r-uniform tree is minimally connected, but not every minimally connected r-uniform hypergraph is an r-uniform tree.

Example 2.1. For example, consider $K_4^{(3)}$, the complete 3-uniform hypergraph of order 4, and denote by P the 3-uniform loose path of size 2. Then if

$$f : \binom{V(K_4^{(3)})}{2} \longrightarrow V(P)$$

is any constant function that maps to a vertex of degree 1 in P, the Sierpiński product $K_4^{(3)} \otimes_f P$ is given in Figure 1. Observe that each copy of gP is isomorphic to P and the hypergraph spanned by the connecting hyperedges (dashed in Figure 1) is isomorphic to $K_4^{(3)}$.

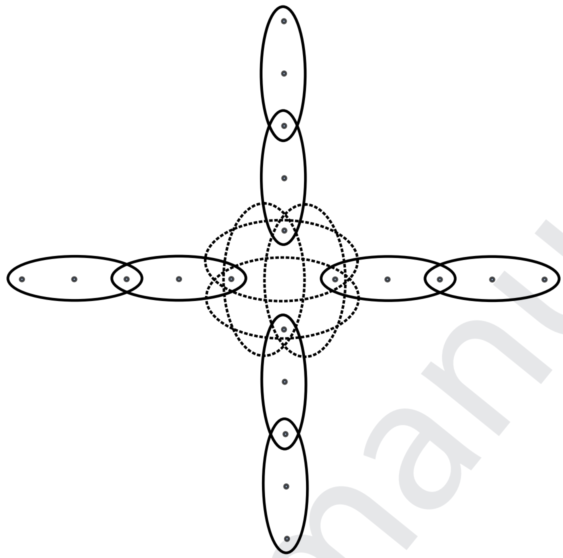

Figure 1: The Sierpiński Product $K_4^{(3)} \otimes_f P$, where P is a 3-uniform loose path of length 2 and f is a constant function whose range consists of a single vertex of degree 1 in P. The inner hyperedges are solid while the connecting hyperedges are dashed.

Example 2.2. Let C denote the 3-uniform Berge cycle of size 2 and order 4 containing exactly two vertices of degree 1. Suppose that $V(C) = \{x_1, x_2, x_3, x_4\}$, where x_1 and x_4 have degree 1. Also, let P be the loose path described in Example 2.1, with vertex set $V(P) = \{y_1, y_2, y_3, y_4, y_5\}$ such that y_3 is the unique vertex of degree 2. Define the function $f : \binom{V(C)}{2} \longrightarrow V(P)$ by

$$f(\{x_1, x_2\}) = y_1, \qquad f(\{x_1, x_3\}) = y_2, \qquad f(\{x_1, y_4\}) = y_3,$$
$$f(\{x_2, x_3\}) = y_3, \qquad f(\{x_2, x_4\}) = y_4, \qquad f(\{x_3, x_4\}) = y_5.$$

Then the connecting hyperedges in $C \otimes_f P$ are given by

$$e_1 = (x_1, y_3)(x_2, y_2)(x_3, y_1) \quad \text{and} \quad e_2 = (x_2, y_5)(x_3, y_4)(x_4, y_3).$$

Since f is nonconstant, such a hypergraph becomes more difficult to illustrate. So in Figure 2, we represent the connecting hyperedges by drawing segments from each hyperedge to the vertices they include. Also, note that the vertex (x_i, y_j) is labeled ij in this figure.

Examples 2.1 and 2.2 provide illustrations of some 3-uniform Sierpiński products when the underlying hypergraphs are connected. We note that when f is a constant function (as

