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Cayley graphs of order $6pq$ and $7pq$ are Hamiltonian

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Abstract

Assume G is a finite group, such that $|G| = 6pq$ or $7pq$, where p and q are distinct prime numbers, and let S be a generating set of G. We prove there is a Hamiltonian cycle in the corresponding connected Cayley graph $\operatorname{Cay}(G;S)$.

Keywords: Cayley graphs, Hamiltonian cycles.

Math. Subj. Class.: 05C25, 05C45

1 Introduction

Arthur Cayley [1] introduced the definition of Cayley graph in 1878. All graphs in this paper are undirected (graphs without loops and direction on the edges).

Definition 1.1 ([16, Definition 1.1], cf. [11, p. 34])**.** Let S be a subset of a finite group G. The *Cayley graph* $\operatorname{Cay}(G;S)$ is the graph whose vertices are elements of G, with an edge joining g and gs, for every $g \in G$ and $s \in S$.

Since then, the theory of Cayley graphs has developed into an important branch of algebraic graph theory. It is an interesting topic to work on because not only is it related to pure mathematics problems, but it is connected to fascinating problems studied by computer scientists, molecular biologists, and coding theorists (see [15] for more information).

*Theorem 1.3 and Proposition 1.4 are the main results of this paper. I would like to express my sincere gratitude to my supervisor, professor Joy Morris who always supported me throughout my graduate journey. I am especially grateful to my co-supervisor, professor Dave Morris, for the patient guidance and advice he has provided during my graduate study. I have been extremely lucky to have a co-supervisor who cared so much about my research, and who responded to my questions so promptly. I am also thankful to professor Hadi Kharaghani and professor Amir Akbary and cannot forget their valuable help and motivation during my graduate years. I am truly grateful to my family for their immeasurable love and care.

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Recall that a *Hamiltonian cycle* is a cycle that visits every vertex of a graph. Finding Hamiltonian cycles is a fundamental question in graph theory, but in general, it is extremely difficult. To be precise, it is an NP-complete problem, which means most mathematicians do not believe there exists an efficient algorithm to determine whether an arbitrary graph contains such a cycle. Because the general case is so hard, it is natural to look at special cases.

Cayley graphs are one of these cases that mathematicians are interested in working on. There have been many papers on the topic of Hamiltonian cycles in Cayley graphs, but it is still an open question whether every connected Cayley graph has a Hamiltonian cycle. (See survey papers [5, 21, 24] for more information. We ignore the trivial counterexamples on 1 or 2 vertices.) The following result combines the main result of this paper with the previous work of several authors (C. C. Chen and N. Quimpo [2], S. J. Curran, J. Morris and D. W. Morris [6], E. Ghaderpour and D. W. Morris [9, 10], D. Jungreis and E. Friedman [13], Kutnar et al. [16], K. Keating and D. Witte [14], D. Li [17], D. W. Morris and K. Wilk [20], and D. Witte [23]).

Theorem 1.2 ([16, 20, 23]). *Let G be a finite group. If $|G|$ has any of the forms below (where p, q, and r are distinct primes), then every connected Cayley graph on G has a Hamiltonian cycle.*

1. *kp, where $1 \leqslant k \leqslant 47$,*
2. *kpq, where $1 \leqslant k \leqslant 7$,*
3. *pqr,*
4. *kp^2, where $1 \leqslant k \leqslant 4$,*
5. *kp^3, where $1 \leqslant k \leqslant 2$,*
6. *p^k, where $1 \leqslant k < \infty$.*

Previously, part (2) of Theorem 1.2 was only known for $1 \leqslant k \leqslant 5$, but we improve this condition: we show that 5 can be replaced with 7. This is the new part of the above theorem which is our result. The hard part is when $k = 6$:

Theorem 1.3. *Assume G is a finite group of order $6pq$, where p and q are distinct prime numbers. Then every connected Cayley graph on G contains a Hamiltonian cycle.*

This generalizes [10], which considered only the case where $q = 5$. The proof takes up all of Section 3, after some preliminaries in Section 2.

Unlike Theorem 1.3, the following observation follows easily from known results, and may be known to experts. The proof is on page 8.

Proposition 1.4. *Assume G is a finite group of order $7pq$, where p and q are distinct prime numbers. Then every connected Cayley graph on G contains a Hamiltonian cycle.*

The Introduction of the author's masters thesis [18] provides additional background and a description of the methods that are used in the proof of the main theorem.

2 Preliminaries

This section establishes basic terminology and notation, and proves a number of technical results that will be used in the proof of Theorem 1.3. In particular, it is shown we may assume that $|G|$ is square-free (note $|G| = 6pq$ in Theorem 1.3), so the Sylow subgroups of G are $\mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_p$, and $\mathcal{C}_q$, and that $|G'|$ has precisely 2 prime factors, so G' is either $\mathcal{C}_p \times \mathcal{C}_q$ or $\mathcal{C}_3 \times \mathcal{C}_p$.

2.1 Basic notation and definitions

Throughout the paper, we have used standard terminology of graph theory and group theory that can be found in textbooks, such as [11, 12].

The following notation is used throughout the paper:

- The commutator $ghg^{-1}h^{-1}$ of g and h is denoted by $[g, h]$.
- We will always let $G' = [G, G]$ be the commutator subgroup of G.
- We define $\overline{G} = G/G'$, $\overline{g} = gG'$ for any $g \in G$, and $\overline{S} = \{\overline{g}; g \in S\}$ for any $S \subseteq G$.
- $C_{G'}(S)$ denotes the centralizer of S in G'.
- $G \ltimes H$ denotes a semidirect product of groups G and H, where H is normal.
- D_{2n} denotes the dihedral group of order $2n$.
- e denotes the identity element of G.
- For $S \subseteq G$, a sequence $(s_1, s_2, \ldots, s_n)$ of elements of $S \cup S^{-1}$ specifies the walk in the Cayley graph Cay$(G; S)$ that visits the vertices: $e, s_1, s_1s_2, \ldots, s_1s_2\cdots s_n$. Also, $(s_1, s_2, \ldots, s_n)^{-1} = (s_n^{-1}, s_{n-1}^{-1}, \ldots, s_1^{-1})$.
- We use $(\overline{s_1}, \overline{s_2}, \ldots, \overline{s_n})$ to denote the image of the walk $(s_1, s_2, \ldots, s_n)$ in the Cay$(G/G'; \overline{S})$ = Cay$(\overline{G}; \overline{S})$ which is a Cayley graph on the quotient group G/G'.
- For $k \in \mathbb{Z}^+$, we use $(s_1, s_2, \ldots, s_m)^k$ to denote the concatenation of k copies of the sequence $(s_1, s_2, \ldots, s_m)$.
- p and q are distinct prime numbers.
- $\mathcal{C}_n$ denotes the cyclic group of order n.
- $\widehat{G} = G/\mathcal{C}_p$, when $\mathcal{C}_p$ is a normal subgroup, we also let $\breve{G} = G/\mathcal{C}_q$ when $\mathcal{C}_q$ is a normal subgroup, and let $\overleftrightarrow{G} = G/\mathcal{C}_3$ when $\mathcal{C}_3$ is a normal subgroup. Also, $\widehat{g} = g\mathcal{C}_p$, $\breve{g} = g\mathcal{C}_q$, for any $g \in G$, and $\widehat{S} = \{\widehat{g}; g \in S\}$, $\breve{S} = \{\breve{g}; g \in S\}$ for any $S \subseteq G$.
- We let a_2, a_3, γ_p, and a_q be elements of G that generate $\mathcal{C}_2$, $\mathcal{C}_3$, $\mathcal{C}_p$, and $\mathcal{C}_q$, respectively.

Remark 2.1. When $|G| = 6pq$ and it is square free (as is usually the case in Section 3), the Sylow subgroups are $\mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_p$, and $\mathcal{C}_q$. Also, the commutator subgroup G' will usually be either $\mathcal{C}_p \times \mathcal{C}_q$ or $\mathcal{C}_3 \times \mathcal{C}_p$, so $\mathcal{C}_p$ is a normal subgroup and either $\mathcal{C}_q$ or $\mathcal{C}_3$ is also a normal subgroup.

2.2 Basic methods

In this subsection, we explain some of the key ideas in the proof of our main result (Theorem 1.3).

It is easy to see that $\mathrm{Cay}(G;S)$ is connected if and only if S generates G ([11, Lemma 3.7.4]). Also, if S is a subset of S_0, then $\mathrm{Cay}(G;S)$ is a subgraph of $\mathrm{Cay}(G;S_0)$ that contains all of the vertices. Therefore, in order to show that every connected Cayley graph on G contains a Hamiltonian cycle, it suffices to consider $\mathrm{Cay}(G;S)$, where S is a generating set that is *minimal*, which means that no proper subset of S generates G.

The following well known (and easy) result handles the case of Theorem 1.3 where G is abelian.

Lemma 2.2 ([3, Corollary on page 257])**.** *Assume G is an abelian group. Then every connected Cayley graph on G has a Hamiltonian cycle.*

Note $\mathrm{Cay}(\mathcal{C}_2;\{a\})$ is a Cayley graph with two vertices, where $\mathcal{C}_2 = \langle a \rangle$. We consider (a,a) as its Hamiltonian cycle which is:

$$e \xrightarrow{a} a \xrightarrow{a} a^2 = e.$$

Although graph theorists would not typically consider this a cycle, it satisfies the basic property of visiting each vertex exactly once. In some of our inductive proofs, we require a Hamiltonian cycle in a Cayley graph on a quotient group. When this quotient group is $\mathcal{C}_2$, this Hamiltonian cycle provides the structure we need for our inductive arguments to work.

Theorem 2.3 (Marušič [19], Durnberger [7, 8], and Keating-Witte [14])**.** *If the commutator subgroup G' of G is a cyclic p-group, then every connected Cayley graph on G has a Hamiltonian cycle.*

Theorem 2.4 (Chen-Quimpo [4])**.** *Let v and w be two distinct vertices of a connected Cayley graph $\mathrm{Cay}(G;S)$. Assume G is abelian, $|G|$ is odd, and the valency of $\mathrm{Cay}(G;S)$ is at least 3. Then $\mathrm{Cay}(G;S)$ has a Hamiltonian path that starts at v and ends at w.*

The following lemma (and its corollary) often provide a way to lift a Hamiltonian cycle in $\mathrm{Cay}(G/N;\overline{S})$ to a Hamiltonian cycle in $\mathrm{Cay}(G;S)$. Before stating the results, we introduce a useful piece of notation.

Notation 2.5. Suppose N is a normal subgroup of G, and $C = (s_1, s_2, \ldots, s_n)$ is a walk in $\mathrm{Cay}(G;S)$. If the walk $(s_1N, s_2N, \ldots, s_nN)$ in $\mathrm{Cay}(G/N;SN/N)$ is closed, then its *voltage* is the product $\mathbb{V}(C) = s_1 s_2 \cdots s_n$. This is an element of N. In particular, if $C = (\overline{s}_1, \overline{s}_2, \ldots, \overline{s}_n)$ is a Hamiltonian cycle in $\mathrm{Cay}(\overline{G}, \overline{S})$, then $\mathbb{V}(C) = s_1 s_2 \cdots s_n$.

Factor Group Lemma 2.6 ([24, Section 2.2])**.** *Suppose:*

- *S is a generating set of G,*
- *N is a cyclic normal subgroup of G,*
- *$\overline{G} = G/N$,*
- *$C = (\overline{s_1}, \overline{s_2}, \ldots, \overline{s_n})$ is a Hamiltonian cycle in $\mathrm{Cay}(G/N;\overline{S})$, and*
- *the voltage $\mathbb{V}(C)$ generates N.*

Then there is a Hamiltonian cycle in $\mathrm{Cay}(G; S)$.

Corollary 2.7 ([10, Corollary 2.3]). *Suppose:*

- S *is a generating set of* G*,*
- N *is a normal subgroup of* G*, such that* $|N|$ *is prime,*
- $sN = tN$ *for some* $s, t \in S$ *with* $s \neq t$*, and*
- *there is a Hamiltonian cycle in* $\mathrm{Cay}(G/N; \overline{S})$ *that uses at least one edge labeled* $\overline{s}$.

Then there is a Hamiltonian cycle in $\mathrm{Cay}(G; S)$.

Lemma 2.8. *Assume* $G = H \ltimes (\mathcal{C}_p \times \mathcal{C}_q)$*, where* $G' = \mathcal{C}_p \times \mathcal{C}_q$*, and let* S *be a generating set of* G*. As usual, let* $\overline{G} = G/G' \cong H$*. Assume there is a unique element* c *of* S *that is not in* $H \ltimes \mathcal{C}_q$*, and* C *is a Hamiltonian cycle in* $\mathrm{Cay}(\overline{G}; \overline{S})$ *such that* c *occurs precisely once in* C*. Then the subgroup generated by* $\mathbb{V}(C)$ *contains* $\mathcal{C}_p$.

Proof. Write $C = (\overline{s}_1, \overline{s}_2, \cdots, \overline{s}_n)$, and let $H^+ = H \ltimes \mathcal{C}_q$. By assumption, there is a unique k, such that $s_k = c$, and all other elements of S are in H^+. Therefore,

$$\mathbb{V}(C) = s_1 s_2 \ldots s_n \in H^+ \cdot H^+ \cdots H^+ \cdot c \cdot H^+ \cdot H^+ \cdots H^+ = H^+ c H^+.$$

Since $c \notin H^+$, we conclude that $\mathbb{V}(C) \notin H^+$.

On the other hand, since $\mathbb{V}(C)$ is an element of $G' = \mathcal{C}_p \times \mathcal{C}_q$, we have $\mathbb{V}(C) = a_q^i \gamma_p^j \in H^+ \gamma_p^j$. Since $\mathbb{V}(C) \notin H^+$, this implies $j \not\equiv 0 \pmod{p}$, so $\langle a_q^i \gamma_p^j \rangle$ contains $\mathcal{C}_p$. □

Definition 2.9. The *Cartesian product* $X_1 \square X_2$ of graphs X_1 and X_2 is a graph such that the vertex set of $X_1 \square X_2$ is $V(X_1) \times V(X_2) = \{(v, v'); v \in V(X_1), v' \in V(X_2)\}$, and two vertices (v_1, v_2) and (v_1', v_2') are adjacent in $X_1 \square X_2$ if and only if either

- $v_1 = v_1'$ and v_2 is adjacent to v_2' in X_2 or
- $v_2 = v_2'$ and v_1 is adjacent to v_1' in X_1.

Lemma 2.10 ([4, Lemma 5 on page 28]). *The Cartesian product of a path and a cycle is Hamiltonian.*

Corollary 2.11 (cf. [4, Corollary on page 29]). *The Cartesian product of two Hamiltonian graphs is Hamiltonian.*

Lemma 2.12 ([16, Lemma 2.27]). *Let* S *generate the finite group* G*, and let* $s \in S$*, such that* $\langle s \rangle \lhd G$*. If* $\mathrm{Cay}(G/\langle s \rangle; \overline{S})$ *has a Hamiltonian cycle, and either*

1. $s \in Z(G)$ *, or*
2. $Z(G) \cap \langle s \rangle = \{e\}$*,*

then $\mathrm{Cay}(G; S)$ *has a Hamiltonian cycle.*

2.3 Some facts from group theory

In this subsection, we state some facts in group theory, which are used to prove our main result. The following lemma often makes it possible to use Factor Group Lemma 2.6 for finding Hamiltonian cycles in connected Cayley graphs of G.

Lemma 2.13 ([6, Corollary 4.4]). *Assume $G = \langle a, b\rangle$ and G' is cyclic. Then $G' = \langle [a, b]\rangle$.*

Corollary 2.14. *Assume $G = \langle a, b\rangle$ and $\gcd(k, |a|) = 1$, where $k \in \mathbb{Z}$, and G' is cyclic. Then $G' = \langle [a^k, b]\rangle$.*

Proposition 2.15 ([12, Theorem 9.4.3 on page 146], cf. [10, Lemma 2.11]). *Assume $|G|$ is square-free. Then:*

1. *G' and G/G' are cyclic,*
2. *$Z(G) \cap G' = \{e\}$,*
3. *$G \cong C_n \ltimes G'$, for some $n \in \mathbb{Z}^+$,*
4. *If b and γ are elements of G such that $\langle bG'\rangle = G/G'$ and $\langle\gamma\rangle = G'$, then $\langle b, \gamma\rangle = G$, and there are integers m, n, and τ, such that $|\gamma| = m$, $|b| = n$, $b\gamma b^{-1} = \gamma^\tau$, $mn = |G|$, $\gcd(\tau - 1, m) = 1$, and $\tau^n \equiv 1 \pmod m$.*

Lemma 2.16. *Assume*

- *$G = (\mathcal{C}_p \times \mathcal{C}_q) \ltimes (\mathcal{C}_r \times \mathcal{C}_t)$,*
- *$G' = (\mathcal{C}_r \times \mathcal{C}_t)$,*
- *$\overline{a} \in \overline{G}$,*
- *p, q, r, and t are distinct primes.*

If $|\overline{a}| = pq$, then $|a| = pq$.

Proof. Suppose $|a| \neq pq$. Without loss of generality, assume $|a|$ is divisible by r. Then (after replacing a by a conjugate) the abelian group $\langle a\rangle$ contains $\mathcal{C}_p \times \mathcal{C}_q$ and $\mathcal{C}_r$, so $\mathcal{C}_r$ centralizes $\mathcal{C}_p \times \mathcal{C}_q$. Since $\mathcal{C}_r$ also centralizes $\mathcal{C}_t$, this implies that $\mathcal{C}_r \subseteq Z(G)$. This contradicts the fact that $G' \cap Z(G) = \{e\}$ (see Proposition 2.15(2)). □

Lemma 2.17 ([22, Exercise 19 on page 43]). *Assume $|G| = 2k$, where k is odd. Then G has a subgroup of index 2.*

Corollary 2.18. *Assume $|G| = 2k$, where k is odd. Then $|G'|$ is odd.*

Proof. By Lemma 2.17, there is a normal subgroup H of G such that $[G : H] = 2$. Now since G/H has order 2, then G/H is abelian, so $G' \subseteq H$. Therefore, $|G'|$ is odd. □

Notation 2.19. For τ as defined in Proposition 2.15(4), we use τ^{-1} to denote the inverse of τ modulo m (so $\tau^{-1} \equiv \tau^{n-1} \pmod m$).

2.4 Cayley graphs that contain a Hamiltonian cycle

We show, throughout this subsection, that there exists a Hamiltonian cycle in some connected Cayley graphs with additional assumptions. The following proposition shows that in our proof of Theorem 1.3 we can assume $|G|$ is square-free, since the cases where $|G|$ is not square-free have already been dealt with. At the end of this subsection we prove Proposition 1.4.

Proposition 2.20. *Assume:*

- $|G| = 6pq$, *where p and q are distinct prime numbers, and*
- $|G|$ *is not square-free (i.e. $\{p, q\} \cap \{2, 3\} \neq \varnothing$).*

Then every connected Cayley graph on G has a Hamiltonian cycle.

Proof. Without loss of generality we may assume $q \in \{2, 3\}$. Then $|G| \in \{12p, 18p\}$. Therefore, Theorem 1.2(1) applies. □

Proposition 2.21 ([25, Proposition 5.5])**.** *If n is divisible by at most 3 distinct primes, then every Cayley diagram (directed Cayley graph) in D_{2n} has a Hamiltonian cycle.*

The following proposition demonstrates that we can assume $|G'|$ in Theorem 1.3 is a product of two distinct prime numbers.

Proposition 2.22. *Assume $|G| = 2pqr$, where p, q and r are distinct odd prime numbers. If $|G'| \in \{1, pqr\}$ or $|G'|$ is prime, then every connected Cayley graph on G has a Hamiltonian cycle.*

Proof. If $|G'| = 1$, then $G' = \{e\}$. So G is an abelian group. Therefore, Lemma 2.2 applies. If $|G'|$ is prime, then Theorem 2.3 applies. Finally, if $|G'| = pqr$, then

$$G = \mathcal{C}_2 \ltimes (\mathcal{C}_p \times \mathcal{C}_q \times \mathcal{C}_r) \cong D_{2pqr}.$$

So Proposition 2.21 applies. □

The following lemmas show that some special Cayley graphs have a Hamiltonian cycle, and we use these facts in Section 3 in order to prove our main result.

Lemma 2.23. *Assume $G = (\mathcal{C}_2 \times \mathcal{C}_r) \ltimes G'$, and $G' = \mathcal{C}_p \times \mathcal{C}_q$, where p, q and r are distinct odd prime numbers and let $S = \{a, b\}$ be a generating set of G. Additionally, assume $|\overline{a}| \in \{2, 2r\}$, $|\overline{b}| = r$ and $\gcd(|b|, r-1) = 1$. Then $\mathrm{Cay}(G; S)$ contains a Hamiltonian cycle.*

Proof. We have $C = (\overline{b}^{r-1}, \overline{a}, \overline{b}^{-(r-1)}, \overline{a}^{-1})$ as a Hamiltonian cycle in $\mathrm{Cay}(\overline{G}; \overline{S})$. Now we calculate its voltage

$$\mathbb{V}(C) = b^{r-1} a b^{-(r-1)} a^{-1} = [b^{r-1}, a].$$

Since $\gcd(|b|, r-1) = 1$, then by Lemma 2.14 we have $[b^{r-1}, a] = G'$. Therefore, Factor Group Lemma 2.6 applies. □

Lemma 2.24 (cf. [10, Case 2 of proof of Theorem 1.1, pages 3619-3620])**.** *Assume*

- $G = (\mathcal{C}_2 \times \mathcal{C}_r) \ltimes (\mathcal{C}_p \times \mathcal{C}_q)$,
- $|S| = 3$,
- $\widehat{S}$ *is a minimal generating set of* $\widehat{G} = G/\mathcal{C}_p$,
- $\mathcal{C}_r$ *centralizes* $\mathcal{C}_q$,
- $\mathcal{C}_2$ *inverts* $\mathcal{C}_q$.

Then, $\mathrm{Cay}(G; S)$ *contains a Hamiltonian cycle.*

Lemma 2.25 ([10, Lemma 2.6])**.** *Assume:*

- $G = \langle a \rangle \ltimes \langle S_0 \rangle$, *where* $\langle S_0 \rangle$ *is an abelian subgroup of odd order,*
- $|(S_0 \cup S_0^{-1})| \geqslant 3$, *and*
- $\langle S_0 \rangle$ *has a nontrivial subgroup* H, *such that* $H \lhd G$ *and* $H \cap Z(G) = \{e\}$.

Then $\mathrm{Cay}(G; S_0 \cup \{a\})$ *has a Hamiltonian cycle.*

Lemma 2.26 ([10, Lemma 2.9])**.** *If* $G = D_{2pq} \times \mathcal{C}_r$*, where* p, q *and* r *are distinct odd primes, then every connected Cayley graph on* G *has a Hamiltonian cycle.*

Now we prove Proposition 1.4 which is on page 2.

Proof of Proposition 1.4. If $p \neq 7$ and $q \neq 7$, then Theorem 1.2(3) applies. So we may assume $q = 7$, which means $|G| = 49p$ (and $p \neq 7$). We may also assume that G is not abelian, for otherwise Lemma 2.2 applies.

If a Sylow p-subgroup P of G is normal, then $|G/P| = 49$, so the quotient G/P is abelian. (Because if q is prime, then every group of order q^2 is abelian). Therefore, since P is normal and G/P is abelian, then G' is contained in P. So $|G'| = p$. Therefore, Theorem 2.3 applies.

Now we may assume P is not normal in G. Then by Sylow's Theorem, $n_p | 49$ and $n_p \equiv 1 \pmod{p}$, where n_p is the number of Sylow p-subgroups in G. Thus, $p \in \{2, 3\}$, so $|G| \in \{14q, 21q\}$. Therefore, Theorem 1.2(1) applies. □

2.5 Some specific sets that generate G

This Subsection presents a few results that provide conditions under which certain 2-element subsets generate G. Obviously, no 3-element minimal generating set can contain any of these subsets.