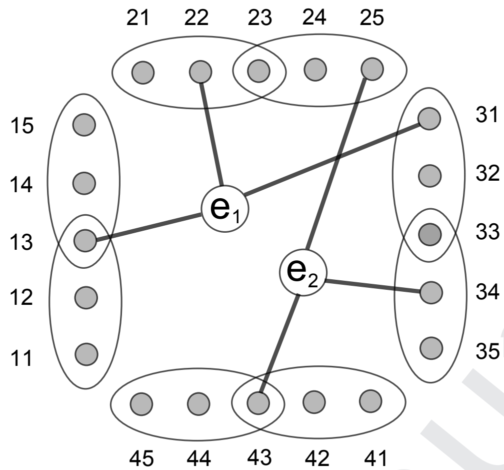

Figure 2: The Sierpiński Product $C \otimes_f P$, where C is a 3-uniform Berge cycle of size 2 and order 4 containing exactly two vertices of degree 1, P is a 3-uniform loose path of length 2, and f is the surjective function described in Example 2.2.

in Example 2.1), the resulting Sierpiński product may be considered a hypergraph generalization of a rooted product graph (for example, see [5]). In Proposition 2.10 of [11], it was shown that when G_1 and G_2 are graphs, then $G_1 \otimes_f G_2$ is connected if and only if G_1 and G_2 are connected. The following theorem considers connectivity for higher uniformity.

Theorem 2.3. *Assume that $r \geq 3$, H_1 and H_2 are r-uniform hypergraphs, and $f : \binom{V(H_1)}{r-1} \longrightarrow V(H_2)$ is a function. If H_1 and H_2 are connected, then $H_1 \otimes_f H_2$ is connected. If $H_1 \otimes_f H_2$ is connected, then H_1 is connected. If $H_1 \otimes_f H_2$ is connected and f is a constant function, then H_2 is connected.*

Proof. First, suppose that H_1 and H_2 are connected and let (g, h) and (g', h') be vertices in $H_1 \otimes_f H_2$. If $g = g'$, then there exists a Berge path that contains both (g, h) and (g, h') since gH_2 is isomorphic to H_2. Otherwise, suppose that $g \neq g'$. Since H_1 is connected, there exists a Berge path

$$P = ge_0g_1e_1g_2e_2 \cdots g_{\ell-1}e_{\ell-1}g'$$

in H_1 (and we may write $g = g_0$ and $g' = g_\ell$). Each hyperedge e_i in P corresponds with a unique hyperedge E_i in $H_1 \otimes_f H_2$. Suppose that $(g_i, h_i) \in E_{i-1}$ while $(g_i, k_i) \in E_i$. If $h_i = k_i$, then E_{i-1} and E_i are adjacent. If $h_i \neq k_i$, then there must exist a Berge path Q_i connecting (g_i, h_i) to (g_i, k_i) in g_iH_2. Thus, we are able to form a Berge path from (g, h) to (g', h') in $H_1 \otimes_f H_2$ by following along the hyperedges $E_0, E_1, \ldots, E_{\ell-1}$ and taking a detour along the Berge path Q_i in g_iH_2 whenever

$$E_{i-1} \cap g_iH_2 \neq E_i \cap g_iH_2.$$

Finally, if $(g',k) \in E_{\ell-1}$ and $k \neq h'$, then we again follow the Berge path connecting (g',k) to (g',h') in $g'H_2$. Thus, $H_1 \otimes_f H_2$ is connected. Now assume that $H_1 \otimes_f H_2$ is connected. For any pair $g,g' \in V(H_1)$, there exists a Berge path from gH_2 to $g'H_2$ in $H_1 \otimes_f H_2$ that corresponds with a Berge path from g to g' in H_1. Thus, H_1 is connected. Now assume that $H_1 \otimes_f H_2$ is connected, f is a constant function, and $k,k' \in E(H_2)$ are distinct. Then there exists a Berge path from (g,k) to (g,k') that does not contains any of the connecting hyperedges in $H_1 \otimes_f H_2$ since all such hyperedges intersect gH_2 at a single vertex. Such a Berge path necessarily corresponds with a Berge path in gH_2, from which it follows that H_2 must be connected. □

Theorem 2.3 is not as strong as in the case of graphs. This is demonstrated in Example 2.4, where a case is given in which $H_1 \otimes_f H_2$ is connected, but H_2 is disconnected.

Example 2.4. Consider the Sierpiński product $K_4^{(3)} \otimes_f 2K_3^{(3)}$, where $2K_3^{(3)}$ is the disjoint union of two 3-uniform hyperedges and $f : \binom{V(K_4^{(3)})}{2} \longrightarrow V(2K_3^{(3)})$ by

$$\begin{array}{lll} f(\{x_1,x_2\}) = y_6, & f(\{x_1,x_3\}) = y_1, & f(\{x_1,y_4\}) = y_1, \\ f(\{x_2,x_3\}) = y_1, & f(\{x_2,x_4\}) = y_6, & f(\{x_3,x_4\}) = y_6. \end{array}$$

Here, $V(K_4^{(3)}) = \{x_1,x_2,x_3,x_4\}$ and $2K_3^{(3)}$ consists of the hyperedges $y_1y_2y_3$ and $y_4y_5y_6$. The connecting hyperedges are given by

$$\begin{aligned} e_1 &= (x_1,y_1)(x_2,y_1)(x_3,y_6) \\ e_2 &= (x_1,y_6)(x_2,y_1)(x_4,y_6) \\ e_3 &= (x_1,y_6)(x_3,y_1)(x_4,y_1) \\ e_4 &= (x_2,y_6)(x_3,y_6)(x_4,y_1). \end{aligned}$$

From Figure 3, it is clear that $K_4^{(3)} \otimes_f 2K_3^{(3)}$ is connected even though $2K_3^{(3)}$ is disconnected.

Consider the case where $H_1 \otimes_f H_2$ is minimally connected and H_2 is assumed to be connected. Then by Theorem 2.3, H_1 is also connected. When an inner hyperedge of $H_1 \otimes_f H_2$ is removed, the removal of the corresponding hyperedge in H_2 disconnects H_2. When a connecting hyperedge is removed, the removal of the corresponding hyperedge in H_1 disconnects H_1. We obtain the following corollary.

Corollary 2.5. *Assume that $r \geq 3$, H_1 and H_2 are r-uniform hypergraphs, and $f : \binom{V(H_1)}{r-1} \longrightarrow V(H_2)$ is a function. If $H_1 \otimes_f H_2$ is minimally connected and H_2 is connected, then H_1 and H_2 are minimally connected.*

In the more restrictive class of r-uniform trees, we obtain the following theorem.

Theorem 2.6. *Assume that $r \geq 3$, H_1 and H_2 are r-uniform hypergraphs, and $f : \binom{V(H_1)}{r-1} \longrightarrow V(H_2)$ is a function. If $H_1 \otimes_f H_2$ is an r-uniform tree and H_2 is connected, then, H_2 is an r-uniform tree.*

Proof. Assume that H_2 is connected. Since $H_1 \otimes_f H_2$ contains a subhypergraph isomorphic to H_2, $H_1 \otimes_f H_2$ will contain a Berge cycle whenever H_2 contains a Berge cycle. It follows that H_2 is an r-uniform tree whenever $H_1 \otimes_f H_2$ is an r-uniform tree. □

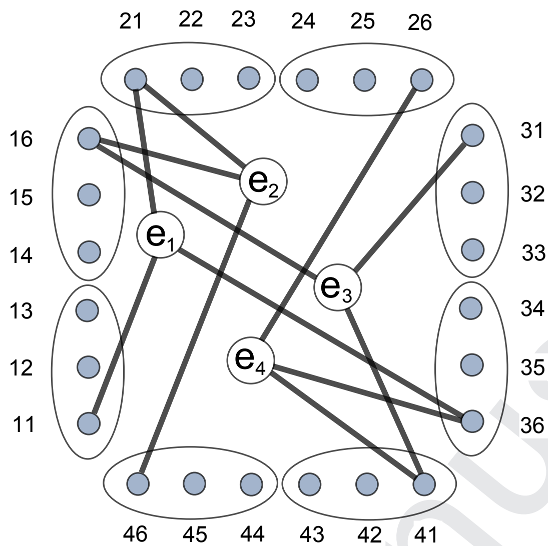

Figure 3: The Sierpiński Product $K_4^{(3)} \otimes_f 2K_3^{(3)}$, where f is the function described in Example 2.4. Observe that $K_4^{(3)} \otimes_f 2K_3^{(3)}$ is connected even though $2K_3^{(3)}$ is disconnected.