Lemma 2.27. *Assume* $G = (\mathcal{C}_2 \times \mathcal{C}_3) \ltimes G'$, *and* $G' = \mathcal{C}_p \times \mathcal{C}_q$. *Also, assume* $C_{G'}(\mathcal{C}_3) = \mathcal{C}_q$ *and* $\mathcal{C}_q \nsubseteq C_{G'}(\mathcal{C}_2)$. *If* (a, b) *is one of the following ordered pairs*

1. $(a_3 a_q, a_2 a_3^j a_q^k \gamma_p)$,
2. $(a_2 a_3, a_3^j a_q^k \gamma_p)$, *where* $k \not\equiv 0 \pmod{q}$,
3. $(a_2 a_3 a_q, a_3^j a_q^k \gamma_p)$, *where* $k \not\equiv 0 \pmod{q}$,
4. $(a_2 a_3 a_q, a_2 a_3^j a_q^k \gamma_p)$, *where* $k \not\equiv 1 \pmod{q}$,

then $\langle a, b\rangle = G$.

Proof. It is easy to see that $(\overline{a}, \overline{b}) = \overline{G}$, so it suffices to show that $\langle a, b\rangle$ contains $\mathcal{C}_p$ and $\mathcal{C}_q$. Thus, it suffices to show that $\breve{G}$ and $\breve{G}$ are nonabelian, where $\breve{G} = G/(\mathcal{C}_3 \ltimes \mathcal{C}_p) \cong D_{2q}$ and $\breve{G} = G/C_q$.

Since a_3 does not centralize $\mathcal{C}_p$, it is clear in each of $(1) - (4)$ that $\breve{a}$ does not centralize γ_p (and γ_p is one of the factors in $\breve{b}$), so $\breve{G}$ is not abelian.

The pair $(\breve{a}, \breve{b})$ is either $(a_q, a_2a_q^k)$, (a_2, a_q^k) where $k \not\equiv 0 \pmod q$, (a_2a_q, a_q^k) where $k \not\equiv 0 \pmod q$, or $(a_2a_q, a_2a_q^k)$ where $k \not\equiv 1 \pmod q$. Each of these is either a reflection and a nontrivial rotation or two different reflections, and therefore generates the (nonabelian) dihedral group $D_{2q} = \breve{G}$. □

Lemma 2.28. *Assume $G = (\mathcal{C}_2 \times \mathcal{C}_3) \ltimes G'$, and $G' = \mathcal{C}_p \times \mathcal{C}_q$. Also, assume $C_{G'}(\mathcal{C}_3) = \{e\}$. If (a, b) is one of the following ordered pairs*

1. $(a_2a_3, a_2^i a_3^j a_q^k \gamma_p)$, *where* $k \not\equiv 0 \pmod q$,
2. $(a_3a_q, a_2a_3^j\gamma_p)$, *where* $j \not\equiv 0 \pmod 3$,
3. $(a_3, a_2a_3^j a_q^k\gamma_p)$, *where* $k \not\equiv 0 \pmod q$,
4. $(a_2a_3a_q, a_2^i a_3^j \gamma_p)$, *where* $j \not\equiv 0 \pmod 3$,

then $\langle a, b\rangle = G$.

Proof. It is easy to see that $(\overline{a}, \overline{b}) = \overline{G}$, so it suffices to show that $\langle a, b\rangle$ contains $\mathcal{C}_p$ and $\mathcal{C}_q$. we need to show that $\widehat{G}$ and $\breve{G}$ are nonabelian, where $\widehat{G} = G/\mathcal{C}_p$ and $\breve{G} = G/\mathcal{C}_q$, as usual.

As in the proof of Lemma 2.27, since a_3 does not centralize $\mathcal{C}_p$, it is clear in each of $(1) - (4)$ that $\breve{a}$ does not centralize γ_p (and γ_p is one of the factors in $\breve{b}$), so $\breve{G}$ is not abelian.

In $(1) - (4)$, a_q appears in one of the generators in $(\hat{a}, \widehat{b})$, but not the other, and the other generator does have an occurrence of a_3. Since a_3 does not centralize a_q, this implies that $\widehat{G}$ is not abelian. □

Lemma 2.29. *Assume $G = (\mathcal{C}_2 \times \mathcal{C}_q) \ltimes G'$, and $G' = \mathcal{C}_3 \times \mathcal{C}_p$. Also, assume $C_{G'}(\mathcal{C}_q) = \mathcal{C}_3$ and $\mathcal{C}_3 \nsubseteq C_{G'}(\mathcal{C}_2)$. If (a, b) is one of the following ordered pairs*

1. $(a_2a_q, a_2^i a_q^j a_3^k \gamma_p)$, *where* $k \not\equiv 0 \pmod q$,
2. $(a_qa_3, a_2a_q^j a_3^k\gamma_p)$,
3. $(a_2^i a_q^m a_3, a_2a_q^j\gamma_p)$, *where* $m \not\equiv 0 \pmod q$,

then $G = \langle a, b\rangle$.

Proof. It is easy to see that $(\overline{a}, \overline{b}) = \overline{G}$, so it suffices to show that $\langle a, b\rangle$ contains $\mathcal{C}_p$ and $\mathcal{C}_3$. We need to show that $\breve{G}$ and $\overleftrightarrow{G}$ are nonabelian, where $\breve{G} = G/(\mathcal{C}_q \ltimes \mathcal{C}_p) \cong D_6$ and $\overleftrightarrow{G} = G/\mathcal{C}_3$.

In each of $(1) - (4)$, a_q appears in $\overleftrightarrow{a}$, and γ_p appears in $\overleftrightarrow{b}$ (but not in $\overleftrightarrow{a}$). Since a_q does not centralize γ_p, this implies that $\overleftrightarrow{G}$ is not abelian.

In each of $(1) - (4)$, $(\overleftrightarrow{a}, \overleftrightarrow{b})$ consists of either a reflection and a nontrivial rotation or two different reflections, so it generates the (nonabelian) dihedral group $D_6 = \overleftrightarrow{G}$. □

3 Proof of the main result

In this section, we prove Theorem 1.3, which is the main result. We are given a generating set S of a finite group G of order $6pq$, where p and q are distinct prime numbers, and we wish to show $\mathrm{Cay}(G;S)$ contains a Hamiltonian cycle. The proof is a long case-by-case analysis (see Figures 1, 2 and 3 for outlines of the many cases that are considered). Here are our main assumptions throughout the whole section.

Assumption 3.1. *We assume:*

1. $p, q > 7$, *otherwise Theorem 1.2(1) applies.*
2. $|G|$ *is square-free, otherwise Proposition 2.20 applies.*
3. $G' \cap Z(G) = \{e\}$, *by Proposition 2.15(2).*
4. $G \cong \mathcal{C}_n \ltimes G'$, *by Proposition 2.15(3).*
5. $|G'| \in \{pq, 3p\}$, *by Corollary 2.18.*
6. *For every element* $\overline{s} \in \overline{S}$, $|\overline{s}| \neq 1$. *Otherwise, if* $|\overline{s}| = 1$, *then* $s \in G'$, *so* $G' = \langle s \rangle$ *or* $|s|$ *is prime. In each case* $\mathrm{Cay}(G/\langle s \rangle; \overline{S})$ *has a Hamiltonian cycle by part 2 or 3 of Theorem 1.2. By Assumption 3.1(3),* $\langle s \rangle \cap Z(G) = \{e\}$, *therefore, Lemma 2.12(2) applies.*
7. S *is a minimal generating set of* G. *Note that* S *must generate* G, *for otherwise* $\mathrm{Cay}(G;S)$ *is not connected. Also, in order to show that every connected Cayley graph on* G *contains a Hamiltonian cycle, it suffices to consider* $\mathrm{Cay}(G;S)$, *where* S *is a generating set that is minimal.*

3.1 Assume $|S| = 2$ and $G' = \mathcal{C}_p \times \mathcal{C}_q$

In this subsection, we prove the part of Theorem 1.3 where, $|S| = 2$ and $G' = \mathcal{C}_p \times \mathcal{C}_q$. Recall $\overline{G} = G/G'$ and $\widehat{G} = G/\mathcal{C}_p$.

I. $|S| = 2$
   - A. $G' = \mathcal{C}_p \times \mathcal{C}_q$ (Section 3.1).
     1. $\overline{S}$ is a minimal generating set.
     2. $\overline{S}$ is not a minimal generating set.
   - B. $G' = \mathcal{C}_3 \times \mathcal{C}_p$ (Section 3.2).
     1. $|\overline{a}| = |\overline{b}| = 2q$.
     2. $|\overline{a}| = q$.
     3. $|\overline{a}| = 2q$ and $|\overline{b}| = 2$.
     4. None of the previous cases apply.

Figure 1: Outline of the cases in the proof of Theorem 1.3 where $|S| = 2$

II. $|S| = 3$.

A. $G' = \mathcal{C}_p \times \mathcal{C}_q$.

a. $C_{G'}(\mathcal{C}_3) \neq \{e\}$ or $\widehat{S}$ is minimal.

i. $C_{G'}(\mathcal{C}_3) \neq \{e\}$ (Section 3.3).

1. $a = a_2$ and $b = a_q a_3$.
2. $a = a_2$ and $b = a_2 a_q a_3$.
3. $a = a_2 a_3$ and $b = a_2 a_q$.
4. $a = a_2 a_3$ and $b = a_q a_3$.
5. $a = a_2 a_3$ and $b = a_2 a_3 a_q$.

ii. $\widehat{S}$ is minimal (Section 3.4).

1. $C_{G'}(\mathcal{C}_2) = \mathcal{C}_p \times \mathcal{C}_q$.
2. $C_{G'}(\mathcal{C}_2) = \mathcal{C}_q$.
3. $C_{G'}(\mathcal{C}_2) = \mathcal{C}_p$.
4. $C_{G'}(\mathcal{C}_2) = \{e\}$.

b. $C_{G'}(\mathcal{C}_3) = \{e\}$ and $\widehat{S}$ is not minimal.

i. $C_{G'}(\mathcal{C}_2) = \mathcal{C}_p \times \mathcal{C}_q$ (Section 3.5).

1. $a = a_3$ and $b = a_2 a_q$.
2. $a = a_3$ and $b = a_2 a_3 a_q$.
3. $a = a_2 a_3$ and $b = a_3 a_q$.
4. $a = a_2 a_3$ and $b = a_2 a_q$.
5. $a = a_2 a_3$ and $b = a_2 a_3 a_q$.

ii. $C_{G'}(\mathcal{C}_2) \neq \{e\}$ (Section 3.6).

1. $a = a_2 a_3$ and $b = a_2 a_3 a_q$.
2. $a = a_2 a_3$ and $b = a_2 a_q$.
3. $a = a_2 a_3$ and $b = a_3 a_q$.
4. $a = a_3$ and $b = a_2 a_q$.

iii. $C_{G'}(\mathcal{C}_2) = \{e\}$ (Section 3.7).

1. $a = a_2 a_3$ and $b = a_2 a_3 a_q$.
2. $a = a_2 a_3$ and $b = a_2 a_q$.
3. $a = a_2 a_3$ and $b = a_3 a_q$.
4. $a = a_3$ and $b = a_2 a_q$.

B. $G' = \mathcal{C}_3 \times \mathcal{C}_p$. (Section 3.8).

1. $a = a_2 a_q$ and $b = a_2 a_q^m a_3$.
2. $a = a_2 a_q$ and $b = a_2 a_3$.
3. $a = a_2 a_q$ and $b = a_q^m a_3$.
4. $a = a_2$ and $b = a_q a_3$.

Figure 2: Outline of the cases in the proof of Theorem 1.3 where $|S| = 3$

III. $|S| \geqslant 4$ (Section 3.9). This part of the proof applies whenever $|G| = pqrt$ with p, q, r, and t distinct primes.

1. $|G'|$ has only two prime factors.
2. $|G'|$ has three prime factors.

Figure 3: Outline of the cases in the proof of Theorem 1.3 where $|S| \geqslant 4$

Proposition 3.2. *Assume*

- $G = (\mathcal{C}_2 \times \mathcal{C}_3) \ltimes (\mathcal{C}_p \times \mathcal{C}_q)$,
- $|S| = 2$.

Then $\mathrm{Cay}(G; S)$ *contains a Hamiltonian cycle.*

Proof. Let $S = \{a, b\}$. For every $s \in S$, $|\overline{s}| \neq 1$, by Assumption 3.1(6).

Case 1. Assume $\overline{S}$ is minimal. Then $|\overline{a}|, |\overline{b}| \in \{2, 3\}$. When $|\overline{a}| = |\overline{b}| = 2$ or $|\overline{a}| = |\overline{b}| = 3$, then $\overline{G} \neq \langle \overline{a}, \overline{b} \rangle$. Therefore, $G \neq \langle a, b \rangle$ which contradicts the fact that $G = \langle a, b \rangle$. So we may assume $|\overline{a}| = 2$ and $|\overline{b}| = 3$. Since $|b| \in \{3, 3p, 3q, 3pq\}$, then $\gcd(|b|, 2) = 1$. Thus, Lemma 2.23 applies.

Case 2. Assume $\overline{S}$ is not minimal. Then $\{|\overline{a}|, |\overline{b}|\}$ is either $\{6, 2\}$, $\{6, 3\}$, or $\{6\}$. We may assume $|\overline{a}| = 6$.

Subcase 2.1. Assume $|\overline{b}| = 2$. So we have $\overline{b} = \overline{a}^3$, then $b = a^3\gamma$, where $G' = \langle\gamma\rangle$ for otherwise $\langle a, b\rangle = \langle a, a^3\gamma\rangle = \langle a, \gamma\rangle \neq G$ which contradicts the fact that $G = \langle a, b\rangle$. Now by Proposition 2.15(4), we have $\tau \in \mathbb{Z}^+$ such that $a\gamma a^{-1} = \gamma^\tau$ and $\tau^6 \equiv 1 \pmod{pq}$, also $\gcd(\tau - 1, pq) = 1$. This implies that $\tau \not\equiv 1 \pmod{p}$ and $\tau \not\equiv 1 \pmod{q}$. We have $C_1 = (\overline{a}^2, \overline{b}, \overline{a}^{-2}, \overline{b}^{-1})$ as a Hamiltonian cycle in $\mathrm{Cay}(\overline{G}; \overline{S})$. Now we calculate its voltage.

$$\mathbb{V}(C_1) = a^2ba^{-2}b^{-1} = a^2a^3\gamma a^{-2}\gamma^{-1}a^{-3} = \gamma^{\tau^5 - \tau^3} = \gamma^{\tau^3(\tau^2-1)}.$$

We may assume $\gcd(\tau^2 - 1, pq) \neq 1$ for otherwise Factor Group Lemma 2.6 applies. Without loss of generality let $\tau^2 \equiv 1 \pmod{q}$, then $\tau \equiv -1 \pmod{q}$. We may assume $\tau \not\equiv -1 \pmod{p}$, for otherwise $G \cong D_{2pq} \times \mathcal{C}_3$, so Lemma 2.26 applies.

Consider $\widehat{G} = G/\mathcal{C}_p = \mathcal{C}_6 \ltimes \mathcal{C}_q$. Since $|\overline{a}| = 6$, then by Lemma 2.16 $|a| = 6$, so $|\widehat{a}| = 6$. We may assume $|\widehat{b}| = 2$, for otherwise Corollary 2.7 applies with $s = b$ and $t = b^{-1}$ since $\langle\widehat{a}\rangle \neq \widehat{G}$, so any Hamiltonian cycle must use an edge labeled $\widehat{b}$. Thus, $\widehat{b} = \widehat{a}^3 a_q$, where $\langle a_q\rangle = \mathcal{C}_q$. Since $\tau \equiv -1 \pmod{q}$, then $\mathcal{C}_3$ centralizes $\mathcal{C}_q$ and $\mathcal{C}_2$ inverts $\mathcal{C}_q$. Therefore, $\widehat{G} \cong D_{2q} \times \mathcal{C}_3$. Now we have

$$C_2 = ((\widehat{a}^5, \widehat{b}, \widehat{a}^{-5}, \widehat{b})^{(q-3)/2}, (\widehat{a}^5, \widehat{b})^3)$$

as a Hamiltonian cycle in $\mathrm{Cay}(\widehat{G}; \widehat{S})$. The picture in Figure 4 on page 13 shows the Hamiltonian cycle when $q = 7$. If in C_2 we change one occurrence of $(\widehat{a}^5, \widehat{b}, \widehat{a}^{-5}, \widehat{b})$ to $(\widehat{a}^{-5}, \widehat{b}, \widehat{a}^5, \widehat{b})$ we have another Hamiltonian cycle. Note that,

$$a^5ba^{-5}b = a^5 \cdot a^3\gamma \cdot a^{-5} \cdot a^3\gamma = a^2\gamma a^{-2}\gamma = \gamma^{\tau^2+1},$$

and

$$a^{-5}ba^5b = a^{-5} \cdot a^3\gamma \cdot a^5 \cdot a^3\gamma = a^{-2}\gamma a^2\gamma = \gamma^{\tau^{-2}+1}.$$

Since $\tau^4 \not\equiv 0 \pmod{p}$ we see that $\tau^2 + 1 \not\equiv \tau^{-2} + 1 \pmod{p}$. Therefore, the voltages of these two Hamiltonian cycles are different, so one of these Hamiltonian cycles has a nontrivial voltage. Thus, Factor Group Lemma 2.6 applies.

Subcase 2.2. Assume $|\overline{b}| = 3$. Since $|\overline{b}| = 3$, then $|b| \in \{3, 3p, 3q, 3pq\}$. Since $|\overline{a}| = 6$, then by 2.16 $|a| = 6$. Since $\gcd(|b|, 2) = 1$, then Lemma 2.23 applies.

Subcase 2.3. Assume $|\overline{b}| = 6$. Then we have $\overline{a} = \overline{b}$ or $\overline{a} = \overline{b}^{-1}$. Additionally, by Lemma 2.16 we have $|a| = |b| = 6$. We may assume $\overline{a} = \overline{b}$ by replacing b with its inverse if necessary. Then $b = a\gamma$, where $G' = \langle\gamma\rangle$, because $G = \langle a, b\rangle$. We have $C = (\overline{a}^5, \overline{b})$ as a Hamiltonian cycle in $\mathrm{Cay}(\overline{G}, \overline{S})$. Now we calculate its voltage

$$\mathbb{V}(C) = a^5b = a^5a\gamma = a^6\gamma = \gamma$$

which generates G'. Therefore, Factor Group Lemma 2.6 applies. □

3.2 Assume $|S| = 2$ and $G' = \mathcal{C}_3 \times \mathcal{C}_p$

In this subsection, we prove the part of Theorem 1.3 where, $|S| = 2$ and $G' = \mathcal{C}_3 \times \mathcal{C}_p$. Recall $\overline{G} = G/G'$ and $\widehat{G} = G/\mathcal{C}_p$.

Proposition 3.3. *Assume*

- $G = (\mathcal{C}_2 \times \mathcal{C}_q) \ltimes (\mathcal{C}_3 \times \mathcal{C}_p)$,
- $|S| = 2$.

Then $\mathrm{Cay}(G; S)$ *contains a Hamiltonian cycle.*

Figure 4: The Hamiltonian cycle C_1: $\hat{a}$ edges are solid and $\hat{b}$ edges are dashed.

Proof. Let $S = \{a, b\}$. Since the only non-trivial automorphism of $\mathcal{C}_3$ is inversion, $\mathcal{C}_q$ centralizes $\mathcal{C}_3$. Since $G' \cap Z(G) = \{e\}$ (see Proposition 2.15(4)), $\mathcal{C}_2$ does not centralize $\mathcal{C}_3$.

Case 1. Assume $|\overline{a}| = |\overline{b}| = 2q$. Then $\overline{b} = \overline{a}^m$, where $1 \leqslant m \leqslant q-1$ by replacing b with its inverse if needed. Therefore, $b = a^m\gamma$, where $G' = \langle \gamma \rangle$. Also, $\gcd(m, 2q) = 1$. So, by Proposition 2.15(4) we have $a\gamma a^{-1} = \gamma^\tau$ where $\tau^{2q} \equiv 1 \pmod{3p}$ and $\gcd(\tau - 1, 3p) = 1$. Consider $\overline{G} = \mathcal{C}_{2q}$.

Subcase 1.1. Assume $m > 3$. Then we have

$$C = (\overline{b}^{-2}, \overline{a}^{-2}, \overline{b}, \overline{a}, \overline{b}, \overline{a}^{-(m-2)}, \overline{b}^{-1}, \overline{a}^{m-4}, \overline{b}^{-1}, \overline{a}^{-(2q-2m-3)})$$

as a Hamiltonian cycle in $\mathrm{Cay}(\overline{G}; \overline{S})$. Now we calculate its voltage.

$$\begin{aligned}
\mathbb{V}(C) &= b^{-2}a^{-2}baba^{-(m-2)}b^{-1}a^{m-4}b^{-1}a^{-(2q-2m-3)} \\
&= \gamma^{-1}a^{-m}\gamma^{-1}a^{-m}a^{-2}a^{m}\gamma a a^{m}\gamma a^{-m+2}\gamma^{-1}a^{-m}a^{m-4}\gamma^{-1}a^{-m}a^{-2q+2m+3} \\
&= \gamma^{-1}a^{-m}\gamma^{-1}a^{-2}\gamma a^{m+1}\gamma a^{-m+2}\gamma^{-1}a^{-4}\gamma^{-1}a^{m+3} \\
&= \gamma^{-1-\tau^{-m}+\tau^{-m-2}+\tau^{-1}-\tau^{-m+1}-\tau^{-m-3}} \\
&= \gamma^{-1+\tau^{-1}-\tau^{-m+1}-\tau^{-m}+\tau^{-m-2}-\tau^{-m-3}}.
\end{aligned}$$

We may assume $\mathbb{V}(C)$ does not generate $G' = \mathcal{C}_3 \times \mathcal{C}_p$. Therefore, the subgroup generated by $\mathbb{V}(C)$ either does not contain $\mathcal{C}_3$, or does not contain $\mathcal{C}_p$. We already know $\tau \equiv -1 \pmod 3$, then we have

$$-1 + \tau^{-1} - \tau^{-m+1} - \tau^{-m} + \tau^{-m-2} - \tau^{-m-3} \equiv -4 = -1 \pmod 3.$$

This implies that the subgroup generated by $\mathbb{V}(C)$ contains $\mathcal{C}_3$. So we may assume the subgroup generated by $\mathbb{V}(C)$ does not contain $\mathcal{C}_p$, then

$$0 \equiv -1 + \tau^{-1} - \tau^{-m+1} - \tau^{-m} + \tau^{-m-2} - \tau^{-m-3} \pmod p. \tag{1.1A}$$

Multiplying by $-\tau^{m+3}$ we have

$$0 \equiv \tau^{m+3} - \tau^{m+2} + \tau^4 + \tau^3 - \tau + 1 \pmod{p}. \tag{1.1B}$$

Replacing $\{\overline{a}, \overline{b}\}$ with $\{\overline{a}^{-1}, \overline{b}^{-1}\}$ replaces τ with τ^{-1}. Therefore, applying the above argument to $\{\overline{a}^{-1}, \overline{b}^{-1}\}$ establishes that 1.1A holds with τ^{-1} in the place of τ, which means we have

$$0 \equiv -\tau^{m+3} + \tau^{m+2} - \tau^m - \tau^{m-1} + \tau - 1 \pmod{p}. \tag{1.1C}$$

By adding 1.1B and 1.1C we have

$$0 \equiv -\tau^m - \tau^{m-1} + \tau^4 + \tau^3 = \tau^3(\tau+1)(1-\tau^{m-4}) \pmod{p}.$$

If $\tau \equiv -1 \pmod{p}$, then $\mathcal{C}_{2q}$ inverts $\mathcal{C}_{3p}$, so $\mathcal{C}_q$ centralizes $\mathcal{C}_p$. This implies that $G \cong D_{6p} \times \mathcal{C}_q$, so Lemma 2.26 applies. The only other possibility is $\tau^{m-4} \equiv 1 \pmod{p}$. Multiplying by τ^4, we have $\tau^m \equiv \tau^4 \pmod{p}$. We also know that $\tau^{2q} \equiv 1 \pmod{p}$. So $\tau^d \equiv 1 \pmod{p}$, where $d = \gcd(m-4, 2q)$. Since m is odd and $m < q$, then $d = 1$. This contradicts the fact that $\gcd(\tau - 1, 3p) = 1$.