3 Cliques and strong chromatic numbers

In this section, we focus on the clique numbers and chromatic numbers of Sierpiński products. These parameters serve as measures of connectivity and they play important roles in various extremal aspects of the study of hypergraphs. If H is any r-uniform hypergraph, then the clique number $\omega(H)$ is the maximum order of a complete subhypergraph of H. When $r = 2$, it is well-known that $\omega(G_1[G_2]) = \omega(G_1)\omega(G_2)$ (e.g., see [4]), and since $G_1 \otimes_f G_2$ is a subgraph of $G_1[G_2]$ for all f, it follows that

$$\omega(G_1 \otimes_f G_2) \leq \omega(G_1)\omega(G_2).$$

When $r \geq 3$, we obtain the following theorem.

Theorem 3.1. *Let $r \geq 3$. If H_1 and H_2 are r-uniform hypergraphs and $f : \binom{V(H_1)}{r-1} \longrightarrow V(H_2)$ is a function, then*

$$\omega(H_1 \otimes_f H_2) \leq \max\{\omega(H_1), \omega(H_2)\}.$$

If $\omega(H_2) \geq \omega(H_1)$, then

$$\omega(H_1 \otimes_f H_2) = \omega(H_2).$$

If f is a constant function, then

$$\omega(H_1 \otimes_f H_2) = \max\{\omega(H_1), \omega(H_2)\}.$$

Proof. The statement

$$\omega(H_1 \otimes_f H_2) \leq \max\{\omega(H_1), \omega(H_2)\}$$

follows from Theorem 3 of [1], where it was proved that the lexicographic product satisfies

$$\omega(H_1[H_2]) = \max\{\omega(H_1), \omega(H_2)\},$$

and the observation that $H_1 \otimes_f H_2$ is a subhypergraph of $H_1[H_2]$. Since each gH_2 contained in $H_1 \otimes_f H_2$ is isomorphic to H_2, we find that $H_1 \otimes_f H_2$ contains a complete subhypergraph at least as large as a clique in H_2. It follows that

$$\omega(H_1 \otimes_f H_2) = \omega(H_2)$$

whenever $\omega(H_2) \geq \omega(H_1)$. Finally, if f is a constant function, then the subhypergraph induced by

$$H_1 h = \{(g, h) \mid g \in V(H_1)\}$$

is isomorphic to H_1 for the unique vertex h in the image of f. So, $H_1 \otimes_f H_2$ contains complete subgraphs of orders equal to both $\omega(H_1)$ and $\omega(H_2)$, giving us

$$\omega(H_1 \otimes_f H_2) = \max\{\omega(H_1), \omega(H_2)\}$$

in this case. □

In the setting of r-uniform hypergraphs, there are many ways to generalize chromatic numbers. In this paper, we will focus on the strong chromatic number of an r-uniform hypergraph H. First, define a *strong proper vertex coloring* of an r-uniform hypergraph H to be a map

$$c : V(H) \longrightarrow \{1, 2, \ldots, n\}$$

such that no two adjacent vertices in H receive the same color. Then the least n for which a strong proper vertex coloring exists is called the *strong chromatic number* of H, and is denoted $\chi_s(H)$. Our reasoning for focusing on this generalization is due to the relationship between the strong chromatic number and the existence of certain hypergraph homomorphisms. Recall that if H_1 and H_2 are two r-uniform hypergraphs, then a *homomorphism* is a function $\phi : V(H_1) \longrightarrow V(H_2)$ such that if $x_1 x_2 \cdots x_r \in E(H_1)$, then $\phi(x_1)\phi(x_2)\cdots\phi(x_r) \in E(H_2)$.