Subcase 1.2. Assume $m \leqslant 3$. Therefore, either $m = 1$ or $m = 3$. If $m = 1$, then $\overline{a} = \overline{b}$ and $b = a\gamma$. So we have $C_1 = (\overline{a}^{2q-1}, \overline{b})$ as a Hamiltonian cycle in $\mathrm{Cay}(\overline{G}; \overline{S})$. Now we calculate its voltage.

$$\mathbb{V}(C_1) = a^{2q-1}b = a^{2q-1}a\gamma = \gamma$$

which generates G'. Therefore, Factor Group Lemma 2.6 applies. Now if $m = 3$, then $b = a^3\gamma$ and we have

$$C_2 = (\overline{b}^2, \overline{a}^{-1}, \overline{b}^{-1}, \overline{a}^{-1}, \overline{b}^3, \overline{a}^{-2}, \overline{b}, \overline{a}^{2q-11})$$

as a Hamiltonian cycle in $\mathrm{Cay}(\overline{G}; \overline{S})$. We calculate its voltage.

$$\begin{aligned}
\mathbb{V}(C_2) &= b^2a^{-1}b^{-1}a^{-1}b^3a^{-2}ba^{2q-11} \\
&= a^3\gamma a^3\gamma a^{-1}\gamma^{-1}a^{-3}a^{-1}a^3\gamma a^3\gamma a^3\gamma a^{-2}a^3\gamma a^{-11} \\
&= a^3\gamma a^3\gamma a^{-1}\gamma^{-1}a^{-1}\gamma a^3\gamma a^3\gamma a\gamma a^{-11} \\
&= \gamma^{\tau^3+\tau^6-\tau^5+\tau^4+\tau^7+\tau^{10}+\tau^{11}} \\
&= \gamma^{\tau^{11}+\tau^{10}+\tau^7+\tau^6-\tau^5+\tau^4+\tau^3}
\end{aligned}$$

We may assume $\mathbb{V}(C_2)$ does not generate $G' = \mathcal{C}_3 \times \mathcal{C}_p$. Therefore, the subgroup generated by $\mathbb{V}(C)$ does not contain either $\mathcal{C}_3$, or $\mathcal{C}_p$. We already know $\tau \equiv -1 \pmod{3}$, then

$$\tau^{11} + \tau^{10} + \tau^7 + \tau^6 - \tau^5 + \tau^4 + \tau^3 \equiv -1 + 1 - 1 + 1 + 1 + 1 - 1 = 1 \pmod{3}.$$

This implies that the subgroup generated by $\mathbb{V}(C_2)$ contains $\mathcal{C}_3$. So we may assume the subgroup generated by $\mathbb{V}(C_2)$ does not contain $\mathcal{C}_p$, for otherwise Factor Group Lemma 2.6 applies. Then we have

$$0 \equiv \tau^{11} + \tau^{10} + \tau^7 + \tau^6 - \tau^5 + \tau^4 + \tau^3 \pmod{p}$$

$$= \tau^3(\tau^8 + \tau^7 + \tau^4 + \tau^3 - \tau^2 + \tau + 1).$$

This implies that

$$0 \equiv \tau^8 + \tau^7 + \tau^4 + \tau^3 - \tau^2 + \tau + 1 \pmod{p}. \tag{1.2A}$$

We can replace τ with τ^{-1} in the above equation, by replacing $\{\overline{a}, \overline{b}\}$ with $\{\overline{a}^{-1}, \overline{b}^{-1}\}$ if necessary. Then we have

$$0 \equiv \tau^{-8} + \tau^{-7} + \tau^{-4} + \tau^{-3} - \tau^{-2} + \tau^{-1} + 1 \pmod{p}.$$

Multiplying τ^8, then we have

$$\begin{aligned} 0 &\equiv 1 + \tau + \tau^4 + \tau^5 - \tau^6 + \tau^7 + \tau^8 \pmod{p} \\ &= \tau^8 + \tau^7 - \tau^6 + \tau^5 + \tau^4 + \tau + 1. \end{aligned}$$

Now by subtracting the above equation from 1.2A we have

$$\begin{aligned} 0 &\equiv \tau^6 - \tau^5 + \tau^3 - \tau^2 \pmod{p} \\ &= \tau^2(\tau - 1)(\tau^3 + 1). \end{aligned}$$

This implies that $\tau \equiv 1 \pmod{p}$ or $\tau^3 \equiv -1 \pmod{p}$. If $\tau \equiv 1 \pmod{p}$, then it contradicts the fact that $\gcd(\tau - 1, 3p) = 1$. Now if $\tau^3 \equiv -1 \pmod{p}$, then $\tau^6 \equiv 1 \pmod{p}$. We already know $\tau^{2q} \equiv 1 \pmod{p}$. Then $\tau^d \equiv 1 \pmod{p}$, where $d = \gcd(2q, 6)$. Since $\gcd(2, 6) = 2$ and $\gcd(q, 6) = 1$, then $d = 2$. This implies that $\tau^2 \equiv 1 \pmod{p}$, which means $\mathcal{C}_q$ centralizes $\mathcal{C}_p$. Then we have

$$G = \mathcal{C}_q \times (\mathcal{C}_2 \ltimes \mathcal{C}_{3p}) \cong \mathcal{C}_q \times D_{6p}.$$

So Lemma 2.26 applies.

Case 2. Assume $|\overline{a}| = q$. Then $|\overline{b}| \in \{2, 2q\}$. Thus $|b| \in \{2, 2q, 2p, 2pq\}$. If $|b| = 2pq$, then $\mathcal{C}_q$ centralizes $\mathcal{C}_p$. This implies that

$$G = \mathcal{C}_q \times (\mathcal{C}_2 \ltimes \mathcal{C}_{3p}) \cong \mathcal{C}_q \times D_{6p}$$

so, Lemma 2.26 applies. Therefore, we may assume $\mathcal{C}_q$ does not centralize $\mathcal{C}_p$, so $|a|$ is not divisible by p. If $|b| = 2p$, then Corollary 2.7 applies with $s = b$ and $t = b^{-1}$, because we have a Hamiltonian cycle in $\operatorname{Cay}(\widehat{G}; \widehat{S})$ by Theorem 1.2(3). Since b is the only generator whose order is even, then any Hamiltonian cycle in $\operatorname{Cay}(\widehat{G}; \widehat{S})$ must use some edge labeled $\widehat{b}$.

We may now assume $|b| \in \{2, 2q\}$. We have $C = (\overline{a}^{q-1}, \overline{b}, \overline{a}^{-(q-1)}, \overline{b}^{-1})$ as a Hamiltonian cycle in $\operatorname{Cay}(\overline{G}; \overline{S})$. Now if $|a| = q$, then by Lemma 2.14 we have $G' = \langle [a^{q-1}, b] \rangle$. Therefore, Factor Group Lemma 2.6 applies. So, we may assume $|a| = 3q$. Since $\mathcal{C}_q$ does not centralize $\mathcal{C}_p$, then after conjugation we can assume $a = a_3 a_q$ and $b = a_2 a_q^j \gamma_p$, where $0 \leqslant j \leqslant q - 1$. We already know that C is a Hamiltonian cycle in $\operatorname{Cay}(\overline{G}; \overline{S})$. So we can assume $\gcd(3q, q - 1) \neq 1$ for otherwise Lemma 2.14 applies, which implies that Factor Group Lemma 2.6 applies. This implies that $\gcd(3, q - 1) \neq 1$ which means $q \equiv 1 \pmod{3}$.

Figure 5: The Hamiltonian cycle C_1: $\hat{a}$ edges are solid and $\hat{b}$ edges are dashed.

Figure 6: The Hamiltonian cycle C_2: $\hat{a}$ edges are solid and $\hat{b}$ edges are dashed.

Consider $\widehat{G} = G/\mathcal{C}_p$. Then $\hat{a} = a_3 a_q$ and $\hat{b} = a_2 a_q^j$. Therefore, there exists $0 \leqslant k \leqslant 3q-1$ such that $\hat{b}^{-1}\hat{a}\hat{b} = \hat{a}^k$. Since $\hat{b}$ inverts a_3 and centralizes a_q, then we must have $\hat{a} = \hat{b}\hat{a}^k\hat{b}^{-1} = a_3^{-k}a_q^k$, so $k \equiv -1 \pmod 3$ and $k \equiv 1 \pmod q$. Since $q \equiv 1 \pmod 3$, then $k = q+1$. Additionally, we have $a\gamma_p a^{-1} = \gamma_p^{\hat{\tau}}$, where $\hat{\tau}^q \equiv 1 \pmod p$. We also have $\hat{\tau} \not\equiv 1 \pmod p$, because $\mathcal{C}_q$ does not centralize $\mathcal{C}_p$. Now we have

$$b^{-1}ab = \gamma_p^{-1}a_q^{-j}a_2 a a_2 a_q^j \gamma_p = \gamma_p^{-1}a^{q+1}\gamma_p.$$

This implies that

$$b^{-1}a^i b = (b^{-1}ab)^i = (\gamma_p^{-1}a^{q+1}\gamma_p)^i = \gamma_p^{-1}a^{i(q+1)}\gamma_p.$$

Therefore,

$$b^{-1}a^i b = \gamma_p^{-1}a^{i(q+1)}\gamma_p \equiv \gamma_p^{-1}a^i\gamma_p \pmod{\mathcal{C}_3}.$$

We have

$$\begin{aligned} C_1 = (&\hat{a}^{q-3}, \hat{b}^{-1}, \hat{a}^{-(q-2)}, \hat{b}, \hat{a}^{-1}, \hat{b}^{-1}, \hat{a}, \hat{b}, \hat{a}^{q-2}, \hat{b}^{-1}, \\ &\hat{a}^{-(q-3)}, \hat{b}, \hat{a}^{q-2}, \hat{b}^{-1}, \hat{a}, \hat{b}, \hat{a}^{-1}, \hat{b}^{-1}, \hat{a}^{-(q-2)}, \hat{b}) \end{aligned}$$

as our first Hamiltonian cycle in $\mathrm{Cay}(\widehat{G}; \widehat{S})$. The picture in Figure 5 on page 16 shows the Hamiltonian cycle. In addition,

$$\begin{aligned} C_2 = (&\hat{a}^{q-1}, \hat{b}^{-1}, \hat{a}^{-(q-3)}, \hat{b}, \hat{a}^{-1}, \hat{b}^{-1}, \hat{a}^{q-2}, \hat{b}, \hat{a}, \hat{b}^{-1}, \hat{a}^2, \hat{b}, \\ &\hat{a}^{q-4}, \hat{b}^{-1}, \hat{a}^{-(q-5)}, \hat{b}, \hat{a}^{q-4}, \hat{b}^{-1}, \hat{a}, \hat{b}, \hat{a}, \hat{b}^{-1}, \hat{a}^{-1}, \hat{b}) \end{aligned}$$

is the second Hamiltonian cycle in $\mathrm{Cay}(\widehat{G}; \widehat{S})$. The picture in Figure 6 on page 16 shows the Hamiltonian cycle. We calculate the voltage of C_1 in $\overleftrightarrow{G} = G/\mathcal{C}_3$. Since $a^q \equiv e \pmod{\mathcal{C}_3}$, we have

$$\begin{aligned} \mathbb{V}(C_1) &\equiv a^{-3}(b^{-1}a^2b)a^{-1}(b^{-1}ab)a^{-2}(b^{-1}a^3b)a^{-2}(b^{-1}ab)a^{-1}(b^{-1}a^2b) \pmod{\mathcal{C}_3} \\ &= a^{-3}(\gamma_p^{-1}a^2\gamma_p)a^{-1}(\gamma_p^{-1}a\gamma_p)a^{-2}(\gamma_p^{-1}a^3\gamma_p)a^{-2}(\gamma_p^{-1}a\gamma_p)a^{-1}(\gamma_p^{-1}a^2\gamma_p) \\ &= a^{-3}(\gamma_p^{\hat{\tau}^2-1}a^2)a^{-1}(\gamma_p^{\hat{\tau}-1}a)a^{-2}(\gamma_p^{\hat{\tau}^3-1}a^3)a^{-2}(\gamma_p^{\hat{\tau}-1}a)a^{-1}(\gamma_p^{\hat{\tau}^2-1}a^2) \\ &= a^{-3}\gamma_p^{\hat{\tau}^2-1}a\gamma_p^{\hat{\tau}-1}a^{-1}\gamma_p^{\hat{\tau}^3-1}a\gamma_p^{\hat{\tau}^2+\hat{\tau}-2}a^2 \\ &= \gamma_p^{\hat{\tau}^{-3}(\hat{\tau}^2-1)+\hat{\tau}^{-2}(\hat{\tau}-1)+\hat{\tau}^{-3}(\hat{\tau}^3-1)+\hat{\tau}^{-2}(\hat{\tau}^2+\hat{\tau}-2)} \\ &= \gamma_p^{-2\hat{\tau}^{-3}-3\hat{\tau}^{-2}+3\hat{\tau}^{-1}+2}. \end{aligned}$$

We may assume this does not generate $\mathcal{C}_p$, so

$$0 \equiv -2\hat{\tau}^{-3} - 3\hat{\tau}^{-2} + 3\hat{\tau}^{-1} + 2 \pmod p.$$

Multiplying by $\hat{\tau}^3$, we have

$$0 \equiv 2\hat{\tau}^3 + 3\hat{\tau}^2 - 3\hat{\tau} - 2 = (\hat{\tau} - 1)(\hat{\tau} + 2)(2\hat{\tau} + 1) \pmod{p}.$$

Since $\hat{\tau} \not\equiv 1 \pmod{p}$, then we may assume $\hat{\tau} \equiv -2 \pmod{p}$, by replacing $\hat{a}$ with $\hat{a}^{-1}$ if needed.

Now we calculate the voltage of C_2 in $\overleftrightarrow{G} = G/\mathcal{C}_3$.

$$\begin{aligned}
\mathbb{V}(C_2) &\equiv a^{-1}(b^{-1}a^3b)a^{-1}(b^{-1}a^{-2}b)a(b^{-1}a^2b) \\
&\qquad \cdot a^{-4}(b^{-1}a^5b)a^{-4}(b^{-1}ab)a(b^{-1}a^{-1}b) \pmod{\mathcal{C}_3} \\
&= a^{-1}(\gamma_p^{-1}a^3\gamma_p)a^{-1}(\gamma_p^{-1}a^{-2}\gamma_p)a(\gamma_p^{-1}a^2\gamma_p) \\
&\qquad \cdot a^{-4}(\gamma_p^{-1}a^5\gamma_p)a^{-4}(\gamma_p^{-1}a\gamma_p)a(\gamma_p^{-1}a^{-1}\gamma_p) \\
&= a^{-1}(\gamma_p^{\hat{\tau}^3-1}a^3)a^{-1}(\gamma_p^{\hat{\tau}^{-2}-1}a^{-2})a(\gamma_p^{\hat{\tau}^2-1}a^2) \\
&\qquad \cdot a^{-4}(\gamma_p^{\hat{\tau}^5-1}a^5)a^{-4}(\gamma_p^{\hat{\tau}-1}a)a(\gamma_p^{\hat{\tau}^{-1}-1}a^{-1}) \\
&= a^{-1}\gamma_p^{\hat{\tau}^3-1}a^2\gamma_p^{\hat{\tau}^{-2}-1}a^{-1}\gamma_p^{\hat{\tau}^2-1}a^{-2}\gamma_p^{\hat{\tau}^5-1}a\gamma_p^{\hat{\tau}-1}a^2\gamma_p^{\hat{\tau}^{-1}-1}a^{-1} \\
&= \gamma_p^{\hat{\tau}^{-1}(\hat{\tau}^3-1)+\hat{\tau}(\hat{\tau}^{-2}-1)+\hat{\tau}^2-1+\hat{\tau}^{-2}(\hat{\tau}^5-1)+\hat{\tau}^{-1}(\hat{\tau}-1)+\hat{\tau}(\hat{\tau}^{-1}-1)} \\
&= \gamma_p^{\hat{\tau}^3+2\hat{\tau}^2-2\hat{\tau}+1-\hat{\tau}^{-1}-\hat{\tau}^{-2}}.
\end{aligned}$$

We may assume this does not generate $\mathcal{C}_p$, so

$$0 \equiv \hat{\tau}^3 + 2\hat{\tau}^2 - 2\hat{\tau} + 1 - \hat{\tau}^{-1} - \hat{\tau}^{-2} \pmod{p}.$$

Multiplying by $\hat{\tau}^2$, we have

$$0 \equiv \hat{\tau}^5 + 2\hat{\tau}^4 - 2\hat{\tau}^3 + \hat{\tau}^2 - \hat{\tau} - 1 \pmod{p}.$$

We already know $\hat{\tau} \equiv -2 \pmod{p}$. By substituting this in the equation above, we have

$$0 \equiv (-2)^5 + 2(-2)^4 - 2(-2)^3 + (-2)^2 - (-2) - 1 = 21 = 3 \cdot 7 \pmod{p}.$$

Since $p > 7$, then $21 \not\equiv 0 \pmod{p}$. This is a contradiction.

Case 3. Assume $|\overline{a}| = 2q$ and $|\overline{b}| = 2$. Since $|\overline{a}| = 2q$, then by Lemma 2.16 $|a| = 2q$. We have $b = a^q\gamma$ where $G' = \langle\gamma\rangle$.

By Proposition 2.15(4) we have $a\gamma a^{-1} = \gamma^\tau$, where $\tau^{2q} \equiv 1 \pmod{3p}$ and $\gcd(\tau - 1, 3p) = 1$. This implies that $\tau \not\equiv 0, 1 \pmod{p}$ and $\tau \equiv -1 \pmod{3}$.

Suppose, for the moment, that $\tau \equiv -1 \pmod{p}$. Then $G \cong D_{6p} \times \mathcal{C}_q$, so $\mathrm{Cay}(G; S)$ has a Hamiltonian cycle by Lemma 2.26.

We may now assume that $\tau \not\equiv -1 \pmod{p}$. Recall that $\widehat{G} = G/\mathcal{C}_p = \mathcal{C}_{2q} \ltimes \mathcal{C}_3$. We may assume $\hat{a} = a_2a_q$ and $\hat{b} = a_2a_3$. We have

$$\begin{aligned}
C_1 = ((&\hat{a}, \hat{b}, \hat{a}, \hat{b}, \hat{a}^{-1}, \hat{b}, \hat{a}, \hat{b}, \hat{a}^{-1}, \hat{b}, \hat{a}, \hat{b})^{(q-5)/2}, \hat{a}, \hat{b}, \hat{a}^4, \\
&\hat{b}, \hat{a}^{-3}, \hat{b}, \hat{a}^{-1}, \hat{b}, \hat{a}^2, \hat{b}, \hat{a}^2, \hat{b}, \hat{a}^{-1}, \hat{b}, \hat{a}^{-3}, \hat{b}, \hat{a}^4, \hat{b})
\end{aligned}$$

as the first Hamiltonian cycle in $\mathrm{Cay}(\widehat{G}; \widehat{S})$. The picture in Figure 7 on page 18 shows the Hamiltonian cycle. We also have

$$C_2 = ((\hat{a}, \hat{b}, \hat{a}^{-1}, \hat{b}, \hat{a}, \hat{b})^{q-5}, \hat{a}^3, \hat{b}, \hat{a}^2, \hat{b}, \hat{a}^{-1}, \hat{b}, \hat{a}^{-3}, \hat{b}, \hat{a}^3, \hat{b}, \hat{a}^{-3}, \hat{b}, \hat{a}^{-1}, \hat{b}, \hat{a}^2, \hat{b}, \hat{a}^3, \hat{b})$$

Figure 7: The Hamiltonian cycle C_1: $\hat{a}$ edges are solid and $\hat{b}$ edges are dashed.

as the second Hamiltonian cycle in $\mathrm{Cay}(\widehat{G};\widehat{S})$. The picture in Figure 8 on page 19 shows the Hamiltonian cycle. Now we calculate the voltage of C_1.

$$\begin{aligned}
\mathbb{V}(C_1) &= ((ababa^{-1}b)(aba^{-1}bab))^{(q-5)/2}(aba^4ba^{-3}ba^{-1}ba^2ba^2ba^{-1}ba^{-3}ba^4b) \\
&= ((aa^q\gamma aa^q\gamma a^{-1}a^q\gamma)(aa^q\gamma a^{-1}a^q\gamma aa^q\gamma))^{(q-5)/2} \\
&\qquad \cdot (aa^q\gamma a^4a^q\gamma a^{-3}a^q\gamma a^{-1}a^q\gamma a^2a^q\gamma a^2a^q\gamma a^{-1}a^q\gamma a^{-3}a^q\gamma a^4a^q\gamma) \\
&= ((a^{q+1}\gamma a^{q+1}\gamma a^{q-1}\gamma)(a^{q+1}\gamma a^{q-1}\gamma a^{q+1}\gamma))^{(q-5)/2} \\
&\qquad \cdot (a^{q+1}\gamma a^{q+4}\gamma a^{q-3}\gamma a^{q-1}\gamma a^{q+2}\gamma a^{q+2}\gamma a^{q-1}\gamma a^{q-3}\gamma a^{q+4}\gamma) \\
&= ((\gamma^{\tau^{q+1}+\tau^2+\tau^{q+1}}a^{q+1})(\gamma^{\tau^{q+1}+1+\tau^{q+1}}a^{q+1}))^{(q-5)/2} \\
&\qquad \cdot (\gamma^{\tau^{q+1}+\tau^5+\tau^{q+2}+\tau+\tau^{q+3}+\tau^5+\tau^{q+4}+\tau+\tau^{q+5}}a^{q+5}) \\
&= ((\gamma^{2\tau^{q+1}+\tau^2}a^{q+1})(\gamma^{2\tau^{q+1}+1}a^{q+1}))^{(q-5)/2} \\
&\qquad \cdot (\gamma^{\tau^{q+5}+\tau^{q+4}+\tau^{q+3}+\tau^{q+2}+\tau^{q+1}+2\tau^5+2\tau}a^{q+5}) \\
&= ((\gamma^{2\tau^{q+1}+\tau^2+\tau^{q+1}(2\tau^{q+1}+1)}a^2))^{(q-5)/2} \\
&\qquad \cdot (\gamma^{\tau^{q+5}+\tau^{q+4}+\tau^{q+3}+\tau^{q+2}+\tau^{q+1}+2\tau^5+2\tau}a^{q+5}) \\
&= (\gamma^{3\tau^{q+1}+3\tau^2}a^2)^{(q-5)/2}(\gamma^{\tau^{q+5}+\tau^{q+4}+\tau^{q+3}+\tau^{q+2}+\tau^{q+1}+2\tau^5+2\tau}a^{q+5}) \\
&= (\gamma^{(3\tau^{q+1}+3\tau^2)(\tau^{q-5}-1)/(\tau^2-1)}a^{q-5})(\gamma^{\tau^{q+5}+\tau^{q+4}+\tau^{q+3}+\tau^{q+2}+\tau^{q+1}+2\tau^5+2\tau}a^{q+5}) \\
&= \gamma^{(3\tau^{q+1}+3\tau^2)(\tau^{q-5}-1)/(\tau^2-1)+\tau^{q-5}(\tau^{q+5}+\tau^{q+4}+\tau^{q+3}+\tau^{q+2}+\tau^{q+1}+2\tau^5+2\tau)}.
\end{aligned}$$

Since $\tau^{2q} \equiv 1 \pmod p$, we have $\tau^q \equiv \pm 1 \pmod p$.

Let us now consider the case where $\tau^q \equiv 1 \pmod p$, then by substituting this in the formula for the voltage of C_1 we have

$$\begin{aligned}
\mathbb{V}(C_1) &= \gamma^{(3\tau+3\tau^2)(\tau^{-5}-1)/(\tau^2-1)+\tau^{-5}(\tau^5+\tau^4+\tau^3+\tau^2+\tau+2\tau^5+2\tau)} \\
&= \gamma^{3\tau(1+\tau)(\tau^{-5}-1)/(\tau+1)(\tau-1)+(1+\tau^{-1}+\tau^{-2}+\tau^{-3}+\tau^{-4}+2+2\tau^{-4})} \\
&= \gamma^{3\tau(\tau^{-5}-1)/(\tau-1)+(3+\tau^{-1}+\tau^{-2}+\tau^{-3}+3\tau^{-4})} \\
&= \gamma^{(-2+2\tau^{-3})/(\tau-1)}.
\end{aligned}$$

We may assume this does not generate $\mathcal{C}_p$, then

$$0 \equiv -2 + 2\tau^{-3} \pmod p.$$

Multiplying by τ^3, we have

$$0 \equiv -2\tau^3 + 2 \pmod p.$$

This implies that $\tau^3 \equiv 1 \pmod p$, which contradicts the fact that $\tau^q \equiv 1 \pmod p$ but $\tau \not\equiv 1 \pmod p$.

Figure 8: The Hamiltonian cycle C_2: $\hat{a}$ edges are solid and $\hat{b}$ edges are dashed.

Now we may assume $\tau^q \equiv -1 \pmod p$, then substituting this in the formula for the voltage of C_1 we have

$$\begin{aligned}\mathbb{V}(C_1) &= \gamma^{(-3\tau+3\tau^2)(-\tau^{-5}-1)/(\tau^2-1)-\tau^{-5}(-\tau^5-\tau^4-\tau^3-\tau^2-\tau+2\tau^5+2\tau)} \\ &= \gamma^{3\tau(\tau-1)(-\tau^{-5}-1)/(\tau+1)(\tau-1)+(1+\tau^{-1}+\tau^{-2}+\tau^{-3}+\tau^{-4}-2-2\tau^{-4})} \\ &= \gamma^{3\tau(-\tau^{-5}-1)/(\tau+1)+(-1+\tau^{-1}+\tau^{-2}+\tau^{-3}-\tau^{-4})} \\ &= \gamma^{(-4\tau+2\tau^{-1}+2\tau^{-2}-4\tau^{-4})/(\tau+1)}.\end{aligned}$$

We may assume this does not generate $\mathcal{C}_p$, then

$$0 \equiv -4\tau + 2\tau^{-1} + 2\tau^{-2} - 4\tau^{-4} \pmod p.$$

Multiplying by $(-\tau^4)/2$, we have

$$\begin{aligned}0 &\equiv 2\tau^5 - \tau^3 - \tau^2 + 2 \\ &= (\tau+1)(2\tau^4 - 2\tau^3 + \tau^2 - 2\tau + 2) \pmod p.\end{aligned}$$

Since we assumed $\tau \not\equiv -1 \pmod p$, then the above equation implies that

$$0 \equiv 2\tau^4 - 2\tau^3 + \tau^2 - 2\tau + 2 \pmod p. \tag{3A}$$

Now we calculate the voltage of C_2.

$$\begin{aligned}\mathbb{V}(C_2) &= (aba^{-1}bab)^{(q-5)}(a^3ba^2ba^{-1}ba^{-3}ba^3ba^{-3}ba^{-1}ba^2ba^3b) \\ &= (aa^q\gamma a^{-1}a^q\gamma aa^q\gamma)^{(q-5)}(a^3a^q\gamma a^2a^q\gamma \\ &\qquad \cdot a^{-1}a^q\gamma a^{-3}a^q\gamma a^3a^q\gamma a^{-3}a^q\gamma a^{-1}a^q\gamma a^2a^q\gamma a^3a^q\gamma) \\ &= (a^{q+1}\gamma a^{q-1}\gamma a^{q+1}\gamma)^{(q-5)}(a^{q+3}\gamma a^{q+2}\gamma a^{q-1}\gamma a^{q-3}\gamma a^{q+3}\gamma a^{q-3}\gamma a^{q-1}\gamma a^{q+2}\gamma a^{q+3}\gamma) \\ &= (\gamma^{\tau^{q+1}+1+\tau^{q+1}}a^{q+1})^{(q-5)}(\gamma^{\tau^{q+3}+\tau^5+\tau^{q+4}+\tau+\tau^{q+4}+\tau+\tau^q+\tau^2+\tau^{q+5}}a^{q+5}) \\ &= (\gamma^{2\tau^{q+1}+1}a^{q+1})^{(q-5)}(\gamma^{\tau^{q+5}+2\tau^{q+4}+\tau^{q+3}+\tau^q+\tau^5+\tau^2+2\tau}a^{q+5}) \\ &= (\gamma^{(2\tau^{q+1}+1)((\tau^{q+1})^{(q-5)}-1)/(\tau^{q+1}-1)}a^{(q+1)(q-5)}) \\ &\qquad \cdot (\gamma^{\tau^{q+5}+2\tau^{q+4}+\tau^{q+3}+\tau^q+\tau^5+\tau^2+2\tau}a^{q+5}) \\ &= \gamma^{(2\tau^{q+1}+1)((\tau^{q+1})^{(q-5)}-1)/(\tau^{q+1}-1)+\tau^{(q+1)(q-5)}(\tau^{q+5}+2\tau^{q+4}+\tau^{q+3}+\tau^q+\tau^5+\tau^2+2\tau)}.\end{aligned}$$

Since we are assuming $\tau^q \equiv -1 \pmod p$, then by substituting this in the above formula we have

$$\begin{aligned}\mathbb{V}(C_2) &= \gamma^{(-2\tau+1)((-\tau)^{-5}-1)/(-\tau-1)-\tau^{-5}(-\tau^5-2\tau^4-\tau^3-1+\tau^5+\tau^2+2\tau)} \\ &= \gamma^{(2\tau^{-4}+2\tau-\tau^{-5}-1)/(-\tau-1)+1+2\tau^{-1}+\tau^{-2}+\tau^{-5}-1-\tau^{-3}-2\tau^{-4}} \\ &= \gamma^{(2\tau-3-3\tau^{-1}+3\tau^{-3}+3\tau^{-4}-2\tau^{-5})/(-\tau-1)}.\end{aligned}$$

We may assume this does not generate $\mathcal{C}_p$, then

$$2\tau - 3 - 3\tau^{-1} + 3\tau^{-3} + 3\tau^{-4} - 2\tau^{-5} \equiv 0 \pmod{p}.$$

Multiplying by τ^5, we have

$$0 \equiv 2\tau^6 - 3\tau^5 - 3\tau^4 + 3\tau^2 + 3\tau - 2 = (\tau^2 - 1)(2\tau^4 - 3\tau^3 - \tau^2 - 3\tau + 2) \pmod{p}.$$

Since $\tau^2 \not\equiv 1 \pmod{p}$, then the above equation implies that

$$0 \equiv 2\tau^4 - 3\tau^3 - \tau^2 - 3\tau + 2 \pmod{p}.$$

Therefore, by subtracting the above equation from 3A, we have

$$0 \equiv (\tau^3 + 2\tau^2 + \tau) = \tau(\tau + 1)^2 \pmod{p}.$$

This is a contradiction.

Case 4. Assume none of the previous cases apply. Since $\langle \overline{a}, \overline{b} \rangle = \overline{G}$, we may assume $|\overline{a}|$ is divisible by q, which means $|\overline{a}|$ is either q or $2q$. Since Case 2 applies when $|\overline{a}| = q$, we must have $|\overline{a}| = 2q$. Then $|\overline{b}| = q$, since Cases 1 and 3 do not apply. So Case 2 applies after interchanging a and b. □

3.3 Assume $|S| = 3$, $G' = \mathcal{C}_p \times \mathcal{C}_q$ and $C_{G'}(\mathcal{C}_3) \neq \{e\}$

In this subsection, we prove the part of Theorem 1.3 where, $|S| = 3$, $G' = \mathcal{C}_p \times \mathcal{C}_q$ and $C_{G'}(\mathcal{C}_3) \neq \{e\}$. Recall $\overline{G} = G/G'$, $\check{G} = G/\mathcal{C}_q$ and $\widehat{G} = G/\mathcal{C}_p$.

Proposition 3.4. *Assume*

- $G = (\mathcal{C}_2 \times \mathcal{C}_3) \ltimes (\mathcal{C}_p \times \mathcal{C}_q)$,
- $|S| = 3$,
- $C_{G'}(\mathcal{C}_3) \neq \{e\}$.

Then $\mathrm{Cay}(G; S)$ *contains a Hamiltonian cycle.*

Proof. Let $S = \{a, b, c\}$. If $C_{G'}(\mathcal{C}_3) = \mathcal{C}_p \times \mathcal{C}_q$, then since $G' \cap Z(G) = \{e\}$ (see Proposition 2.15(2)), we conclude that $C_{G'}(\mathcal{C}_2) = \{e\}$. So we have

$$G = \mathcal{C}_3 \times (\mathcal{C}_2 \ltimes \mathcal{C}_{pq}) \cong \mathcal{C}_3 \times D_{2pq}.$$

Therefore, Lemma 2.26 applies.

Since $C_{G'}(\mathcal{C}_3) \neq \{e\}$, then we may assume $C_{G'}(\mathcal{C}_3) = \mathcal{C}_q$ by interchanging q and p if necessary. Since $G' \cap Z(G) = \{e\}$, then $\mathcal{C}_2$ inverts $\mathcal{C}_q$. Since $\mathcal{C}_3$ centralizes $\mathcal{C}_q$ and $Z(G) \cap G' = \{e\}$ (by Proposition 2.15(2)), then $\mathcal{C}_2$ inverts $\mathcal{C}_q$. Thus,

$$\widehat{G} = (\mathcal{C}_2 \times \mathcal{C}_3) \ltimes \mathcal{C}_q \cong (\mathcal{C}_2 \ltimes \mathcal{C}_q) \times \mathcal{C}_3 = D_{2q} \times \mathcal{C}_3.$$

Now if $\widehat{S}$ is minimal, then Lemma 2.24 applies. Therefore, we may assume $\widehat{S}$ is not minimal. Choose a 2-element subset $\{a, b\}$ of S that generates $\widehat{G}$. From the minimality of S, we see that $\langle a, b \rangle = D_{2q} \times \mathcal{C}_3$ after replacing a and b by conjugates. The projection of (a, b) to D_{2q} must be of the form (a_2, a_q) or $(a_2, a_2 a_q)$, where a_2 is reflection and a_q is a rotation. Also note that $\hat{b} \neq a_q$ because $S \cap G' = \varnothing$ by Assumption 3.1(6). Therefore, (a, b) must have one of the following forms:

1. (a_2, a_3a_q),
2. $(a_2, a_2a_3a_q)$,
3. (a_2a_3, a_2a_q),
4. (a_2a_3, a_3a_q),
5. $(a_2a_3, a_2a_3a_q)$.

Let c be the third element of S. We may write $c = a_2^i a_3^j a_q^k \gamma_p$ with $0 \leqslant i \leqslant 1, 0 \leqslant j \leqslant 2$ and $0 \leqslant k \leqslant q-1$. Note since $S \cap G' = \varnothing$, we know that i and j cannot both be equal to 0. Additionally, we have $a_3\gamma_p a_3^{-1} = \gamma_p^{\hat{\tau}}$ where $\hat{\tau}^3 \equiv 1 \pmod{\mathcal{C}_p}$. Also, $\hat{\tau} \not\equiv 1 \pmod{p}$ since $C_{G'}(\mathcal{C}_3) = \mathcal{C}_q$. Therefore, we conclude that $\hat{\tau}^2 + \hat{\tau} + 1 \equiv 0 \pmod{p}$. Note that this implies $\hat{\tau} \not\equiv -1 \pmod{p}$.

Case 1. Assume $a = a_2$ and $b = a_3a_q$.

Subcase 1.1. Assume $i \neq 0$. Then, $c = a_2a_3^j a_q^k \gamma_p$. Thus, by Lemma 2.27(1) $\langle b, c\rangle = G$ which contradicts the minimality of S.

Subcase 1.2. Assume $i = 0$. Then $j \neq 0$. We may assume $j = 1$, by replacing c with c^{-1} if necessary. Thus $c = a_3a_q^k\gamma_p$. Consider $\overline{G} = \mathcal{C}_2 \times \mathcal{C}_3$. We have $\overline{a} = a_2$, $\overline{b} = a_3$ and $\overline{c} = a_3$. Therefore, $\overline{b} = \overline{c} = a_3$. We have $(\overline{a}, \overline{b}^2, \overline{a}, \overline{b}^{-2})$ as a Hamiltonian cycle in $\mathrm{Cay}(\overline{G}; \overline{S})$. Since we can replace each $\overline{b}$ by $\overline{c}$, then we consider $C_1 = (\overline{a}, \overline{b}^2, \overline{a}, \overline{b}^{-1}, \overline{c}^{-1})$ and $C_2 = (\overline{a}, \overline{b}^2, \overline{a}, \overline{c}^{-2})$ as Hamiltonian cycles in $\mathrm{Cay}(\overline{G}; \overline{S})$. Now since there is one occurrence of c in C_1, then by Lemma 2.8 the subgroup generated by $\mathbb{V}(C_1)$ contains $\mathcal{C}_p$. Also,

$$\begin{aligned}
\mathbb{V}(C_1) &= ab^2ab^{-1}c^{-1} \\
&\equiv a_2 \cdot a_3^2a_q^2 \cdot a_2 \cdot a_q^{-1}a_3^{-1} \cdot a_q^{-k}a_3^{-1} \pmod{\mathcal{C}_p} \\
&= a_q^{-2}a_3a_q^{-1-k}a_3^{-1} \\
&= a_q^{-3-k}.
\end{aligned}$$

We can assume this does not generate $\mathcal{C}_q$, for otherwise Factor Group Lemma 2.6 applies. Therefore,

$$-3 - k \equiv 0 \pmod{q}.$$

Thus, $k \equiv -3 \pmod{q}$.

Now we calculate the voltage of C_2.

$$\begin{aligned}
\mathbb{V}(C_2) &= ab^2ac^{-2} \\
&\equiv a_2 \cdot a_3^2 \cdot a_2 \cdot \gamma_p^{-1}a_3^{-1}\gamma_p^{-1}a_3^{-1} \pmod{\mathcal{C}_q} \\
&= a_3^2\gamma_p^{-1}a_3^{-1}\gamma_p^{-1}a_3^{-1} \\
&= \gamma_p^{-\hat{\tau}^2-\hat{\tau}}.
\end{aligned}$$

Since $\hat{\tau}^2 + \hat{\tau} + 1 \equiv 0 \pmod{p}$, then $-\hat{\tau}^2 - \hat{\tau} \equiv 1 \pmod{p}$. Thus, $\gamma_p^{-\hat{\tau}^2-\hat{\tau}} = \gamma_p$ generates $\mathcal{C}_p$.

$$\begin{aligned}\mathbb{V}(C_2) &= ab^2ac^{-2}\\ &\equiv a_2 \cdot a_3^2a_q^2 \cdot a_2 \cdot a_q^{-k}a_3^{-1}a_q^{-k}a_3^{-1} \pmod{\mathcal{C}_p}\\ &= a_q^{-2}a_3^2a_q^{-k}a_3^{-1}a_q^{-k}a_3^{-1}\\ &= a_q^{-2(k+1)}.\end{aligned}$$

We know $k \equiv -3 \pmod{q}$, therefore, $-2(k+1) \equiv 4 \pmod{q}$, so Factor Group Lemma 2.6 applies.

Case 2. Assume $a = a_2$ and $b = a_2a_3a_q$.

Subcase 2.1. Assume $i = 0$, then $j \neq 0$. If $k \neq 0$, then $c = a_3^ja_q^k\gamma_p$. Thus, by Lemma 2.27(3) $\langle b, c\rangle = G$ which contradicts the minimality of S. Therefore, we may assume $k = 0$. We may also assume $j = 1$, by replacing c with c^{-1} if necessary. Then $c = a_3\gamma_p$.

Consider $\overline{G} = \mathcal{C}_2 \times \mathcal{C}_3$, thus $\overline{a} = a_2$, $\overline{b} = a_2a_3$ and $\overline{c} = a_3$. Therefore, $|\overline{a}| = 2$, $|\overline{b}| = 6$ and $|\overline{c}| = 3$. Consider $C = (\overline{b}^2, \overline{c}, \overline{b}, \overline{c}^{-1}, \overline{a})$ as a Hamiltonian cycle in $\mathrm{Cay}(\overline{G}; \overline{S})$. Now we calculate its voltage.

$$\begin{aligned}\mathbb{V}(C) &= b^2cbc^{-1}a\\ &\equiv a_2a_3a_qa_2a_3a_q \cdot a_3 \cdot a_2a_3a_q \cdot a_3^{-1} \cdot a_2 \pmod{\mathcal{C}_p}\\ &= a_q^{-1}\end{aligned}$$

which generates $\mathcal{C}_q$. By considering the fact that $\mathcal{C}_2$ might centralize $\mathcal{C}_p$ or not, we have

$$\begin{aligned}\mathbb{V}(C) &= b^2cbc^{-1}a\\ &\equiv a_2a_3a_2a_3 \cdot a_3\gamma_p \cdot a_2a_3 \cdot \gamma_p^{-1}a_3^{-1} \cdot a_2 \pmod{\mathcal{C}_q}\\ &= \gamma_pa_3\gamma_p^{\mp 1}a_3^{-1}\\ &= \gamma_p^{1\mp\hat{\tau}}.\end{aligned}$$

which generates $\mathcal{C}_p$. Therefore, the subgroup generated by $\mathbb{V}(C)$ is G'. So, Factor Group Lemma 2.6 applies.

Subcase 2.2. Assume $j = 0$. Then $i \neq 0$. If $k \neq 1$, then $c = a_2a_q^k\gamma_p$. Thus, by Lemma 2.27(4) $\langle b, c\rangle = G$ which contradicts the minimality of S. We may therefore assume $k = 1$. Then $c = a_2a_q\gamma_p$.

Consider $\overline{G} = \mathcal{C}_2 \times \mathcal{C}_3$, then $\overline{a} = \overline{c} = a_2$ and $\overline{b} = a_2a_3$. Thus, $|\overline{a}| = |\overline{c}| = 2$ and $|\overline{b}| = 6$. We have $C = (\overline{b}^2, \overline{c}, \overline{b}^{-2}, \overline{a})$ as a Hamiltonian cycle in $\mathrm{Cay}(\overline{G}; \overline{S})$. Since there is one occurrence of c in C, and it is the only generator of G that contains γ_p, then by Lemma 2.8 we conclude that the subgroup generated by $\mathbb{V}(C)$ contains $\mathcal{C}_p$. Also,

$$\begin{aligned}\mathbb{V}(C) &= b^2cb^{-2}a\\ &\equiv a_2a_3a_qa_2a_3a_q \cdot a_2a_q \cdot a_q^{-1}a_3^{-1}a_2a_q^{-1}a_3^{-1}a_2 \cdot a_2 \pmod{\mathcal{C}_p}\\ &= a_q^{-1}a_3a_qa_3a_q^{-1}a_3^{-1}a_qa_3^{-1}a_q^{-1}\end{aligned}$$

$$= a_q^{-1}.$$

which generates $\mathcal{C}_q$. Therefore, the subgroup generated by $\mathbb{V}(C)$ is G'. So, Factor Group Lemma 2.6 applies.

Subcase 2.3. Assume $i \neq 0$ and $j \neq 0$. We may assume $j = 1$, by replacing c with c^{-1} if necessary. So $c = a_2 a_3 a_q^k \gamma_p$. If $k \neq 1$, then by Lemma 2.27(4) $\langle b, c\rangle = G$ which contradicts the minimality of S. We may now assume $k = 1$. Then $c = a_2 a_3 a_q \gamma_p$.

Consider $\overline{G} = \mathcal{C}_2 \times \mathcal{C}_3$. Then $\overline{a} = a_2$ and $\overline{b} = \overline{c} = a_2 a_3$. Therefore, $|\overline{b}| = |\overline{c}| = 6$ and $|\overline{a}| = 2$. We have $C = (\overline{c}, \overline{a}, (\overline{b}, \overline{a})^2)$ as a Hamiltonian cycle in $\mathrm{Cay}(\overline{G}; \overline{S})$. Since there is one occurrence of c in C, and it is the only generator of G that contains γ_p, then by Lemma 2.8 we conclude that the subgroup generated by $\mathbb{V}(C)$ is $\mathcal{C}_p$. Also,

$$\begin{aligned}
\mathbb{V}(C) &= ca(ba)^2 \\
&\equiv a_2 a_3 a_q \cdot a_2 \cdot a_2 a_3 a_q \cdot a_2 \cdot a_2 a_3 a_q \cdot a_2 \pmod{\mathcal{C}_p} \\
&= a_3 a_q^{-2} a_3 a_q^{-1} a_3 \\
&= a_q^{-3}
\end{aligned}$$

which generates $\mathcal{C}_q$. Therefore, the subgroup generated by $\mathbb{V}(C)$ is G'. So, Factor Group Lemma 2.6 applies.

Case 3. Assume $a = a_2 a_3$ and $b = a_2 a_q$. Since $b = a_2 a_q$ is conjugate to a_2 via an element of $\mathcal{C}_q$ (which centralizes $\mathcal{C}_3$), then $\{a, b\}$ is conjugate to $\{a_2 a_3 a_q^m, a_2\}$ for some nonzero m. So Case 2 applies (after replacing a_q with a_q^m).

Case 4. Assume $a = a_2 a_3$ and $b = a_3 a_q$.

Subcase 4.1. Assume $i \neq 0$. Then $c = a_2 a_3^j a_q^k \gamma_p$. Thus, by Lemma 2.27(1) $\langle b, c\rangle = G$ which contradicts the minimality of S.

Subcase 4.2. Assume $i = 0$. Then $j \neq 0$ and $c = a_3^j a_q^k \gamma_p$. If $k \neq 0$, then by Lemma 2.27(2) $\langle a, c\rangle = G$ which contradicts the minimality of S. So we may assume $k = 0$. We may also assume $j = 1$, by replacing c with c^{-1} if necessary. Then $c = a_3 \gamma_p$.

Consider $\overline{G} = \mathcal{C}_2 \times \mathcal{C}_3$. Therefore, $\overline{a} = a_2 a_3$ and $\overline{b} = \overline{c} = a_3$. In addition, $|\overline{a}| = 6$ and $|\overline{b}| = |\overline{c}| = 3$. We have $C = (\overline{c}, \overline{b}, \overline{a}, \overline{b}^{-2}, \overline{a}^{-1})$ as a Hamiltonian cycle in $\mathrm{Cay}(\overline{G}; \overline{S})$. Since there is one occurrence of c in C, and it is the only generator of G that contains γ_p, then by Lemma 2.8 we conclude that the subgroup generated by $\mathbb{V}(C)$ contains $\mathcal{C}_p$. Also,

$$\begin{aligned}
\mathbb{V}(C) &= cbab^{-2}a^{-1} \\
&\equiv a_3 \cdot a_3 a_q \cdot a_2 a_3 \cdot a_q^{-2} a_3^{-2} \cdot a_3^{-1} a_2 \pmod{\mathcal{C}_p} \\
&= a_3 a_q a_3^2 a_q^2 \\
&= a_q^3
\end{aligned}$$

which generates $\mathcal{C}_q$. Therefore, the subgroup generated by $\mathbb{V}(C)$ is G'. Thus, Factor Group Lemma 2.6 applies.