For any strong proper vertex coloring $c : V(H) \longrightarrow \{1, 2, \ldots, n\}$, there is a natural homomorphism $\phi : V(H) \longrightarrow V(K_n^{(r)})$ given by mapping each vertex $h \in V(H)$ to a vertex $\phi(h) \in V(K_n^{(r)})$ identified with the color class of h under c. This identification of strong proper vertex colorings of r-uniform hypergraphs with homomorphisms leads to a useful property. Specifically, if H_1 and H_2 are any r-uniform hypergraphs and if $\phi : V(H_1) \longrightarrow V(H_2)$ is a homomorphism, then

$$\chi_s(H_1) \leq \chi_s(H_2), \tag{3.1}$$

since any strong proper vertex coloring of H_2 can be applied to H_1 under ϕ.

Theorem 3.2. *For $r \geq 2$, let H_1 and H_2 be r-uniform hypergraphs such that $\chi_s(H_2) = n$. Let $\phi : V(H_2) \longrightarrow V(K_n^{(r)})$ be a homomorphism. For any function $f : \binom{V(H_1)}{r-1} \longrightarrow V(H_2)$,*

$$\chi_s(H_1 \otimes_f H_2) \leq \chi_s(H_1 \otimes_{f^*} K_n^{(r)}),$$

where $f^* := \phi \circ f$.

Proof. Let $c : V(H_2) \longrightarrow \{1, 2, \ldots, n\}$ be a strong proper vertex coloring of H_2 such that $\chi_s(H_2) = n$. Note that c is necessarily surjective. Such a coloring naturally extends to the surjective homomorphism $\phi : V(H_2) \longrightarrow K_n^{(r)}$ given by sending each vertex in $h \in V(H_2)$ to a vertex in $K_n^{(r)}$ identified with the color class of h under c. Consider the map

$$\phi^* : V(H_1 \otimes_f H_2) \longrightarrow V(H_1 \otimes_{f^*} K_n^{(r)})$$

given by $\phi^*(g, h) = (g, \phi(h))$. We claim that ϕ^* is a homomorphism. To prove this claim, let

$$(g_1, h_1)(g_2, h_2) \cdots (g_r, h_r) \in E(H_1 \otimes_f H_2)$$

and consider

$$(g_1, \phi(h_1))(g_2, \phi(h_2)) \cdots (g_r, \phi(h_r)) \in E(H_1 \otimes_{f^*} K_n^{(r)}).$$

Then either $g_1 = g_2 = \cdots = g_r$ (in which case, $\phi(h_1)\phi(h_2)\cdots\phi(h_r) \in E(K_n^{(r)})$ since $h_1 h_2 \cdots h_r \in E(H_2)$ and ϕ is a homomorphism) or $g_1 g_2 \cdots g_r \in E(H_1)$ and

$$\begin{aligned}
\phi(h_1) &= \phi(f(\{g_2, g_3, \ldots, g_r\})) = f^*(\{g_2, g_3, \ldots, g_r\}) \\
\phi(h_2) &= \phi(f(\{g_1, g_3, \ldots, g_r\})) = f^*(\{g_1, g_3, \ldots, g_r\}) \\
&\vdots \\
\phi(h_r) &= \phi(f(\{g_1, g_2, \ldots, g_{r-1}\})) = f^*(\{g_1, g_2, \ldots, g_{r-1}\}).
\end{aligned}$$

It follows that ϕ^* is a homomorphism, from which we conclude that

$$\chi_s(H_1 \otimes_f H_2) \le \chi_s(H_1 \otimes_{f^*} K_n^{(r)})$$

by (3.1). □

Note that in the previous theorem, the case $r = 2$ is included. In this case, χ_s is the usual chromatic number for graphs. We find that in general, Theorem 3.2 is the strongest statement that can be made, as the following example demonstrates a case where a strict inequality is satisfied.