Case 5. Assume $a = a_2 a_3$, $b = a_2 a_3 a_q$.

Subcase 5.1. Assume $i = 0$. Then $j \neq 0$ and $c = a_3^j a_q^k \gamma_p$. If $k \neq 0$, then by Lemma 2.27(3) $\langle b, c\rangle = G$ which contradicts the minimality of S. So we may assume $k = 0$. We may also assume $j = 1$, by replacing c with c^{-1} if necessary. Then $c = a_3\gamma_p$.

Consider $\overline{G} = \mathcal{C}_2 \times \mathcal{C}_3$. Therefore, $\overline{a} = \overline{b} = a_2a_3$ and $\overline{c} = a_3$. Thus, $|\overline{a}| = |\overline{b}| = 6$ and $|\overline{c}| = 3$. We have $C = (\overline{a}, \overline{c}^2, \overline{b}^{-1}, \overline{c}^{-2})$ as a Hamiltonian cycle in $\mathrm{Cay}(\overline{G}; \overline{S})$. Now we calculate its voltage.

$$\begin{aligned}\mathbb{V}(C) &= ac^2b^{-1}c^{-2}\\ &\equiv a_2a_3 \cdot a_3^2 \cdot a_q^{-1}a_3^{-1}a_2 \cdot a_3^{-2} \pmod{\mathcal{C}_p}\\ &= a_3^{-1}a_qa_3^{-2}\\ &= a_q\end{aligned}$$

which generates $\mathcal{C}_q$. Also

$$\begin{aligned}\mathbb{V}(C) &= ac^2b^{-1}c^{-2}\\ &\equiv ac^2a^{-1}c^{-2} \pmod{\mathcal{C}_q} \text{ (because } a \equiv b \pmod{\mathcal{C}_q})\\ &= ac^{-1}a^{-1}c \text{ (because } |c| = 3)\\ &= [a, c^{-1}].\end{aligned}$$

This generates $\mathcal{C}_p$, because $\{a, c\}$ generates $G/\mathcal{C}_q$. Therefore, the subgroup generated by $\mathbb{V}(C)$ is G'. So, Factor Group Lemma 2.6 applies.

Subcase 5.2. Assume $i \neq 0$. Then $c = a_2a_3^ja_q^k\gamma_p$. If $k \neq 1$, then by Lemma 2.27(4) $\langle b, c\rangle = G$ which contradicts the minimality of S. So we may assume $k = 1$. Then $c = a_2a_3^ja_q\gamma_p$. We show that $\langle a, c\rangle = G$. Now, we have

$$\begin{aligned}\langle a, c\rangle &= \langle a_2, a_3, c\rangle \text{ (because } \langle a\rangle = \langle a_2a_3\rangle = \langle a_2, a_3\rangle)\\ &= \langle a_2, a_3, a_2a_3^ja_q\gamma_p\rangle\\ &= \langle a_2, a_3, a_q\gamma_p\rangle\\ &= \langle a_2, a_3, a_q, \gamma_p\rangle\\ &= G,\end{aligned}$$

which contradicts the minimality of S. □

3.4 Assume $|S| = 3$, $G' = \mathcal{C}_p \times \mathcal{C}_q$ and $\widehat{S}$ is minimal

In this subsection, we prove the part of Theorem 1.3 where, $|S| = 3$, $G' = \mathcal{C}_p \times \mathcal{C}_q$ and $C_{G'}(\mathcal{C}_3) = \{e\}$. Recall $\overline{G} = G/G'$ and $\widehat{G} = G/\mathcal{C}_p$.

Proposition 3.5. *Assume*

- $G = (\mathcal{C}_2 \times \mathcal{C}_3) \ltimes (\mathcal{C}_p \times \mathcal{C}_q)$,
- $|S| = 3$,
- $\widehat{S}$ *is minimal.*

Then $\mathrm{Cay}(G; S)$ *contains a Hamiltonian cycle.*

Proof. Let $S = \{a, b, c\}$. If $C_{G'}(\mathcal{C}_3) \neq \{e\}$, then Proposition 3.4 applies. Hence we may assume $C_{G'}(\mathcal{C}_3) = \{e\}$. Then we have four different cases.

Case 1. Assume $C_{G'}(\mathcal{C}_2) = \mathcal{C}_p \times \mathcal{C}_q$, thus $G = \mathcal{C}_2 \times (\mathcal{C}_3 \ltimes \mathcal{C}_{pq})$. Since $\widehat{S}$ is minimal, then all three elements belonging to $\widehat{S}$ must have prime order. There is an element $\hat{a} \in \widehat{S}$, such that $|\hat{a}| = 2$, otherwise all elements of S belong to a subgroup of index 2 of G, so $\langle a, b, c\rangle \neq G$ which is a contradiction. If $|a| = 2p$, then Corollary 2.7 applies with $s = a$ and $t = a^{-1}$, because there is a Hamiltonian cycle in $\mathrm{Cay}(\widehat{G}; \widehat{S})$ (see Theorem 1.2(3)) which uses at least one labeled edge $\hat{a}$ because $\widehat{S}$ is minimal.

Now we may assume $|a| = 2$. Replacing a by a conjugate we may assume $\langle a\rangle = \mathcal{C}_2$. Thus, $\langle b, c\rangle = \mathcal{C}_3 \ltimes \mathcal{C}_{pq}$. By Theorem 1.2(3), there is a Hamiltonian path L in $\mathrm{Cay}(\mathcal{C}_3 \ltimes \mathcal{C}_{pq}, \{b, c\})$. Therefore, $LaL^{-1}a^{-1}$ is a Hamiltonian cycle in $\mathrm{Cay}(G; S)$.

Case 2. Assume $C_{G'}(\mathcal{C}_2) = \mathcal{C}_q$. Therefore,

$$\widehat{G} = G/\mathcal{C}_p = \mathcal{C}_6 \ltimes \mathcal{C}_q \cong \mathcal{C}_2 \times (\mathcal{C}_3 \ltimes \mathcal{C}_q).$$

There is some $a \in S$ such that $|\hat{a}| = 2$. Thus, we can assume $|a| = 2$, for otherwise Corollary 2.7 applies with $s = a$ and $t = a^{-1}$. (Note since $\widehat{S}$ is minimal, then $\hat{a}$ must be used in any Hamiltonian cycle in $\mathrm{Cay}(\widehat{G}; \widehat{S})$.) We may assume $a = a_2$. Since $\widehat{S}$ is minimal, $S \cap G' = \varnothing$ (see Assumption 3.1(6)) and each element belonging to $\widehat{S}$ has prime order, then $|\hat{b}| = |\hat{c}| = 3$. We may assume $\hat{a} = a_2$, $\hat{b} = a_3$ and $\hat{c} = a_3 a_q$. We have the following two Hamiltonian paths in $\mathrm{Cay}(\mathcal{C}_3 \ltimes \mathcal{C}_q; \{\hat{b}, \hat{c}\})$:

$$L_1 = ((\hat{c}, \hat{b}^2)^{q-1}, \hat{c}, \hat{b})$$

and

$$L_2 = ((\hat{b}, \hat{c}, \hat{b})^{q-1}, \hat{b}, \hat{c}).$$

These lead to the following two Hamiltonian cycles in $\mathrm{Cay}(\widehat{G}; \widehat{S})$:

$$C_1 = (L_1, \hat{a}, L_1^{-1}, \hat{a})$$

and

$$C_2 = (L_2, \hat{a}, L_2^{-1}, \hat{a}).$$

Then if we let

$$\prod L_1 = (cb^2)^{q-1}cb = (cb^2)^q b^{-1} \in a_3^{-1}\mathcal{C}_p$$

and

$$\prod L_2 = (bcb)^{q-1}bc = (bcb)^q b^{-1} = b(cb^2)^q b^{-2} = b(\prod L_1)b^{-1}$$

then it is clear that $V(C_i) = [\prod L_i, a]$ for $i = 1, 2$. Therefore, we may assume a centralizes $\prod L_1$ and $\prod L_2$, for otherwise Factor Group Lemma 2.6 applies. Now, since a centralizes $\prod L_1$, and $\prod L_1 \in a_3^{-1}\mathcal{C}_p$, we must have $\prod L_1 = a_3^{-1}$. So $\prod L_2 = ba_3^{-1}b^{-1}$. If b does not centralize a_3, then $\mathbb{V}(C_1) \neq \mathbb{V}(C_2)$, so the voltage of C_1 or C_2 cannot both be equal to identity. Therefore, Factor Group Lemma 2.6 applies. Now if b centralizes a_3, then we can assume $b = a_3$. Therefore, $c = a_3 a_q \gamma_p$. We calculate the voltage of C_1. We have

$$\begin{aligned}\mathbb{V}(C_1) &= (cb^2)^q b^{-1} a((cb^2)^q b^{-1})^{-1} a \\ &= (a_3 a_q \gamma_p \cdot a_3^2)^q \cdot a_3^{-1} \cdot a_2 \cdot ((a_3 a_q \gamma_p \cdot a_3^2)^q \cdot a_3^{-1})^{-1} \cdot a_2 \\ &= (a_3 a_q \gamma_p a_3^{-1})^q a_3^{-1} a_2 ((a_3 a_q \gamma_p a_3^{-1}) a_3^{-1})^{-1} a_2 \\ &= a_3 a_q^q \gamma_p^q a_3^{-1} a_3^{-1} a_2 (a_3 a_q^q \gamma_p^q a_3^{-1} a_3^{-1})^{-1} a_2 \\ &= a_3 \gamma_p^q a_3^{-2} a_2 (a_3 \gamma_p^q a_3^{-2})^{-1} a_2 \\ &= a_3 \gamma_p^q a_3^{-2} a_2 a_3^2 \gamma_p^{-q} a_3^{-1} a_2 \\ &= a_3 \gamma_p^{2q} a_3^{-1}\end{aligned}$$

which generates $\mathcal{C}_p$. Thus, Factor Group Lemma 2.6 applies.

Case 3. Assume $C_{G'}(\mathcal{C}_2) = \mathcal{C}_p$. Therefore,

$$\breve{G} = G/\mathcal{C}_q = \mathcal{C}_6 \ltimes \mathcal{C}_p \cong \mathcal{C}_2 \times (\mathcal{C}_3 \ltimes \mathcal{C}_p).$$

Now since $S \cap G' = \varnothing$ (see Assumption 3.1(6)) and $\mathcal{C}_3$ does not centralize $\mathcal{C}_p$, then for all $a \in S$, we have $|\breve{a}| \in \{2, 3, 6, 2p\}$. If $|\breve{a}| = 6$, then $|\hat{a}|$ is divisible by 6 which contradicts the minimality of $\widehat{S}$. Note that every element belong to $\widehat{S}$ has prime order. If $|\breve{a}| = 2p$, then $|\hat{a}| = 2$ (because $\widehat{S}$ is minimal). Therefore, Corollary 2.7 applies with $s = a$ and $t = a^{-1}$. Note that since $\widehat{S}$ is minimal, then there is a Hamiltonian cycle in $\mathrm{Cay}(\widehat{G}; \widehat{S})$ uses at least one labeled edge $\hat{a}$. Thus, $|\breve{a}| \in \{2, 3\}$ for all $a \in S$. This implies that $\breve{S}$ is minimal, because we need an a_2 and an a_3 to generate $\mathcal{C}_2 \times \mathcal{C}_3$ and two elements whose order divisible by 2 or 3 to generate $\mathcal{C}_p$. So by interchanging p and q the proof in Case 2 applies.

Case 4. Assume $C_{G'}(\mathcal{C}_2) = \{e\}$. Consider

$$\widehat{G} = G/\mathcal{C}_p = (\mathcal{C}_2 \times \mathcal{C}_3) \ltimes \mathcal{C}_q.$$

Now since $\widehat{S}$ is minimal, every element of $\widehat{S}$ has prime order. Since $S \cap G' = \varnothing$ (see Assumption 3.1(6)), then for every $\hat{s} \in \widehat{S}$, we have $|\hat{s}| \in \{2, 3\}$. Since $C_{G'}(\mathcal{C}_2) = \{e\}$ and $C_{G'}(\mathcal{C}_3) = \{e\}$, this implies that for every $s \in S$, we have $|s| \in \{2, 3\}$. From our assumption we know that $S = \{a, b, c\}$. Now we may assume $|a| = 2$ and $|b| = 3$. Also, we know that $|c| \in \{2, 3\}$.

If $|c| = 2$, then $c = a\gamma$, where $\gamma \in G'$. Suppose, for the moment, $\langle \gamma \rangle \neq G'$. Since $\langle \gamma \rangle \lhd G$, then we have

$$G = \langle a, b, c \rangle = \langle a, b, \gamma \rangle = \langle a, b \rangle \langle \gamma \rangle.$$

Now since $\widehat{S}$ is minimal, $\langle a, b \rangle$ does not contain $\mathcal{C}_q$. So this implies that $\langle \gamma \rangle$ contains $\mathcal{C}_q$. Since $\langle \gamma \rangle$ does not contain G', then $\langle \gamma \rangle = \mathcal{C}_q$. Thus, we may assume that $a = a_2$ (by conjugation if necessary), $b = a_3 \gamma_p$ and $c = a_2 a_q$. So $\langle b, c \rangle = \langle a_3 \gamma_p, a_2 a_q \rangle = G$ (since $a_3 \gamma_p$ and $a_2 a_q$ clearly generate $\overline{G}$ and do not commute modulo p or modulo q, they must generate G). This contradicts the minimality of S. Therefore, $\langle \gamma \rangle = G'$.

Consider $\overline{G} = \mathcal{C}_2 \times \mathcal{C}_3$. Then $\overline{a} = \overline{c}$. We have $|\overline{a}| = |\overline{c}| = 2$ and $|\overline{b}| = 3$. We also have $C_1 = (\overline{c}^{-1}, \overline{b}^{-2}, \overline{a}, \overline{b}^{2})$ as a Hamiltonian cycle in $\mathrm{Cay}(\overline{G}; \overline{S})$. Now we calculate its voltage.

$$\mathbb{V}(C_1) = c^{-1}b^{-2}ab^2 = \gamma^{-1}a^{-1}b^{-2}ab^2.$$

Now, $a^{-1}b^{-2}ab^2 \in G'$. Since $\langle a, b\rangle \neq G$, we have $a^{-1}b^{-2}ab^2 \in \{e, \gamma_p\}$. If $a^{-1}b^{-2}ab^2 = e$, then a and b^2 commute, so a and b commute. Hence $b = a_3$, so $\langle b, c\rangle = G$, a contradiction. So $a^{-1}b^{-2}ab^2 = \gamma_p$, and $\mathbb{V}(C_1) = \gamma^{-1}\gamma_p$ which generates G'. Therefore, Factor Group Lemma 2.6 applies.

Now we can assume $|c| = 3$. Then $c = b\gamma$, where $\gamma \in G'$ (after replacing c with its inverse if necessary). Suppose, for the moment, $\langle \gamma \rangle \neq G'$. Since $\langle \gamma \rangle \lhd G$, then we have

$$G = \langle a, b, c\rangle = \langle a, b, \gamma\rangle = \langle a, b\rangle\langle \gamma\rangle.$$

Now since $\widehat{S}$ is minimal, then $\langle a, b\rangle$ does not contain $\mathcal{C}_q$. So this implies that $\langle \gamma \rangle$ contains $\mathcal{C}_q$. Since $\langle \gamma \rangle$ does not contain G', then $\langle \gamma \rangle = \mathcal{C}_q$. Therefore, we may assume that $a = a_2\gamma_p$ (by conjugation if necessary), $b = a_3$ and $c = a_3 a_q$. So $\langle a, c\rangle = \langle a_2\gamma_p, a_3 a_q\rangle = G$ (since $a_2\gamma_p$ and $a_3 a_q$ clearly generate $\overline{G}$ and do not commute modulo p or modulo q, they must generate G). This contradicts the minimality of S. So $\langle \gamma \rangle = G'$.

Consider $\overline{G} = \mathcal{C}_2 \times \mathcal{C}_3$. Then $\overline{b} = \overline{c}$. We have $|\overline{a}| = 2$ and $|\overline{b}| = |\overline{c}| = 3$. We also have $C_2 = (\overline{c}^{-1}, \overline{b}^{-1}, \overline{a}^{-1}, \overline{b}^{2}, \overline{a})$ as a Hamiltonian cycle in $\mathrm{Cay}(\overline{G}; \overline{S})$. Now we calculate its voltage.

$$\mathbb{V}(C_2) = c^{-1}b^{-1}a^{-1}b^2a = \gamma^{-1}b^{-1}b^{-1}a^{-1}b^2a.$$

Now, $b^{-2}a^{-1}b^2a \in G'$. Since $\langle a, b\rangle \neq G$, we have $b^{-2}a^{-1}b^2a \in \{e, \gamma_p\}$. If $b^{-2}a^{-1}b^2a = e$, then a and b^2 commute, so a and b commute. Hence $a = a_2$, so $\langle a, c\rangle = G$, a contradiction. So $b^{-2}a^{-1}b^2a = \gamma_p$, and $\mathbb{V}(C_2) = \gamma^{-1}\gamma_p$ which generates G'. Therefore, Factor Group Lemma 2.6 applies. □

3.5 Assume $|S| = 3$, $G' = \mathcal{C}_p \times \mathcal{C}_q$ and $C_{G'}(\mathcal{C}_2) = \mathcal{C}_p \times \mathcal{C}_q$

In this subsection, we prove the part of Theorem 1.3 where, $|S| = 3$, $G' = \mathcal{C}_p \times \mathcal{C}_q$, $C_{G'}(\mathcal{C}_2) = \mathcal{C}_p \times \mathcal{C}_q$, and neither $C_{G'}(\mathcal{C}_3) \neq \{e\}$ nor $\widehat{S}$ is minimal holds. Recall $\overline{G} = G/G'$, $\breve{G} = G/\mathcal{C}_q$ and $\widehat{G} = G/\mathcal{C}_p$.

Proposition 3.6. *Assume*

- $G = (\mathcal{C}_2 \times \mathcal{C}_3) \ltimes (\mathcal{C}_p \times \mathcal{C}_q)$,
- $|S| = 3$,
- $C_{G'}(\mathcal{C}_2) = \mathcal{C}_p \times \mathcal{C}_q$.

Then $\mathrm{Cay}(G; S)$ *contains a Hamiltonian cycle.*

Proof. Let $S = \{a, b, c\}$. If $C_{G'}(\mathcal{C}_3) \neq \{e\}$, then Proposition 3.4 applies. So we may assume $C_{G'}(\mathcal{C}_3) = \{e\}$. Now if $\widehat{S}$ is minimal, then Proposition 3.5 applies. So we may assume $\widehat{S}$ is not minimal. Consider

$$\widehat{G} = G/\mathcal{C}_p = (\mathcal{C}_2 \times \mathcal{C}_3) \ltimes \mathcal{C}_q \cong (\mathcal{C}_3 \ltimes \mathcal{C}_q) \times \mathcal{C}_2.$$

Choose a 2-element $\{a, b\}$ subset of S that generates $\widehat{G}$. From the minimality of S, we see that

$$\langle a, b\rangle = (\mathcal{C}_3 \ltimes \mathcal{C}_q) \times \mathcal{C}_2,$$

after replacing a and b by conjugates. The projection of (a, b) to $\mathcal{C}_3 \ltimes \mathcal{C}_q$ must be of the form (a_3, a_q) or $(a_3, a_3 a_q)$ (perhaps after replacing a and/or b with its inverse; also note that $\widehat{b} \neq a_q$ because $S \cap G' = \varnothing$). Therefore, (a, b) must have one of the following forms:

1. $(a_3, a_2 a_q)$,
2. $(a_3, a_2 a_3 a_q)$,
3. $(a_2 a_3, a_3 a_q)$,
4. $(a_2 a_3, a_2 a_q)$,
5. $(a_2 a_3, a_2 a_3 a_q)$.

Let c be the third element of S. We may write $c = a_2^i a_3^j a_q^k \gamma_p$ with $0 \leqslant i \leqslant 1$, $0 \leqslant j \leqslant 2$ and $0 \leqslant k \leqslant q-1$. Note since $S \cap G' = \varnothing$, we know that i and j cannot both be equal to 0. Additionally, we have $a_3 \gamma_p a_3^{-1} = \gamma_p^{\widehat{\tau}}$ where $\widehat{\tau}^3 \equiv 1 \pmod p$ and $\widehat{\tau} \not\equiv 1 \pmod p$. Thus $\widehat{\tau}^2 + \widehat{\tau} + 1 \equiv 0 \pmod p$. Note that this implies $\widehat{\tau} \not\equiv -1 \pmod p$. Also we have $a_3 a_q a_3^{-1} = a_q^{\check{\tau}}$. By using the same argument we can conclude that $\check{\tau} \not\equiv 1 \pmod q$ and $\check{\tau}^2 + \check{\tau} + 1 \equiv 0 \pmod q$. Note that this implies $\check{\tau} \not\equiv -1 \pmod q$. Combining these facts with $\widehat{\tau}^3 \equiv 1 \pmod p$ and $\check{\tau}^3 \equiv 1 \pmod q$, we conclude that $\widehat{\tau}^2 \not\equiv \pm 1 \pmod p$, and $\check{\tau}^2 \not\equiv \pm 1 \pmod q$.

Case 1. Assume $a = a_3$ and $b = a_2 a_q$.

Subcase 1.1. Assume $i = 0$. Then $j \neq 0$ and $c = a_3^j a_q^k \gamma_p$. For future reference in Subcase 4.1 of Proposition 3.7, we note that the argument here does not require our current assumption that $\mathcal{C}_2$ centralizes $\mathcal{C}_p$. We may assume $j = 1$, by replacing c with c^{-1} if necessary. Then $c = a_3 a_q^k \gamma_p$. Consider $\overline{G} = \mathcal{C}_2 \times \mathcal{C}_3$. Then we have $\overline{a} = \overline{c} = a_3$, $\overline{b} = a_2$. We have $C_1 = (\overline{c}, \overline{a}, \overline{b}, \overline{a}^{-2}, \overline{b})$ and $C_2 = (\overline{c}^2, \overline{b}, \overline{a}^{-2}, \overline{b})$ as Hamiltonian cycles in $\mathrm{Cay}(\overline{G}; \overline{S})$. Since there is one occurrence of c in C_1, then by Lemma 2.8 we conclude that the subgroup generated by $\mathbb{V}(C_1)$ contains $\mathcal{C}_p$. Also,

$$\begin{aligned}
\mathbb{V}(C_1) &= caba^{-2}b \\
&\equiv a_3 a_q^k \cdot a_3 \cdot a_2 a_q \cdot a_3^{-2} \cdot a_2 a_q \pmod{\mathcal{C}_p} \\
&= a_q^{k\check{\tau} + \check{\tau}^2 + 1} \\
&= a_q^{\check{\tau}^2 + k\check{\tau} + 1}.
\end{aligned}$$

We may assume this does not generate $\mathcal{C}_q$, for otherwise Factor Group Lemma 2.6 applies. Therefore,

$$0 \equiv \check{\tau}^2 + k\check{\tau} + 1 \pmod q. \tag{1.1A}$$

We also have

$$0 \equiv \check{\tau}^2 + \check{\tau} + 1 \pmod q. \tag{1.1B}$$

By subtracting the above equation from 1.1A, we have $0 \equiv (k-1)\check{\tau} \pmod q$. This implies that $k = 1$.