Example 3.3. Consider the complete 3-uniform hypergraph $K_4^{(3)}$ with vertex set $V(K_4^{(3)}) = \{x_1, x_2, x_3, x_4\}$ and the 3-uniform loose path P of length 2 with vertex set $V(P) = \{y_1, y_2, y_3, y_4, y_5\}$, where y_3 is the unique vertex in P with degree 2. Let $f : \binom{V(K_4^{(3)})}{2} \longrightarrow V(P)$ be the function

$$\begin{aligned}
&f(\{x_1, x_2\}) = y_1, && f(\{x_1, x_3\}) = y_2, && f(\{x_1, y_4\}) = y_3, \\
&f(\{x_2, x_3\}) = y_4, && f(\{x_2, x_4\}) = y_5, && f(\{x_3, x_4\}) = y_5.
\end{aligned}$$

Then the Sierpiński product $K_4^{(3)} \otimes_f P$ is given in Figure 4, with vertex (x_i, y_i) labelled ij. The connecting hyperedges are given by

$$\begin{aligned}
e_1 &= (x_1, y_4)(x_2, y_2)(x_3, y_1) \\
e_2 &= (x_1, y_5)(x_2, y_3)(x_4, y_1) \\
e_3 &= (x_1, y_5)(x_3, y_3)(x_4, y_2) \\
e_4 &= (x_2, y_5)(x_3, y_5)(x_4, y_4).
\end{aligned}$$

Since every strong proper vertex coloring of a hypergraph containing at least one hyperedge requires at least 3 colors, the coloring given in Figure 4 implies that $\chi_s(K_4^{(3)} \otimes_f P) = 3$. Note that $\chi_s(P) = 3$, and we can identify a strong proper vertex coloring of P with the

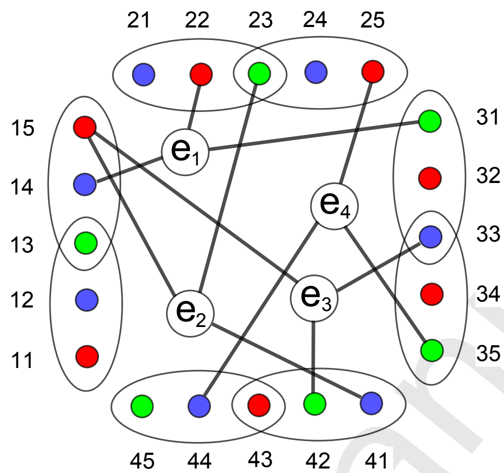

Figure 4: The Sierpiński product $K_4^{(3)} \otimes_f P$, where P is a 3-uniform loose path of length 2 and f is the function given in Example 3.3. The strong proper vertex coloring given shows that this hypergraph has a strong chromatic number of 3.

homomorphism $\phi : V(P) \longrightarrow K_3^{(3)}$ that maps

$$\phi(y_1) = \phi(y_5) = z_1, \quad \phi(y_2) = \phi(y_4) = z_2, \quad \text{and} \quad \phi(y_3) = z_3,$$

where $V(K_3^{(3)}) = \{z_1, z_2, z_3\}$. Then $f^* := \phi \circ f$ and the connecting hyperedges in $K_4^{(3)} \otimes_{f^*} K_3^{(3)}$ are given by

$$\begin{aligned} e'_1 &= (x_1, z_2)(x_2, z_2)(x_3, z_1) \\ e'_2 &= (x_1, z_1)(x_2, z_3)(x_4, z_1) \\ e'_3 &= (x_1, z_1)(x_3, z_3)(x_4, z_2) \\ e'_4 &= (x_2, z_1)(x_3, z_1)(x_4, z_2). \end{aligned}$$

The resulting hypergraph $K_4^{(3)} \otimes_{f^*} K_3^{(3)}$ is given in Figure 5.