Now we calculate the voltage of C_2.

$$\begin{aligned}\mathbb{V}(C_2) &= c^2ba^{-2}b \\ &\equiv a_3\gamma_p a_3\gamma_p \cdot a_2 \cdot a_3^{-2} \cdot a_2 \pmod{\mathcal{C}_q} \\ &= \gamma_p^{\hat{\tau}+\hat{\tau}^2}\end{aligned}$$

which generates $\mathcal{C}_p$. Also

$$\begin{aligned}\mathbb{V}(C_2) &= c^2ba^{-2}b \\ &\equiv a_3a_q \cdot a_3a_q \cdot a_2a_q \cdot a_3^{-2} \cdot a_2a_q \pmod{\mathcal{C}_p} \\ &= a_q^{\check{\tau}+\check{\tau}^2+\check{\tau}^2+1} \\ &= a_q^{2\check{\tau}^2+\check{\tau}+1}.\end{aligned}$$

We may assume this does not generate $\mathcal{C}_q$, for otherwise Factor Group Lemma 2.6 applies. Then

$$0 \equiv 2\check{\tau}^2 + \check{\tau} + 1 \pmod q.$$

By subtracting 1.1B from the above equation we have

$$0 \equiv \check{\tau}^2 \pmod q$$

which is a contradiction.

Subcase 1.2. Assume $j = 0$. Then $i \neq 0$ and $c = a_2a_q^k\gamma_p$. For future reference in Subcase 4.2 of Proposition 3.7, we note that the argument here does not require our current assumption that $\mathcal{C}_2$ centralizes $\mathcal{C}_p$. If $k \neq 0$, then by Lemma 2.28(3) $\langle a, c\rangle = G$ which contradicts the minimality of S.

So we can assume $k = 0$. Then $c = a_2\gamma_p$. Consider $\overline{G} = \mathcal{C}_2 \times \mathcal{C}_3$. Then we have $\overline{a} = a_3$ and $\overline{b} = \overline{c} = a_2$. This implies that $|\overline{a}| = 3$ and $|\overline{b}| = |\overline{c}| = 2$. We have $C = (\overline{c}^{-1}, \overline{a}^2, \overline{b}, \overline{a}^{-2})$ as a Hamiltonian cycle in $\mathrm{Cay}(\overline{G}; \overline{S})$. Since there is one occurrence of c in C, and it is the only generator of G that contains γ_p, then by Lemma 2.8 we conclude that the subgroup generated by $\mathbb{V}(C)$ contains $\mathcal{C}_p$. Similarly, since there is one occurrence of b in C, and it is the only generator of G that contains a_q, then by Lemma 2.8 we conclude that the subgroup generated by $\mathbb{V}(C)$ contains $\mathcal{C}_q$. Therefore, the subgroup generated by $\mathbb{V}(C)$ is G'. So, Factor Group Lemma 2.6 applies.

Subcase 1.3. Assume $i \neq 0$ and $j \neq 0$. Then $c = a_2a_3^ja_q^k\gamma_p$. If $k \neq 0$, then by Lemma 2.28(3) $\langle a, c\rangle = G$ which contradicts the minimality of S.

So we can assume $k = 0$. We may also assume $j = 1$, by replacing c with c^{-1} if necessary. Then $c = a_2a_3\gamma_p$. Consider $\overline{G} = \mathcal{C}_2 \times \mathcal{C}_3$. Then we have $\overline{a} = a_3$, $\overline{b} = a_2$ and $\overline{c} = a_2a_3$. This implies that $|\overline{a}| = 3$, $|\overline{b}| = 2$ and $|\overline{c}| = 6$. We have $C = (\overline{c}, \overline{b}, \overline{a}, \overline{c}, \overline{a}^{-1}, \overline{c})$ as a Hamiltonian cycle in $\mathrm{Cay}(\overline{G}; \overline{S})$. Now we calculate its voltage.

$$\begin{aligned}\mathbb{V}(C) &= cbaca^{-1}c \\ &\equiv a_2a_3 \cdot a_2a_q \cdot a_3 \cdot a_2a_3 \cdot a_3^{-1} \cdot a_2a_3 \pmod{\mathcal{C}_p}\end{aligned}$$

$$= a_3 a_q a_3^2$$
$$= a_q^{\check{\tau}}$$

which generates $\mathcal{C}_q$. Also

$$\begin{aligned}\mathbb{V}(C) &= cbaca^{-1}c \\ &\equiv a_2a_3\gamma_p \cdot a_2 \cdot a_3 \cdot a_2a_3\gamma_p \cdot a_3^{-1} \cdot a_2a_3\gamma_p \pmod{\mathcal{C}_q} \\ &= a_3\gamma_p a_3^2\gamma_p^2 \\ &= \gamma_p^{\hat{\tau}+2}.\end{aligned}$$

We may assume this does not generate $\mathcal{C}_p$, for otherwise Factor Group Lemma 2.6 applies. Then $\hat{\tau} \equiv -2 \pmod{p}$. By substituting this in

$$0 \equiv \hat{\tau}^2 + \hat{\tau} + 1 \pmod{p},$$

we have

$$\begin{aligned}0 &\equiv 4 - 2 + 1 \pmod{p} \\ &= 3.\end{aligned}$$

This contradicts the fact that $p > 3$.

Case 2. Assume $a = a_3$ and $b = a_2a_3a_q$.

Subcase 2.1. Assume $i \neq 0$ and $j \neq 0$. Then $c = a_2a_3^ja_q^k\gamma_p$. If $k \neq 0$, then by Lemma 2.28(3) $\langle a, c\rangle = G$ which contradicts the minimality of S. So we can assume $k = 0$. Then $c = a_2a_3^j\gamma_p$. Thus, by Lemma 2.28(4) $\langle b, c\rangle = G$ which contradicts the minimality of S.

Subcase 2.2. Assume $i = 0$. Then $j \neq 0$. We may assume $j = 1$, by replacing c with c^{-1} if necessary. Then $c = a_3a_q^k\gamma_p$.

Suppose, for the moment, that $k \neq 1$. Then $c = a_3a_q^k\gamma_p$. We have $\langle \overline{b}, \overline{c}\rangle = \langle \overline{a}_2\overline{a}_3, \overline{a}_3\rangle = \overline{G}$. Consider $\{\hat{b}, \hat{c}\} = \{a_2a_3a_q, a_3a_q^k\}$. Since $\mathcal{C}_2$ centralizes $\mathcal{C}_q$, then

$$[a_2a_3a_q, a_3a_q^k] = [a_3a_q, a_3a_q^k] = a_3a_qa_3a_q^ka_q^{-1}a_3^{-1}a_q^{-k}a_3^{-1} = a_q^{\check{\tau}+k\check{\tau}^2-\check{\tau}^2-k\check{\tau}} = a_q^{\check{\tau}(k-1)(\check{\tau}-1)}$$

which generates $\mathcal{C}_q$. Now consider $\{\check{b}, \check{c}\} = \{a_2a_3, a_3\gamma_p\}$. Since $\mathcal{C}_2$ centralizes $\mathcal{C}_p$, then

$$[a_2a_3, a_3\gamma_p] = [a_3, a_3\gamma_p] = a_3a_3\gamma_pa_3^{-1}\gamma_p^{-1}a_3^{-1} = \gamma_p^{\hat{\tau}^2-\hat{\tau}} = \gamma_p^{\hat{\tau}(\hat{\tau}-1)}$$

which generates $\mathcal{C}_p$. Therefore, $\langle b, c\rangle = G$ which contradicts the minimality of S.

Now we can assume $k = 1$. Then $c = a_3a_q\gamma_p$. Consider $\overline{G} = \mathcal{C}_2 \times \mathcal{C}_3$. We have $\overline{a} = \overline{c} = a_3$ and $\overline{b} = a_2a_3$. This implies that $|\overline{a}| = |\overline{c}| = 3$ and $|\overline{b}| = 6$. We have $C = (\overline{c}, \overline{b}, \overline{a}^2, \overline{b}, \overline{a})$ as a Hamiltonian cycle in $\mathrm{Cay}(\overline{G}; \overline{S})$. Since there is one occurrence of c in C, and it is the only generator of G that contains γ_p, then by Lemma 2.8 we conclude that the subgroup generated by $\mathbb{V}(C)$ is $\mathcal{C}_p$. Also,

$$\mathbb{V}(C) = cba^2ba$$

$$\begin{aligned}
&\equiv a_3a_q \cdot a_2a_3a_q \cdot a_3^2 \cdot a_2a_3a_q \cdot a_3 \pmod{\mathcal{C}_p} \\
&= a_3a_qa_3a_q^2a_3 \\
&= a_q^{\breve{\tau}+2\breve{\tau}^2} \\
&= a_q^{\breve{\tau}(1+2\breve{\tau})}.
\end{aligned}$$

We may assume this does not generate $\mathcal{C}_q$, for otherwise Factor Group Lemma 2.6 applies. Therefore, $1+2\breve{\tau} \equiv 0 \pmod q$. This implies that $\breve{\tau} \equiv -1/2 \pmod q$. By substituting $\breve{\tau} \equiv -1/2 \pmod q$ in

$$\breve{\tau}^2 + \breve{\tau} + 1 \equiv 0 \pmod q,$$

then we have $3/4 \equiv 0 \pmod q$, which contradicts Assumption 3.1(1).

Subcase 2.3. Assume $j = 0$. Then $i \neq 0$ and $c = a_2a_q^k\gamma_p$. If $k \neq 0$, then by Lemma 2.28(3) $\langle a, c\rangle = G$ which contradicts the minimality of S.

So we can assume $k = 0$. Then $c = a_2\gamma_p$. Consider $\overline{G} = \mathcal{C}_2 \times \mathcal{C}_3$. Then we have $\overline{a} = a_3$, $\overline{b} = a_2a_3$ and $\overline{c} = a_2$. This implies that $|\overline{a}| = 3$, $|\overline{b}| = 6$ and $|\overline{c}| = 2$. We have $C = (\overline{c}, \overline{a}, \overline{b}, \overline{a}^{-1}, \overline{b}^2)$ as a Hamiltonian cycle in $\mathrm{Cay}(\overline{G}; \overline{S})$. Since there is one occurrence of c in C, and it is the only generator of G that contains γ_p, then by Lemma 2.8 we conclude that the subgroup generated by $\mathbb{V}(C)$ contains $\mathcal{C}_p$. Also,

$$\begin{aligned}
\mathbb{V}(C) &= caba^{-1}b^2 \\
&\equiv a_2 \cdot a_3 \cdot a_2a_3a_q \cdot a_3^{-1} \cdot a_2a_3a_qa_2a_3a_q \pmod{\mathcal{C}_p} \\
&= a_3^2a_q^2a_3a_q \\
&= a_q^{2\breve{\tau}^2+1}.
\end{aligned}$$

We may assume this does not generate $\mathcal{C}_q$, for otherwise Factor Group Lemma 2.6 applies. Thus, $\breve{\tau}^2 \equiv -1/2 \pmod q$. By substituting this in

$$\breve{\tau}^2 + \breve{\tau} + 1 \equiv 0 \pmod q,$$

we have $\breve{\tau} \equiv -1/2 \pmod q$ which contradicts $\breve{\tau}^2 \equiv -1/2 \pmod q$.

Case 3. Assume $a = a_2a_3$ and $b = a_3a_q$. Since $b = a_3a_q$ is conjugate to a_3 via an element of $\mathcal{C}_q$, then $\{a, b\}$ is conjugate to $\{a_2a_3a_q^m, a_3\}$ for some nonzero m. So Case 2 applies (after replacing a_q with a_q^m).

Case 4. Assume $a = a_2a_3$ and $b = a_2a_q$.

Subcase 4.1. Assume $i = 0$. Then $j \neq 0$ and $c = a_3^ja_q^k\gamma_p$. If $k \neq 0$, then by Lemma 2.28(1) $\langle a, c\rangle = G$ which contradicts the minimality of S.

So we can assume $k = 0$. We may also assume $j = 1$, by replacing c with c^{-1} if necessary. Then $c = a_3\gamma_p$. Consider $\overline{G} = \mathcal{C}_2 \times \mathcal{C}_3$. Thus, $\overline{a} = a_2a_3$, $\overline{b} = a_2$ and $\overline{c} = a_3$. This implies that $|\overline{a}| = 6$, $|\overline{b}| = 2$ and $|\overline{c}| = 3$. We have $C = (\overline{a}^2, \overline{b}, \overline{c}, \overline{a}, \overline{c}^{-1})$ as a Hamiltonian cycle in $\mathrm{Cay}(\overline{G}; \overline{S})$. Since there is one occurrence of b in C, and it is the only generator of G that contains a_q, then by Lemma 2.8 we conclude that the subgroup generated by $\mathbb{V}(C)$ contains $\mathcal{C}_q$. Also,

$$\mathbb{V}(C) = a^2bcac^{-1}$$

$$\begin{aligned}&\equiv a_3^2 \cdot a_2 \cdot a_3\gamma_p \cdot a_2a_3 \cdot \gamma_p^{-1}a_3^{-1} \pmod{\mathcal{C}_q}\\&= \gamma_p a_3 \gamma_p^{-1} a_3^{-1}\\&= \gamma_p^{1-\hat{\tau}}\end{aligned}$$

which generates $\mathcal{C}_p$. Therefore, the subgroup generated by $\mathbb{V}(C)$ is G'. So, Factor Group Lemma 2.6 applies.

Subcase 4.2. Assume $j = 0$. Then $i \neq 0$ and $c = a_2a_q^k\gamma_p$. If $k \neq 0$, then by Lemma 2.28(1) $\langle a, c\rangle = G$ which contradicts the minimality of S.

So we can assume $k = 0$. Then $c = a_2\gamma_p$. Consider $\overline{G} = \mathcal{C}_2 \times \mathcal{C}_3$, then $\overline{a} = a_2a_3$ and $\overline{b} = \overline{c} = a_2$. This implies that $|\overline{a}| = 6$ and $|\overline{b}| = |\overline{c}| = 2$. We have $C = ((\overline{a}, \overline{b})^2, \overline{a}, \overline{c})$ as a Hamiltonian cycle in $\mathrm{Cay}(\overline{G}; \overline{S})$. Since there is one occurrence of c in C, and it is the only generator of G that contains γ_p, then by Lemma 2.8 we conclude that the subgroup generated by $\mathbb{V}(C)$ contains $\mathcal{C}_p$. Also,

$$\begin{aligned}\mathbb{V}(C) &= (ab)^2ac\\&\equiv (a_2a_3 \cdot a_2a_q)^2 \cdot a_2a_3 \cdot a_2 \pmod{\mathcal{C}_p}\\&= a_3a_qa_3a_qa_3\\&= a_q^{\check{\tau}+\check{\tau}^2}.\end{aligned}$$

which generates $\mathcal{C}_q$. Therefore, the subgroup generated by $\mathbb{V}(C)$ is G'. Thus, Factor Group Lemma 2.6 applies.

Subcase 4.3. Assume $i \neq 0$ and $j \neq 0$. Then $c = a_2a_3^ja_q^k\gamma_p$. If $k \neq 0$, then by Lemma 2.28(1) $\langle a, c\rangle = G$ which contradicts the minimality of S.

So we can assume $k = 0$. We may also assume $j = 1$, by replacing c with c^{-1} if necessary. Then $c = a_2a_3\gamma_p$. Consider $\overline{G} = \mathcal{C}_2 \times \mathcal{C}_3$. Thus, $\overline{a} = \overline{c} = a_2a_3$ and $\overline{b} = a_2$. This implies that $|\overline{a}| = |\overline{c}| = 6$ and $|\overline{b}| = 2$. We have $C = (\overline{a}, \overline{c}, \overline{b}, \overline{a}^{-2}, \overline{b})$ as a Hamiltonian cycle in $\mathrm{Cay}(\overline{G}; \overline{S})$. Since there is one occurrence of c in C, and it is the only generator of G that contains γ_p, then by Lemma 2.8 we conclude that the subgroup generated by $\mathbb{V}(C)$ contains $\mathcal{C}_p$. Also,

$$\begin{aligned}\mathbb{V}(C) &= acba^{-2}b\\&\equiv a_2a_3 \cdot a_2a_3 \cdot a_2a_q \cdot a_3^{-2} \cdot a_2a_q \pmod{\mathcal{C}_p}\\&= a_3^2a_qa_3^{-2}a_q\\&= a_q^{\check{\tau}^2+1}\end{aligned}$$

which generates $\mathcal{C}_q$, because $\check{\tau}^2 \not\equiv -1 \pmod{q}$. Therefore, the subgroup generated by $\mathbb{V}(C)$ is G'. So, Factor Group Lemma 2.6 applies.

Case 5. Assume $a = a_2a_3$ and $b = a_2a_3a_q$. If $k \neq 0$, then by Lemma 2.28(1) $\langle a, c\rangle = G$ which contradicts the minimality of S. So we can assume $k = 0$. Also, if $j \neq 0$, then by Lemma 2.28(4) $\langle b, c\rangle = G$ which contradicts the minimality of S. So we may also assume $j = 0$. Then $i \neq 0$. Therefore, $c = a_2\gamma_p$. So Case 4 applies, after interchanging b and c, and interchanging p and q. □

3.6 Assume $|S| = 3$, $G' = \mathcal{C}_p \times \mathcal{C}_q$ and $C_{G'}(\mathcal{C}_2) \neq \{e\}$

In this subsection, we prove the part of Theorem 1.3 where, $|S| = 3$, $G' = \mathcal{C}_p \times \mathcal{C}_q$, $C_{G'}(\mathcal{C}_2) \neq \{e\}$, and neither $C_{G'}(\mathcal{C}_2) = \mathcal{C}_p \times \mathcal{C}_q$ nor $C_{G'}(\mathcal{C}_3) \neq \{e\}$ nor $\widehat{S}$ is minimal holds. Recall $\overline{G} = G/G'$, $\check{G} = G/\mathcal{C}_q$ and $\widehat{G} = G/\mathcal{C}_p$.

Proposition 3.7. *Assume*

- $G = (\mathcal{C}_2 \times \mathcal{C}_3) \ltimes (\mathcal{C}_p \times \mathcal{C}_q)$,
- $|S| = 3$,
- $C_{G'}(\mathcal{C}_2) \neq \{e\}$.

Then $\mathrm{Cay}(G; S)$ *contains a Hamiltonian cycle.*

Proof. Let $S = \{a, b, c\}$. If $C_{G'}(\mathcal{C}_3) \neq \{e\}$, then Proposition 3.4 applies. Therefore, we may assume $C_{G'}(\mathcal{C}_3) = \{e\}$. Now if $C_{G'}(\mathcal{C}_2) = \mathcal{C}_p \times \mathcal{C}_q$, then Proposition 3.6 applies. Since $C_{G'}(\mathcal{C}_2) \neq \{e\}$, then we may assume $C_{G'}(\mathcal{C}_2) = \mathcal{C}_q$, by interchanging q and p if necessary. This implies that $\mathcal{C}_2$ inverts $\mathcal{C}_p$. Now if $\widehat{S}$ is minimal, then Proposition 3.5 applies. So we may assume $\widehat{S}$ is not minimal. Consider

$$\widehat{G} = G/\mathcal{C}_p = (\mathcal{C}_2 \times \mathcal{C}_3) \ltimes \mathcal{C}_q.$$

Choose a 2-element subset $\{a, b\}$ in S that generates $\widehat{G}$. From the minimality of S, we see that

$$\langle a, b \rangle = (\mathcal{C}_2 \times \mathcal{C}_3) \ltimes \mathcal{C}_q$$

after replacing a and b by conjugates. We may assume $|\overline{a}| \geqslant |\overline{b}|$ and (by conjugating if necessary) a is an element of $\mathcal{C}_2 \times \mathcal{C}_3$. Then the projection of (a, b) to $\mathcal{C}_2 \times \mathcal{C}_3$ has one of the following forms after replacing a and b with their inverses if necessary.

- $(a_2 a_3, a_2 a_3)$,
- $(a_2 a_3, a_2)$,
- $(a_2 a_3, a_3)$,
- (a_3, a_2).

So there are four possibilities for (a, b):

1. $(a_2 a_3, a_2 a_3 a_q)$,
2. $(a_2 a_3, a_2 a_q)$,
3. $(a_2 a_3, a_3 a_q)$,
4. $(a_3, a_2 a_q)$.

Let c be the third element of S. We may write $c = a_2^i a_3^j a_q^k \gamma_p$ with $0 \leqslant i \leqslant 1$, $0 \leqslant j \leqslant 2$ and $0 \leqslant k \leqslant q-1$. Note since $S \cap G' = \varnothing$, we know that i and j cannot both be equal to 0. Additionally, we have $a_3 \gamma_p a_3^{-1} = \gamma_p^{\hat{\tau}}$ where $\hat{\tau}^3 \equiv 1 \pmod p$ and $\hat{\tau} \not\equiv 1 \pmod p$. Thus $\hat{\tau}^2 + \hat{\tau} + 1 \equiv 0 \pmod p$. Note that this implies $\hat{\tau} \not\equiv -1 \pmod p$. Also we have $a_3 a_q a_3^{-1} = a_q^{\check{\tau}}$. By using the same argument we can conclude that $\check{\tau} \not\equiv 1 \pmod q$ and $\check{\tau}^2 + \check{\tau} + 1 \equiv 0 \pmod q$. Note that this implies $\check{\tau} \not\equiv -1 \pmod q$. Therefore, we conclude that $\hat{\tau}^2 \not\equiv \pm 1 \pmod p$, and $\check{\tau}^2 \not\equiv \pm 1 \pmod q$.

Case 1. Assume $a = a_2 a_3$ and $b = a_2 a_3 a_q$. If $k \neq 0$, then by Lemma 2.28(1), $\langle a, c\rangle = G$ which contradicts the minimality of S. So we can assume $k = 0$. Now if $j \neq 0$, then by Lemma 2.28(4), $\langle b, c\rangle = G$ which contradicts the minimality of S. Therefore, we may assume $j = 0$. Then $i \neq 0$ and $c = a_2 \gamma_p$. Consider $\overline{G} = \mathcal{C}_2 \times \mathcal{C}_3$. Thus $\overline{a} = \overline{b} = a_2 a_3$ and $\overline{c} = a_2$. Therefore, $|\overline{a}| = |\overline{b}| = 6$ and $|\overline{c}| = 2$. We have $C = (\overline{a}, \overline{b}, \overline{c}, \overline{a}^{-2}, \overline{c})$ as a Hamiltonian cycle in $\mathrm{Cay}(\overline{G}; \overline{S})$. Since there is one occurrence of b in C, and it is the only generator of G that contains a_q, then by Lemma 2.8 we conclude that the subgroup generated by $\mathbb{V}(C)$ contains $\mathcal{C}_q$. Also,

$$\begin{aligned}
\mathbb{V}(C) &= abca^{-2}c \\
&\equiv a_2 a_3 \cdot a_2 a_3 \cdot a_2 \gamma_p \cdot a_3^{-2} \cdot a_2 \gamma_p \pmod{\mathcal{C}_q} \\
&= a_3^2 \gamma_p^{-1} a_3^{-2} \gamma_p \\
&= \gamma_p^{-\hat{\tau}^2 + 1}
\end{aligned}$$

which generates $\mathcal{C}_p$. Therefore, the subgroup generated by $\mathbb{V}(C)$ is G'. So, Factor Group Lemma 2.6 applies.

Case 2. Assume $a = a_2 a_3$ and $b = a_2 a_q$.

Subcase 2.1. Assume $i = 0$. Then $j \neq 0$ and $c = a_3^j a_q^k \gamma_p$. If $k \neq 0$, then by Lemma 2.28(1), $\langle a, c\rangle = G$ which contradicts the minimality of S.

So we can assume $k = 0$. We may also assume $j = 1$, by replacing c with c^{-1} if necessary. Then $c = a_3 \gamma_p$. Consider $\overline{G} = \mathcal{C}_2 \times \mathcal{C}_3$. Thus, $\overline{a} = a_2 a_3$, $\overline{b} = a_2$ and $\overline{c} = a_3$. Therefore, $|\overline{a}| = 6$, $|\overline{b}| = 2$ and $|\overline{c}| = 3$. We have $C = (\overline{a}^2, \overline{b}, \overline{c}, \overline{a}, \overline{c}^{-1})$ as a Hamiltonian cycle in $\mathrm{Cay}(\overline{G}; \overline{S})$. Since there is one occurrence of b in C, and it is the only generator of G that contains a_q, then by Lemma 2.8 we conclude that the subgroup generated by $\mathbb{V}(C)$ contains $\mathcal{C}_q$. Also,

$$\begin{aligned}
\mathbb{V}(C) &= a^2 bc^{-1} ac \\
&\equiv a_3^2 \cdot a_2 \cdot a_3 \gamma_p \cdot a_2 a_3 \cdot \gamma_p^{-1} a_3^{-1} \pmod{\mathcal{C}_q} \\
&= \gamma_p^{-1} a_3 \gamma_p^{-1} a_3^{-1} \\
&= \gamma_p^{-1-\hat{\tau}}
\end{aligned}$$

which generates $\mathcal{C}_p$. Therefore, the subgroup generated by $\mathbb{V}(C)$ is G'. So, Factor Group Lemma 2.6 applies.