To obtain a strong proper vertex coloring, we begin by focusing on the connecting hyperedges e'_2 and e'_3. Without loss of generality, suppose that (x_1, z_1) is red and (x_4, z_2) is blue. This forces (x_4, z_1) and (x_3, z_3) to be green and (x_2, z_3) to be blue. Then (x_3, z_1) must be red and (x_2, z_1) must be green. At this point, no color is available for (x_2, z_2) as (x_2, z_1) and (x_2, z_3) require it to be different from blue and green, but e'_1 already contains a red vertex. So, $\chi_s(K_4^{(3)} \otimes_{f^*} K_3^{(3)}) \geq 4$, and one can continue with this process to obtain

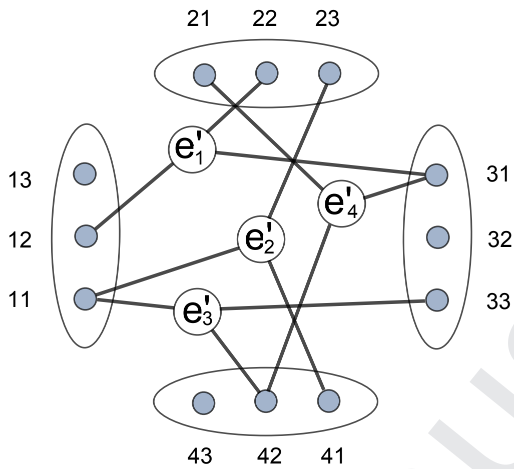

Figure 5: The Sierpiński product $K_4^{(3)} \otimes_{f^*} K_3^{(3)}$ given in Example 3.3.

a strong proper vertex 4-coloring of $K_4^{(3)} \otimes_{f^*} K_3^{(3)}$, showing that $\chi_s(K_4^{(3)} \otimes_{f^*} K_3^{(3)}) = 4$. Thus, our example demonstrates that there are cases where equality is not obtained in Theorem 3.2.

While we can not be precise with the evaluation of the strong chromatic number in general, an exact evaluation can be found when f is assumed to be constant.

Theorem 3.4. *Let $r \geq 2$ and suppose that H_1 and H_2 are r-uniform hypergraphs. If $f : \binom{V(H_1)}{r-1} \longrightarrow V(H_2)$ is a constant function, then*

$$\chi_s(H_1 \otimes_f H_2) = \max\{\chi_s(H_1), \chi_s(H_2)\}.$$

Proof. When $f : \binom{V(H_1)}{r-1} \longrightarrow V(H_2)$ is a constant function, the subhypergraph spanned by the connecting hyperedges is isomorphic to H_1 and each gH_2 is isomorphic to H_2. It follows that

$$\chi_s(H_1 \otimes_f H_2) \geq \max\{\chi_s(H_1), \chi_s(H_2)\}.$$

To prove the opposite inequality, observe that all connecting hyperedges include at most one vertex from each gH_2. Begin with a strong proper vertex coloring of the vertices spanned by the connecting hyperedges using at most $\chi_s(H_1)$. The specific color assigned for at most one vertex in each gH_2 does not affect the number of colors needed to form a strong proper vertex coloring of each gH_2. Hence, it is possible to color $H_1 \otimes_f H_2$ using $\max\{\chi_s(H_1), \chi_s(H_2)\}$ colors, completing the proof. □

An immediate consequence of this theorem is the following corollary.

Corollary 3.5. *Let $r \geq 2$ and suppose that H_1 and H_2 are r-uniform hypergraphs with $\chi_s(H_2) = n$. If $f : \binom{V(H_1)}{r-1} \longrightarrow V(H_2)$ is a constant function, then*

$$\chi_s(H_1 \otimes_f H_2) = \chi_s(H_1 \otimes_{f^*} K_n^{(r)}).$$

4 Conclusion

We conclude our investigation of Sierpiński products of r-uniform hypergraphs by identifying numerous directions for future study. Our primary focus has been on measures of connectivity, but there are many additional parameters (e.g., independence numbers, diameters, etc...) and applications worthy of inquiry. As subhypergraphs of lexicographic products, Sierpiński products may offer new results in Ramsey theory or bounds for certain Turán numbers (e.g., see [1]). Several of the topics studied in Kovič, Pisanski, Zemljič, and Žitnik's paper [11], such as automorphism groups and products involving more than two factors, have not been considered here and should be considered for hypergraphs.