Subcase 2.2. Assume $j = 0$. Then $i \neq 0$ and $c = a_2 a_q^k \gamma_p$. If $k \neq 0$, then by Lemma 2.28(1), $\langle a, c\rangle = G$ which contradicts the minimality of S.

So we can assume $k = 0$. Then $c = a_2 \gamma_p$. Consider $\overline{G} = \mathcal{C}_2 \times \mathcal{C}_3$, then $\overline{a} = a_2 a_3$ and $\overline{b} = \overline{c} = a_2$. We have $C = ((\overline{a}, \overline{b})^2, \overline{a}, \overline{c})$ as a Hamiltonian cycle in $\mathrm{Cay}(\overline{G}; \overline{S})$. Since there is one occurrence of c in C, and it is the only generator of G that contains γ_p, then by Lemma 2.8 we conclude that the subgroup generated by $\mathbb{V}(C)$ contains $\mathcal{C}_p$. Now we calculate its voltage. Also,

$$\begin{aligned}\mathbb{V}(C) &= (ab)^2ac \\ &\equiv (a_2a_3 \cdot a_2a_q)^2 \cdot a_2a_3 \cdot a_2 \pmod{\mathcal{C}_p} \\ &= a_3a_qa_3a_qa_3 \\ &= a_q^{\breve{\tau}+\breve{\tau}^2}.\end{aligned}$$

which generates $\mathcal{C}_q$. Therefore, the subgroup generated by $\mathbb{V}(C)$ generates G'. So, Factor Group Lemma 2.6 applies.

Subcase 2.3. Assume $i \neq 0$ and $j \neq 0$. If $k \neq 0$, then $c = a_2a_3^ja_q^k\gamma_p$. Thus, by Lemma 2.28(1), $\langle a, c\rangle = G$ which contradicts the minimality of S.

So we can assume $k = 0$. We may also assume $j = 1$, by replacing c with c^{-1} if necessary. Then $c = a_2a_3\gamma_p$. Consider $\overline{G} = \mathcal{C}_2 \times \mathcal{C}_3$. Thus, $\overline{a} = \overline{c} = a_2a_3$ and $\overline{b} = a_2$. Therefore, $|\overline{a}| = |\overline{c}| = 6$ and $|\overline{b}| = 2$. We have $C = (\overline{a}, \overline{c}, \overline{b}, \overline{a}^{-2}, \overline{b})$ as a Hamiltonian cycle in $\mathrm{Cay}(\overline{G}; \overline{S})$. Since there is one occurrence of c in C, and it is the only generator of G that contains γ_p, then by Lemma 2.8 we conclude that the subgroup generated by $\mathbb{V}(C)$ contains $\mathcal{C}_p$. Also,

$$\begin{aligned}\mathbb{V}(C) &= acba^{-2}b \\ &\equiv a_2a_3 \cdot a_2a_3 \cdot a_2a_q \cdot a_3^{-2} \cdot a_2a_q \pmod{\mathcal{C}_p} \\ &= a_3^2a_qa_3^{-2}a_q \\ &= a_q^{\breve{\tau}^2+1}.\end{aligned}$$

Since $\breve{\tau}^2 \not\equiv -1 \pmod{q}$, Factor Group Lemma 2.6 applies.

Case 3. Assume $a = a_2a_3$ and $b = a_3a_q$.

Subcase 3.1. Assume $i \neq 0$ and $j \neq 0$. If $k = 0$, then $c = a_2a_3^j\gamma_p$. Thus, by Lemma 2.28(2), $\langle b, c\rangle = G$ which contradicts the minimality of S. So we can assume $k \neq 0$. Then $c = a_2a_3^ja_q^k\gamma_p$. Thus, by Lemma 2.28(1), $\langle a, c\rangle = G$ which contradicts the minimality of S.

Subcase 3.2. Assume $i = 0$. Then $j \neq 0$ and $c = a_3^ja_q^k\gamma_p$. If $k \neq 0$, then by Lemma 2.28(1), $\langle a, c\rangle = G$ which contradicts the minimality of S.

So we can assume $k = 0$. We may also assume $j = 1$, by replacing c with c^{-1} if necessary. Then $c = a_3\gamma_p$. Consider $\overline{G} = \mathcal{C}_2 \times \mathcal{C}_3$, then $\overline{a} = a_2a_3$, $\overline{b} = \overline{c} = a_3$. Therefore, $|\overline{a}| = 6$ and $|\overline{b}| = |\overline{c}| = 3$. We have $C = (\overline{c}, \overline{b}, \overline{a}, \overline{b}^{-2}, \overline{a}^{-1})$ as a Hamiltonian cycle in $\mathrm{Cay}(\overline{G}; \overline{S})$. Since there is one occurrence of c in C, and it is the only generator of G that contains γ_p, then by Lemma 2.8 we conclude that the subgroup generated by $\mathbb{V}(C)$ contains $\mathcal{C}_p$. Also,

$$\begin{aligned}\mathbb{V}(C) &= cbab^{-2}a^{-1} \\ &\equiv a_3 \cdot a_3a_q \cdot a_2a_3 \cdot a_q^{-1}a_3^{-1}a_q^{-1}a_3^{-1} \cdot a_3^{-1}a_2 \pmod{\mathcal{C}_p} \\ &= a_3^2a_qa_3a_q^{-1}a_3^{-1}a_q^{-1}a_3^{-2}\end{aligned}$$

$$= a_q^{\check{\tau}^2-1-\check{\tau}^{-1}}$$
$$= a_q^{\check{\tau}^2-1-\check{\tau}^2}$$
$$= a_q^{-1}$$

which generates $\mathcal{C}_q$. Therefore, the subgroup generated by $\mathbb{V}(C)$ is G'. So, Factor Group Lemma 2.6 applies.

Subcase 3.3. Assume $j = 0$. Then $i \neq 0$ and $c = a_2 a_q^k \gamma_p$. If $k \neq 0$, then by Lemma 2.28(1), $\langle a, c \rangle = G$ which contradicts the minimality of S.

So we can assume $k = 0$. Then $c = a_2\gamma_p$. Consider $\overline{G} = \mathcal{C}_2 \times \mathcal{C}_3$, then $\overline{a} = a_2 a_3$, $\overline{b} = a_3$ and $\overline{c} = a_2$. Therefore, $|\overline{a}| = 6$, $|\overline{b}| = 3$ and $|\overline{c}| = 2$. We have $C = (\overline{a}, \overline{c}, \overline{b}, \overline{a}, \overline{b}^{-1}, \overline{a})$ as a Hamiltonian cycle in $\mathrm{Cay}(\overline{G}; \overline{S})$. Since there is one occurrence of c in C, and it is the only generator of G that contains γ_p, then by Lemma 2.8 we conclude that the subgroup generated by $\mathbb{V}(C)$ contains $\mathcal{C}_p$. Also,

$$\begin{aligned}
\mathbb{V}(C) &= acbab^{-1}a \\
&\equiv a_2a_3 \cdot a_2 \cdot a_3a_q \cdot a_2a_3 \cdot a_q^{-1}a_3^{-1} \cdot a_2a_3 \pmod{\mathcal{C}_p} \\
&= a_3^2 a_q a_3 a_q^{-1} \\
&= a_q^{\check{\tau}^2-1}.
\end{aligned}$$

Since $\check{\tau}^2 \not\equiv 1 \pmod{q}$, Factor Group Lemma 2.6 applies.

Case 4. Assume $a = a_3$ and $b = a_2 a_q$.

Subcase 4.1. Assume $i = 0$. Then $j \neq 0$ and $c = a_3^j a_q^k \gamma_p$. Thus, the argument in Subcase 1.1 of Proposition 3.6 applies.

Subcase 4.2. Assume $j = 0$. Then $i \neq 0$ and $c = a_2 a_q^k \gamma_p$. Thus, the argument in Subcase 1.2 of Proposition 3.6 applies.

Subcase 4.3. Assume $i \neq 0$ and $j \neq 0$. Then $c = a_2 a_3^j a_q^k \gamma_p$. If $k \neq 0$, then by Lemma 2.28(3) $\langle a, c \rangle = G$ which contradicts the minimality of S.

So we can assume $k = 0$. We may also assume $j = 1$, by replacing c with c^{-1} if necessary. Then $c = a_2 a_3 \gamma_p$. Consider $\overline{G} = \mathcal{C}_2 \times \mathcal{C}_3$. Then we have $\overline{a} = a_3$, $\overline{b} = a_2$ and $\overline{c} = a_2 a_3$. This implies that $|\overline{a}| = 3$, $|\overline{b}| = 2$ and $|\overline{c}| = 6$. We have $C = (\overline{c}, \overline{b}, \overline{a}, \overline{c}, \overline{a}^{-1}, \overline{c})$ as a Hamiltonian cycle in $\mathrm{Cay}(\overline{G}; \overline{S})$. Since there is one occurrence of b in C, and it is the only generator of G that contains a_q, then by Lemma 2.8 we conclude that the subgroup generated by $\mathbb{V}(C)$ contains $\mathcal{C}_q$. Also, since a_2 inverts $\mathcal{C}_p$

$$\begin{aligned}
\mathbb{V}(C) &= cbaca^{-1}c \\
&\equiv a_2a_3\gamma_p \cdot a_2 \cdot a_3 \cdot a_2a_3\gamma_p \cdot a_3^{-1} \cdot a_2a_3\gamma_p \pmod{\mathcal{C}_q} \\
&= a_3\gamma_p^{-1}a_3^2 \\
&= \gamma_p^{-\hat{\tau}}
\end{aligned}$$

which generates $\mathcal{C}_p$. Therefore, the subgroup generated by $\mathbb{V}(C)$ is G'. So, Factor Group Lemma 2.6 applies. □

3.7 Assume $|S| = 3$, $G' = \mathcal{C}_p \times \mathcal{C}_q$ and $C_{G'}(\mathcal{C}_2) = \{e\}$

In this subsection, we prove the part of Theorem 1.3 where, $|S| = 3$, $G' = \mathcal{C}_p \times \mathcal{C}_q$, $C_{G'}(\mathcal{C}_2) = \{e\}$, and neither $C_{G'}(\mathcal{C}_3) \neq \{e\}$ nor $\widehat{S}$ is minimal holds. Recall $\overline{G} = G/G'$, $\check{G} = G/\mathcal{C}_q$ and $\widehat{G} = G/\mathcal{C}_p$.

Proposition 3.8. *Assume*

- $G = (\mathcal{C}_2 \times \mathcal{C}_3) \ltimes (\mathcal{C}_p \times \mathcal{C}_q)$,
- $|S| = 3$,
- $C_{G'}(\mathcal{C}_2) = \{e\}$.

Then $\mathrm{Cay}(G; S)$ *contains a Hamiltonian cycle.*

Proof. Let $S = \{a, b, c\}$. If $C_{G'}(\mathcal{C}_3) \neq \{e\}$, then Proposition 3.4 applies. So we may assume $C_{G'}(\mathcal{C}_3) = \{e\}$. Now if $\widehat{S}$ is minimal, then Proposition 3.5 applies. So we may assume $\widehat{S}$ is not minimal. Consider

$$\widehat{G} = G/\mathcal{C}_p = (\mathcal{C}_2 \times \mathcal{C}_3) \ltimes \mathcal{C}_q.$$

Choose a 2-element subset $\{a, b\}$ in S that generates $\widehat{G}$. From the minimality of S, we see

$$\langle a, b\rangle = (\mathcal{C}_2 \times \mathcal{C}_3) \ltimes \mathcal{C}_q.$$

after replacing a and b by conjugates. We may assume $|a| \geqslant |b|$ and (by conjugating if necessary) a is in $\mathcal{C}_2 \times \mathcal{C}_3$. Then the projection of (a, b) to $\mathcal{C}_2 \times \mathcal{C}_3$ is one of the following forms after replacing a and b with their inverses if necessary.

- (a_2a_3, a_2a_3),
- (a_2a_3, a_2),
- (a_2a_3, a_3),
- (a_3, a_2).

There are four possibilities for (a, b):

1. $(a_2a_3, a_2a_3a_q)$,
2. (a_2a_3, a_2a_q),
3. (a_2a_3, a_3a_q),
4. (a_3, a_2a_q).

Let c be the third element of S. We may write $c = a_2^i a_3^j a_q^k \gamma_p$ with $0 \leqslant i \leqslant 1$, $0 \leqslant j \leqslant 2$ and $0 \leqslant k \leqslant q-1$. Note since $S \cap G' = \varnothing$, we know that i and j cannot both be equal to 0. Additionally, we have $a_3\gamma_p a_3^{-1} = \gamma_p^{\hat{\tau}}$ where $\hat{\tau}^3 \equiv 1 \pmod{p}$ and $\hat{\tau} \not\equiv 1 \pmod{p}$. Thus $\hat{\tau}^2 + \hat{\tau} + 1 \equiv 0 \pmod{p}$. Note that this implies $\hat{\tau} \not\equiv -1 \pmod{p}$. We have $a_3 a_q a_3^{-1} = a_q^{\check{\tau}}$. By using the same argument we can conclude that $\check{\tau} \not\equiv 1 \pmod{q}$ and $\check{\tau}^2 + \check{\tau} + 1 \equiv 0 \pmod{q}$. Note that this implies $\check{\tau} \not\equiv -1 \pmod{q}$. Therefore, we conclude that $\hat{\tau}^2 \not\equiv \pm 1 \pmod{p}$, and $\check{\tau}^2 \not\equiv \pm 1 \pmod{q}$.

Case 1. Assume $a = a_2a_3$ and $b = a_2a_3a_q$. If $k \neq 0$, then by Lemma 2.28(1) $\langle a, c\rangle = G$ which contradicts the minimality of S. So we can assume $k = 0$. Now if $j \neq 0$, then by Lemma 2.28(4), $\langle b, c\rangle = G$ which contradicts the minimality of S. Therefore, we may assume $j = 0$. Then $i \neq 0$ and $c = a_2\gamma_p$. We have $\langle \overline{b}, \overline{c}\rangle = \langle \overline{a}_2\overline{a}_3, \overline{a}_2\rangle = \overline{G}$. Consider $\{\check{b}, \check{c}\} = \{a_2a_3, a_2\gamma_p\}$. Therefore,

$$[a_2a_3, a_2\gamma_p] = a_2a_3a_2\gamma_pa_3^{-1}a_2\gamma_p^{-1}a_2 = a_3\gamma_pa_3^{-1}\gamma_p = \gamma_p^{\hat{\tau}+1}.$$

which generates $\mathcal{C}_p$. Now consider $\{\hat{b}, \hat{c}\} = \{a_2a_3a_q, a_2\}$, then

$$[a_2a_3a_q, a_2] = a_2a_3a_qa_2a_q^{-1}a_3^{-1}a_2a_2 = a_3a_q^{-2}a_3^{-1} = a_q^{-2\check{\tau}}$$

which generates $\mathcal{C}_q$. Therefore, $\langle b, c\rangle = G$ which contradicts the minimality of S.

Case 2. Assume $a = a_2a_3$ and $b = a_2a_q$. If $k \neq 0$, then by Lemma 2.28(1), $\langle a, c\rangle = G$ which contradicts the minimality of S. So we can assume $k = 0$.

Subcase 2.1. Assume $j \neq 0$. We may also assume $j = 1$, by replacing c with c^{-1} if necessary. Then $c = a_2^ia_3\gamma_p$. We have $\langle \overline{b}, \overline{c}\rangle = \langle \overline{a}_2, \overline{a}_2^i\overline{a}_3\rangle = \overline{G}$. Consider $\{\hat{b}, \hat{c}\} = \{a_2a_q, a_2^ia_3\}$. We have

$$\begin{aligned}[a_2a_q, a_2^ia_3] &= a_2a_qa_2^ia_3a_q^{-1}a_2a_3^{-1}a_2^i = a_q^{-1}a_2^{i+1}a_3a_q^{-1}a_3^{-1}a_2^{i+1}\\ &= a_q^{-1}a_3a_q^{\mp 1}a_3^{-1} = a_q^{-1\mp\check{\tau}}\end{aligned}$$

which generates $\mathcal{C}_q$. Now consider $\{\check{b}, \check{c}\} = \{a_2, a_2^ia_3\gamma_p\}$. We have

$$[a_2, a_2^ia_3\gamma_p] = a_2a_2^ia_3\gamma_pa_2\gamma_p^{-1}a_3^{-1}a_2^i = a_2^{i+1}a_3\gamma_p^2a_3^{-1}a_2^{i+1} = \gamma_p^{\pm 2\hat{\tau}}$$

which generates $\mathcal{C}_p$. Therefore, $\langle b, c\rangle = G$ which contradicts the minimality of S.

Subcase 2.2. Assume $j = 0$. Then $i \neq 0$ and $c = a_2\gamma_p$. Consider $\overline{G} = \mathcal{C}_2 \times \mathcal{C}_3$, then $\overline{a} = a_2a_3$ and $\overline{b} = \overline{c} = a_2$. Thus, $|\overline{a}| = 6$ and $|\overline{b}| = |\overline{c}| = 2$. We have $C = ((\overline{a}, \overline{b})^2, \overline{a}, \overline{c})$ as a Hamiltonian cycle in $\mathrm{Cay}(\overline{G}; \overline{S})$. Since there is one occurrence of c in C, and it is the only generator of G that contains γ_p, then by Lemma 2.8 we conclude that the subgroup generated by $\mathbb{V}(C)$ contains $\mathcal{C}_p$. Also,

$$\begin{aligned}\mathbb{V}(C) &= (ab)^2(ac)\\ &\equiv a_2a_3 \cdot a_2a_q \cdot a_2a_3 \cdot a_2a_q \cdot a_2a_3 \cdot a_2 \pmod{\mathcal{C}_p}\\ &= a_3a_qa_3a_qa_3\\ &= a_q^{\check{\tau}+\check{\tau}^2}\end{aligned}$$

which generates $\mathcal{C}_q$. Therefore, the subgroup generated by $\mathbb{V}(C)$ is G'. So, Factor Group Lemma 2.6 applies.

Case 3. Assume $a = a_2a_3$ and $b = a_3a_q$. If $k \neq 0$, then by Lemma 2.28(1), $\langle a, c\rangle = G$ which contradicts the minimality of S. So we can assume $k = 0$.

Subcase 3.1. Assume $i \neq 0$ and $j \neq 0$. Then $c = a_2a_3^j\gamma_p$. Thus, by Lemma 2.28(2), $\langle b, c\rangle = G$ which contradicts the minimality of S.

Subcase 3.2. Assume $j = 0$. Then $i \neq 0$ and $c = a_2\gamma_p$. We have $\langle \overline{b}, \overline{c} \rangle = \langle \overline{a}_3, \overline{a}_2 \rangle = \overline{G}$. Consider $\{\check{b}, \check{c}\} = \{a_3, a_2\gamma_p\}$. Then we have

$$[a_3, a_2\gamma_p] = a_3a_2\gamma_p a_3^{-1}\gamma_p^{-1}a_2 = a_3\gamma_p^{-1}a_3^{-1}\gamma_p = \gamma_p^{-\hat{\tau}+1}$$

which generates $\mathcal{C}_p$. Now consider $\{\hat{b}, \hat{c}\} = \{a_3a_q, a_2\}$. Thus,

$$[a_3a_q, a_2] = a_3a_qa_2a_q^{-1}a_3^{-1}a_2 = a_3a_q^2a_3^{-1} = a_q^{2\check{\tau}}$$

which generates $\mathcal{C}_q$. Therefore, $\langle b, c \rangle = G$ which contradicts the minimality of S.

Subcase 3.3. Assume $i = 0$. Then $j \neq 0$. We may also assume $j = 1$, by replacing c with c^{-1} if necessary. Then $c = a_3\gamma_p$. Consider $\overline{G} = \mathcal{C}_2 \times \mathcal{C}_3$, then we have $\overline{a} = a_2a_3$, $\overline{b} = \overline{c} = a_3$. Thus, $|\overline{a}| = 6$ and $|\overline{b}| = |\overline{c}| = 3$. We have $C = (\overline{c}, \overline{b}, \overline{a}, \overline{b}^{-2}, \overline{a}^{-1})$ as a Hamiltonian cycle in $\mathrm{Cay}(\overline{G}; \overline{S})$. Since there is one occurrence of c in C, and it is the only generator of G that contains γ_p, then by Lemma 2.8 we conclude that the subgroup generated by $\mathbb{V}(C)$ contains $\mathcal{C}_p$. Also,

$$\begin{aligned}
\mathbb{V}(C) &= cbab^{-2}a^{-1} \\
&\equiv a_3 \cdot a_3a_q \cdot a_2a_3 \cdot a_q^{-1}a_3^{-1}a_q^{-1}a_3^{-1} \cdot a_3^{-1}a_2 \pmod{\mathcal{C}_p} \\
&= a_3^2a_qa_3a_qa_3^{-1}a_qa_3^{-2} \\
&= a_q^{\check{\tau}^2+1+\check{\tau}^{-1}} \\
&= a_q^{\check{\tau}^2+1-\check{\tau}^2} \\
&= a_q
\end{aligned}$$

which generates $\mathcal{C}_q$. Therefore, the subgroup generated by $\mathbb{V}(C)$ is G'. So, Factor Group Lemma 2.6 applies.

Case 4. Assume $a = a_3$ and $b = a_2a_q$.

Subcase 4.1. Assume $i = 0$. Then $j \neq 0$. We may also assume $j = 1$, by replacing c with c^{-1} if necessary. Then $c = a_3a_q^k\gamma_p$. Consider $\overline{G} = \mathcal{C}_2 \times \mathcal{C}_3$. Then we have $\overline{a} = \overline{c} = a_3$ and $\overline{b} = a_2$. This implies that $|\overline{a}| = |\overline{c}| = 3$ and $|\overline{b}| = 2$. We have $C = (\overline{c}^{-2}, \overline{b}, \overline{a}^2, \overline{b})$ as a Hamiltonian cycle in $\mathrm{Cay}(\overline{G}; \overline{S})$. Now we calculate its voltage.

$$\begin{aligned}
\mathbb{V}(C) &= c^{-2}ba^2b \\
&\equiv \gamma_p^{-1}a_3^{-1}\gamma_p^{-1}a_3^{-1} \cdot a_2 \cdot a_3^2 \cdot a_2 \pmod{\mathcal{C}_q} \\
&= \gamma_p^{-1}a_3^{-1}\gamma_p^{-1}a_3 \\
&= \gamma_p^{-1-\hat{\tau}^{-1}}
\end{aligned}$$

which generates $\mathcal{C}_p$. Also

$$\begin{aligned}
\mathbb{V}(C) &= c^{-2}ba^2b \\
&\equiv a_q^{-k}a_3^{-1}a_q^{-k}a_3^{-1} \cdot a_2a_q \cdot a_3^2 \cdot a_2a_q \pmod{\mathcal{C}_p} \\
&= a_q^{-k}a_3^{-1}a_q^{-k}a_3^{-1}a_q^{-1}a_3^2a_q
\end{aligned}$$

$$= a_q^{-k-k\check{\tau}^{-1}-\check{\tau}^{-2}+1}.$$

If $k = 2$, then

$$a_q^{-k-k\check{\tau}^{-1}-\check{\tau}^{-2}+1} = a_q^{-2-2\check{\tau}^{-1}-\check{\tau}^{-2}+1} = a_q^{-(\check{\tau}^{-1}+1)^2}$$

which generates $\mathcal{C}_q$. So we may assume $k \neq 2$ and the subgroup generated by $\mathbb{V}(C)$ does not contain $\mathcal{C}_q$, for otherwise Factor Group Lemma 2.6 applies. Therefore,

$$\begin{aligned} 0 &\equiv -k - k\check{\tau}^{-1} - \check{\tau}^{-2} + 1 \pmod{q} \\ &= (1-k) - k\check{\tau}^{-1} - \check{\tau}^{-2}. \end{aligned}$$

Multiplying by $\check{\tau}^2$, we have

$$0 \equiv (1-k)\check{\tau}^2 - k\check{\tau} - 1 \pmod{q}. \tag{4.1A}$$

We can replace $\check{\tau}$ with $\check{\tau}^{-1}$ in the above equation, by replacing a_3,a and c with their inverses.

$$0 \equiv (1-k)\check{\tau}^{-2} - k\check{\tau}^{-1} - 1 \pmod{q}.$$

Multiplying by $\check{\tau}^2$, then

$$0 \equiv (1-k) - k\check{\tau} - \check{\tau}^2 \pmod{q}.$$

By subtracting 4.1A from the above equation, we have

$$0 \equiv (k-2)\check{\tau}^2 + (2-k) \pmod{q}.$$

This implies that $\check{\tau}^2 \equiv 1 \pmod{q}$, a contradiction.