Finally, the generalization of Sierpiński products to r-uniform hypergraphs that we have used seemed like the natural choice, but there are other ways in which one can make such a generalization. For example, let H_1 and H_2 be r-uniform hypergraphs. For a function $f : V(H_1) \longrightarrow V(H_2)$, define the product $H_1 \otimes^f H_2$ to have vertex set $V(H_1) \times V(H_2)$. The hyperedges in $E(H_1 \otimes^f H_2)$ have the following forms:

$$(g, h_1)(g, h_2) \cdots (g, h_r), \quad \text{such that } g \in V(H_1) \text{ and } h_1 h_2 \cdots h_r \in E(H_2),$$

and

$$(g_1, f(\pi(g_1)))(g_2, f(\pi(g_2))) \cdots (g_r, f(\pi(g_r))),$$

such that $g_1 g_2 \cdots g_r \in E(H_1)$ and π is any nontrivial permutation on $\{g_1, g_2, \ldots, g_r\}$. Observe that we have denoted this generalization of the Sierpiński product by writing f as a superscript rather than a subscript. Perhaps many of the results proved in this paper hold for this product as well. We leave its investigation for future work.

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References

[1] M. Bruce, M. Budden and J. Hiller, Lexicographic products of r-uniform hypergraphs and some applications to hypergraph Ramsey theory, *Australas. J. Combin.* **70** (2018), 390–401, https://ajc.maths.uq.edu.au/?page=get_volumes&volume=70.

[2] M. Budden, J. Hiller and A. Penland, Minimally connected hypergraphs, 2019, arXiv:1901.04560 [math.CO].

[3] M. Budden and A. Penland, Trees and n-good hypergraphs, *Australas. J. Combin.* **72** (2018), 329–349, https://ajc.maths.uq.edu.au/?page=get_volumes&volume=72.

[4] D. Geller and S. Stahl, The chromatic number and other functions of the lexicographic product, *J. Combinatorial Theory Ser. B* **19** (1975), 87–95, doi:10.1016/0095-8956(75)90076-3.

[5] C. D. Godsil and B. D. McKay, A new graph product and its spectrum, *Bull. Austral. Math. Soc.* **18** (1978), 21–28, doi:10.1017/s0004972700007760.

[6] A. M. Hinz, S. Klavžar and S. S. Zemljič, Sierpiński graphs as spanning subgraphs of Hanoi graphs, *Cent. Eur. J. Math.* **11** (2013), 1153–1157, doi:10.2478/s11533-013-0227-7.

[7] M. Jakovac and S. Klavžar, Vertex-, edge-, and total-colorings of sierpiński-like graphs, *Discrete Math.* **309** (2009), 1548–1556, doi:10.1016/j.disc.2008.02.026.

[8] S. Klavžar and U. Milutinović, Graphs $S(n, k)$ and a variant of the Tower of Hanoi problem, *Czechoslovak Math. J.* **47(122)** (1997), 95–104, doi:10.1023/a:1022444205860.

[9] S. Klavžar, U. Milutinović and C. Petr, 1-perfect codes in Sierpiński graphs, *Bull. Austral. Math. Soc.* **66** (2002), 369–384, doi:10.1017/s0004972700040235.

[10] S. Klavžar and S. S. Zemljič, On distances in Sierpiński graphs: almost-extreme vertices and metric dimension, *Appl. Anal. Discrete Math.* **7** (2013), 72–82, doi:10.2298/aadm130109001k.

[11] J. Kovič, T. Pisanski, S. Zemljič and A. Žitnik, The sierpiński product of graphs, 2019, arXiv:1904.04180 [math.CO].

[12] D. Parisse, On some metric properties of the Sierpiński graphs $S(n, k)$, *Ars Combin.* **90** (2009), 145–160.

[13] B. Xue, L. Zuo and G. Li, The Hamiltonicity and path t-coloring of Sierpiński-like graphs, *Discrete Appl. Math.* **160** (2012), 1822–1836, doi:10.1016/j.dam.2012.03.022.

[14] B. Xue, L. Zuo, G. Wang and G. Li, Shortest paths in Sierpiński graphs, *Discrete Appl. Math.* **162** (2014), 314–321, doi:10.1016/j.dam.2013.08.029.