Subcase 4.2. Assume $j = 0$. Then $i \neq 0$. If $k \neq 0$, then $c = a_2 a_q^k \gamma_p$. Thus, by Lemma 2.28(3), $\langle a, c\rangle = G$ which contradicts the minimality of S. So we can assume $k = 0$. Then $c = a_2\gamma_p$. Consider $\overline{G} = \mathcal{C}_2 \times \mathcal{C}_3$, then $\overline{a} = a_3$ and $\overline{b} = \overline{c} = a_2$. We have $C = (\overline{a}^2, \overline{b}, \overline{a}^{-2}, \overline{c})$ as a Hamiltonian cycle in $\mathrm{Cay}(\overline{G}; \overline{S})$. Since there is one occurrence of c in C, and it is the only generator of G that contains γ_p, then by Lemma 2.8 we conclude that the subgroup generated by $\mathbb{V}(C)$ contains $\mathcal{C}_p$. Similarly, since there is one occurrence of b in C, and it is the only generator of G that contains a_q, then by Lemma 2.8 we conclude that the subgroup generated by $\mathbb{V}(C)$ contains $\mathcal{C}_q$. Therefore, the subgroup generated by $\mathbb{V}(C)$ is G'. So, Factor Group Lemma 2.6 applies.

Subcase 4.3. Assume $i \neq 0$ and $j \neq 0$. If $k \neq 0$, then $c = a_2 a_3^j a_q^k \gamma_p$. Thus, by Lemma 2.28(3), $\langle a, c\rangle = G$ which contradicts the minimality of S. So we can assume $k = 0$. We may also assume $j = 1$, by replacing c with c^{-1} if necessary. Then $c = a_2 a_3 \gamma_p$. We have $\langle \overline{b}, \overline{c}\rangle = \langle \overline{a}_2, \overline{a}_2\overline{a}_3\rangle = \overline{G}$. Consider $\{\hat{b}, \hat{c}\} = \{a_2 a_q, a_2 a_3\}$. Then we have

$$[a_2 a_q, a_2 a_3] = a_2 a_q a_2 a_3 a_q^{-1} a_2 a_3^{-1} a_2 = a_q^{-1} a_3 a_q^{-1} a_3^{-1} = a_q^{-1-\check{\tau}}$$

which generates $\mathcal{C}_q$. Now consider $\{\check{b}, \check{c}\} = \{a_2, a_2 a_3 \gamma_p\}$. Then

$$[a_2, a_2 a_3 \gamma_p] = a_2 a_2 a_3 \gamma_p a_2 \gamma_p^{-1} a_3^{-1} a_2 = a_3 \gamma_p^2 a_3^{-1} = \gamma_p^{2\hat{\tau}}$$

which generates $\mathcal{C}_p$. Therefore, $\langle b, c\rangle = G$ which contradicts the minimality of S. □

3.8 Assume $|S| = 3$ and $G' = \mathcal{C}_3 \times \mathcal{C}_p$

In this subsection, we prove the part of Theorem 1.3 where, $|S| = 3$ and $G' = \mathcal{C}_3 \times \mathcal{C}_p$. Recall $\overline{G} = G/G'$, $\widehat{G} = G/\mathcal{C}_p$ and $\overleftrightarrow{G} = G/\mathcal{C}_3$.

Proposition 3.9. *Assume*

- $G = (\mathcal{C}_2 \times \mathcal{C}_q) \ltimes (\mathcal{C}_3 \times \mathcal{C}_p)$,
- $|S| = 3$.

Then $\mathrm{Cay}(G; S)$ *contains a Hamiltonian cycle.*

Proof. Let $S = \{a, b, c\}$. Since $\mathcal{C}_q$ centralizes $\mathcal{C}_3$ and $Z(G) \cap G' = \{e\}$ (by Proposition 2.15(2)), then $\mathcal{C}_2$ inverts $\mathcal{C}_3$. Now if $\widehat{S}$ is minimal, then Lemma 2.24 applies. So we may assume $\widehat{S}$ is not minimal. Consider

$$\widehat{G} = G/\mathcal{C}_p = (\mathcal{C}_2 \times \mathcal{C}_q) \ltimes \mathcal{C}_3.$$

Choose a 2-element subset $\{a, b\}$ in S that generates $\widehat{G}$. From the minimality of S we see

$$\langle a, b\rangle = (\mathcal{C}_2 \times \mathcal{C}_q) \ltimes \mathcal{C}_3.$$

after replacing a and b with conjugates. Then the projection of (a, b) to $\mathcal{C}_2 \times \mathcal{C}_q$ has one of the following forms:

- $(a_2a_q, a_2a_q^m)$, where $1 \leqslant m \leqslant q-1$,
- (a_2a_q, a_2),
- (a_2a_q, a_q^m), where $1 \leqslant m \leqslant q-1$,
- (a_2, a_q).

Thus, there are four different possibilities for (a, b) after assuming, without loss of generality, that $a \in \mathcal{C}_2 \times \mathcal{C}_q$:

1. $(a_2a_q, a_2a_q^m a_3)$,
2. (a_2a_q, a_2a_3),
3. $(a_2a_q, a_q^m a_3)$,
4. (a_2, a_qa_3).

Let c be the third element of S. We may write $c = a_2^i a_q^j a_3^k \gamma_p$ with $0 \leqslant i \leqslant 1, 0 \leqslant j \leqslant q-1$ and $0 \leqslant k \leqslant 2$. Since $\mathcal{C}_q$ centralizes $\mathcal{C}_3$, we may assume $\mathcal{C}_q$ does not centralize $\mathcal{C}_p$, for otherwise Lemma 2.26 applies. Now we have $a_q\gamma_p a_q^{-1} = \gamma_p^{\hat{\tau}}$, where $\hat{\tau}^q \equiv 1 \pmod p$. We also have $\hat{\tau} \not\equiv 1 \pmod p$. Since $\hat{\tau}^q \equiv 1 \pmod p$, this implies

$$\hat{\tau}^{q-1} + \hat{\tau}^{q-2} + \cdots + 1 \equiv 0 \pmod p.$$

Note that this implies $\hat{\tau} \not\equiv -1 \pmod p$.

Case 1. Assume $a = a_2 a_q$ and $b = a_2 a_q^m a_3$. If $k \neq 0$, then by Lemma 2.29(1) $\langle a, c\rangle = G$ which contradicts the minimality of S. So we can assume $k = 0$. Now if $i \neq 0$, then by Lemma 2.29(3) $\langle b, c\rangle = G$ which contradicts the minimality of S. Therefore, we may assume $i = 0$. Then $j \neq 0$ and $c = a_q^j \gamma_p$.

Consider $\overline{G} = \mathcal{C}_2 \times \mathcal{C}_q$. Then we have $\overline{a} = a_2 a_q$, $\overline{b} = a_2 a_q^m$ and $\overline{c} = a_q^j$. We may assume m is odd by replacing b with b^{-1} (and m with $q - m$) if necessary. Note that this implies $\overline{b} = \overline{a}^m$. Also, we have $|\overline{a}| = |\overline{b}| = 2q$ and $|\overline{c}| = q$.

Subcase 1.1. Assume $m = 1$. Then $\overline{a} = \overline{b}$. We have

$$C = (\overline{c}^{q-1}, \overline{b}, \overline{c}^{-(q-1)}, \overline{a}^{-1})$$

as a Hamiltonian cycle in $\mathrm{Cay}(\overline{G}; \overline{S})$. Since there is one occurrence of b in C, and it is the only generator of G that contains a_3, then by Lemma 2.8 we conclude that the subgroup generated by $\mathbb{V}(C)$ contains $\mathcal{C}_3$. Now by considering the fact that $\mathcal{C}_2$ might centralize $\mathcal{C}_p$ or not we have

$$\begin{aligned}
\mathbb{V}(C) &= c^{q-1} b c^{-(q-1)} a^{-1} \\
&\equiv (a_q^j \gamma_p)^{q-1} \cdot a_2 a_q \cdot (a_q^j \gamma_p)^{-(q-1)} \cdot a_q^{-1} a_2 \pmod{\mathcal{C}_3} \\
&= \gamma_p^{\hat{\tau}^j + \hat{\tau}^{2j} + \cdots + \hat{\tau}^{(q-1)j}} a_q^{(q-1)j} a_2 a_q a_q^{-(q-1)j} \gamma_p^{-(\hat{\tau}^j + \hat{\tau}^{2j} + \cdots + \hat{\tau}^{(q-1)j})} a_q^{-1} a_2 \\
&= \gamma_p^{\hat{\tau}^j (1 + \hat{\tau}^j + \cdots + \hat{\tau}^{(q-2)j})} a_q \gamma_p^{\mp \hat{\tau}^j (1 + \hat{\tau}^j + \cdots + \hat{\tau}^{(q-2)j})} a_q^{-1}.
\end{aligned}$$

Now if $\hat{\tau}^j \not\equiv 1 \pmod{p}$, then

$$\begin{aligned}
\mathbb{V}(C) &= \gamma_p^{\hat{\tau}^j (1 + \hat{\tau}^j + \cdots + \hat{\tau}^{(q-2)j})} a_q \gamma_p^{\mp \hat{\tau}^j (1 + \hat{\tau}^j + \cdots + \hat{\tau}^{(q-2)j})} a_q^{-1} \\
&= \gamma_p^{\hat{\tau}^j ((\hat{\tau}^j)^{q-1} - 1)/(\hat{\tau}^j - 1) \mp \hat{\tau}^{j+1} ((\hat{\tau}^j)^{q-1} - 1)/(\hat{\tau}^j - 1)} \\
&= \gamma_p^{\hat{\tau}^j ((\hat{\tau}^{-j}) - 1)/(\hat{\tau}^j - 1) \mp \hat{\tau}^{j+1} ((\hat{\tau}^{-j}) - 1)/(\hat{\tau}^j - 1)} \\
&= \gamma_p^{(1 - \hat{\tau}^j)(1 \mp \hat{\tau})/(\hat{\tau}^j - 1)} \\
&= \gamma_p^{-(1 \mp \hat{\tau})}.
\end{aligned}$$

We may assume this does not generate $\mathcal{C}_p$, for otherwise Factor Group Lemma 2.6 applies. Therefore, $\hat{\tau}^j \equiv 1 \pmod{p}$ or $\hat{\tau} \equiv \pm 1 \pmod{p}$. The second case is impossible. So we must have $\hat{\tau}^j \equiv 1 \pmod{p}$. We also know that $\hat{\tau}^q \equiv 1 \pmod{p}$. So $\hat{\tau}^d \equiv 1 \pmod{p}$, where $d = \gcd(j, q)$. Since $1 \leqslant j \leqslant q - 1$, then $d = 1$, which contradicts the fact that $\hat{\tau} \not\equiv 1 \pmod{p}$.

Subcase 1.2. Assume $m \neq 1$ and $j = 2$. Then $c = a_q^2 \gamma_p$. We have

$$C = (\overline{b}, \overline{c}^{-(m-1)/2}, \overline{a}, \overline{c}^{(m-1)/2}, \overline{a}^{2q-m-1})$$

as a Hamiltonian cycle in $\mathrm{Cay}(\overline{G}; \overline{S})$. Since there is one occurrence of b in C, and it is the only generator of G that contains a_3, then by Lemma 2.8 we conclude that the subgroup generated by $\mathbb{V}(C)$ contains $\mathcal{C}_3$. Considering the fact that $\mathcal{C}_2$ might centralize $\mathcal{C}_p$ or not we have

$$\mathbb{V}(C) = b c^{-(m-1)/2} a c^{(m-1)/2} a^{2q-m-1}$$

$$\begin{aligned}
&\equiv a_2 a_q^m \cdot (a_q^2\gamma_p)^{-(m-1)/2} \cdot a_2 a_q \cdot (a_q^2\gamma_p)^{(m-1)/2} \cdot a_q^{2q-m-1} \pmod{\mathcal{C}_3}\\
&= a_2 a_q^m (\gamma_p^{\hat\tau^2+(\hat\tau^2)^2+\cdots+(\hat\tau^2)^{(m-1)/2}} a_q^{(m-1)})^{-1} a_2 a_q (\gamma_p^{\hat\tau^2+(\hat\tau^2)^2+\cdots+(\hat\tau^2)^{(m-1)/2}} a_q^{(m-1)}) a_q^{-m-1}\\
&= a_2 a_q^m a_q^{-m+1} \gamma_p^{-\hat\tau^2(1+\hat\tau^2+\cdots+(\hat\tau^2)^{(m-3)/2})} a_2 a_q \gamma_p^{\hat\tau^2(1+\hat\tau^2+\cdots+(\hat\tau^2)^{(m-3)/2})} a_q^{-2}\\
&= a_q \gamma_p^{\pm\hat\tau^2(1+\hat\tau^2+\cdots+\hat\tau^2)^{(m-3)/2}} a_q \gamma_p^{\hat\tau^2(1+\hat\tau^2+\cdots+\hat\tau^2)^{(m-3)/2}} a_q^{-2}\\
&= \gamma_p^{\pm\hat\tau^3(\hat\tau^{m-1}-1)/(\hat\tau^2-1)+\hat\tau^4(\hat\tau^{m-1}-1)/(\hat\tau^2-1)}\\
&= \gamma_p^{\hat\tau^3(\hat\tau^{m-1}-1)(\pm1+\hat\tau)/(\hat\tau^2-1)}.
\end{aligned}$$

We may assume this does not generate $\mathcal{C}_p$, for otherwise Factor Group Lemma 2.6 applies. Therefore, $\hat\tau^{m-1} \equiv 1 \pmod p$. We also know that $\hat\tau^q \equiv 1 \pmod p$. So $\hat\tau^d \equiv 1 \pmod p$, where $d = \gcd(m-1, q)$. Since $2 \leqslant m \leqslant q-1$, then $d = 1$, which contradicts the fact that $\hat\tau \not\equiv 1 \pmod p$.

Subcase 1.3. Assume $m \neq 1$ and $j \neq 2$. We may also assume j is an even number, by replacing c with its inverse and j with $q-j$ if necessary. This implies that $\overline{c} = \overline{a}^j$. We have

$$C = (\overline{b}, \overline{c}, \overline{a}, \overline{c}^{-1}, \overline{b}^{-1}, \overline{a}^{m-2}, \overline{c}, \overline{a}^{-(j-3)}, \overline{c}, \overline{a}^{2q-m-j-2})$$

as a Hamiltonian cycle in $\mathrm{Cay}(\overline{G}; \overline{S})$. Now we calculate its voltage.

$$\begin{aligned}
\mathbb{V}(C) &= bcac^{-1}b^{-1}a^{m-2}ca^{-(j-3)}ca^{2q-m-j-2}\\
&\equiv a_2 a_3 \cdot a_2 \cdot a_3^{-1} a_2 \cdot a_2^{m-2} \cdot a_2^{-(j-3)} \cdot a_2^{2q-m-j-2} \pmod{\mathcal{C}_q \ltimes \mathcal{C}_p}\\
&= a_2 a_3 a_2 a_3^{-1}\\
&= a_3^{-2}
\end{aligned}$$

which generates $\mathcal{C}_3$. Also considering the fact that $\mathcal{C}_2$ might centralize $\mathcal{C}_p$ or not we have

$$\begin{aligned}
\mathbb{V}(C) &= bcac^{-1}b^{-1}a^{m-2}ca^{-(j-3)}ca^{2q-m-j-2}\\
&\equiv a_2 a_q^m \cdot a_q^j \gamma_p \cdot a_2 a_q \cdot \gamma_p^{-1} a_q^{-j} \cdot a_q^{-m} a_2\\
&\qquad \cdot a_2 a_q^{m-2} \cdot a_q^j \gamma_p \cdot a_q^{-j+3} a_2 \cdot a_q^j \gamma_p \cdot a_2 a_q^{2q-m-j-2} \pmod{\mathcal{C}_3}\\
&= a_q^{m+j} \gamma_p^{\pm1} a_q \gamma_p^{-1} a_q^{-2} \gamma_p a_q^3 \gamma_p^{\pm1} a_q^{-m-j-2}\\
&= \gamma_p^{\pm\hat\tau^{m+j}-\hat\tau^{m+j+1}+\hat\tau^{m+j-1}\pm\hat\tau^{m+j+2}}\\
&= \gamma_p^{\hat\tau^{m+j-1}(\pm\hat\tau^3-\hat\tau^2\pm\hat\tau+1)}.
\end{aligned}$$

So we may assume this does not generate $\mathcal{C}_p$, for otherwise Factor Group Lemma 2.6 applies. Then we have

$$0 \equiv \pm\hat\tau^3 - \hat\tau^2 \pm \hat\tau + 1 \pmod p.$$

Let $t = \hat\tau$ if $\mathcal{C}_2$ centralizes $\mathcal{C}_p$ and $t = -\hat\tau$ if $\mathcal{C}_2$ inverts $\mathcal{C}_p$. Then

$$0 \equiv t^3 - t^2 + t + 1 \pmod p. \tag{1.3A}$$

We can replace t with t^{-1} in the above equation after replacing $\{a, b, c\}$ with their inverses, then

$$0 \equiv t^{-3} - t^{-2} + t^{-1} + 1 \pmod p.$$

Multiplying by t^3, we have

$$\begin{aligned} 0 &\equiv 1 - t + t^2 + t^3 \pmod{p} \\ &= t^3 + t^2 - t + 1. \end{aligned}$$

By subtracting 1.3A from the above equation, we have

$$\begin{aligned} 0 &\equiv 2t^2 - 2t \pmod{p} \\ &= 2t(t-1) \end{aligned}$$

This implies that $t \equiv 1 \pmod{p}$ which contradicts the fact that $\hat{\tau} \not\equiv \pm 1 \pmod{p}$.

Case 2. Assume $a = a_2 a_q$ and $b = a_2 a_3$. If $k \neq 0$, then by Lemma 2.29(1) $\langle a, c \rangle = G$ which contradicts the minimality of S. So we can assume $k = 0$.

Subcase 2.1. Assume $i = 0$. Then $j \neq 0$ and $c = a_q^j \gamma_p$. We may assume j is an odd number, by replacing c with its inverse and j with $q - j$ if necessary. Consider $\overline{G} = \mathcal{C}_2 \times \mathcal{C}_q$. Then we have $\overline{a} = a_2 a_q$, $\overline{b} = a_2$ and $\overline{c} = a_q^j$. Also, we have $|\overline{a}| = 2q$, $|\overline{b}| = 2$ and $|\overline{c}| = q$. Now if $j \neq 1$, then we have

$$C = (\overline{c}, \overline{a}^{-1}, \overline{b}, \overline{a}^2, \overline{b}, \overline{c}^{-1}, \overline{a}^{j-3}, \overline{b}, \overline{a}^{-(q-4)}, \overline{b}, \overline{a}^{q-j-2})$$

as a Hamiltonian cycle in $\mathrm{Cay}(\overline{G}; \overline{S})$. Now we calculate the voltage of C.

$$\begin{aligned} \mathbb{V}(C) &= ca^{-1}ba^2bc^{-1}a^{j-3}ba^{-(q-4)}ba^{q-j-2} \\ &\equiv a_2 \cdot a_2a_3 \cdot a_2^2 \cdot a_2a_3 \cdot a_2^{j-3} \cdot a_2a_3 \cdot a_2^{-(q-4)} \cdot a_2a_3 \cdot a_2^{q-j-2} \pmod{\mathcal{C}_q \ltimes \mathcal{C}_p} \\ &= a_3a_2a_3a_2a_3a_2a_2a_3 \\ &= a_3^2 \end{aligned}$$

which generates $\mathcal{C}_3$. By considering the fact that $\mathcal{C}_2$ might centralize $\mathcal{C}_p$ or not, we have

$$\begin{aligned} \mathbb{V}(C) &= ca^{-1}ba^2bc^{-1}a^{j-3}ba^{-(q-4)}ba^{q-j-2} \\ &\equiv a_q^j\gamma_p \cdot a_q^{-1}a_2 \cdot a_2 \cdot a_q^2 \cdot a_2 \cdot \gamma_p^{-1}a_q^{-j} \cdot a_q^{j-3} \cdot a_2 \cdot a_2a_q^{-q+4} \cdot a_2 \cdot a_q^{q-j-2} \pmod{\mathcal{C}_3} \\ &= a_q^j\gamma_p a_q \gamma_p^{\mp 1} a_q^{-j-1} \\ &= \gamma_p^{\hat{\tau}^j \mp \hat{\tau}^{j+1}} \\ &= \gamma_p^{\hat{\tau}^j(1 \mp \hat{\tau})} \end{aligned}$$

which generates $\mathcal{C}_p$. Therefore, the subgroup generated by $\mathbb{V}(C)$ is G'. Thus, Factor Group Lemma 2.6 applies.

So we may assume $j = 1$, then $c = a_q\gamma_p$ and $\overline{c} = a_q$. We have

$$C_1 = ((\overline{b}, \overline{c})^{q-1}, \overline{b}, \overline{a})$$

as a Hamiltonian cycle in $\mathrm{Cay}(\overline{G}; \overline{S})$. Now we calculate its voltage.

$$\begin{aligned} \mathbb{V}(C_1) &= (bc)^{q-1}ba \\ &\equiv (a_2a_3)^{q-1} \cdot a_2a_3 \cdot a_2 \pmod{\mathcal{C}_q \ltimes \mathcal{C}_p} \end{aligned}$$

$$= a_3^{-1}$$

which generates $\mathcal{C}_3$. If $\mathcal{C}_2$ centralizes $\mathcal{C}_p$, then

$$\begin{aligned}
\mathbb{V}(C_1) &= (bc)^{q-1}ba \\
&\equiv (a_2 \cdot a_q\gamma_p)^{q-1} \cdot a_2 \cdot a_2 a_q \pmod{\mathcal{C}_3} \\
&= (a_q\gamma_p)^{q-1}a_q \\
&= \gamma_p^{\hat{\tau}+\hat{\tau}^2+\cdots+\hat{\tau}^{q-1}} \\
&= \gamma_p^{-1}
\end{aligned}$$

which generates $\mathcal{C}_p$. So in this case, the subgroup generated by $\mathbb{V}(C_1)$ is G'. Thus, Factor Group Lemma 2.6 applies.

Now if $\mathcal{C}_2$ inverts $\mathcal{C}_p$, then

$$\begin{aligned}
\mathbb{V}(C_1) &= (bc)^{q-1}ba \\
&\equiv (a_2 \cdot a_q\gamma_p)^{q-1} \cdot a_2 \cdot a_2 a_q \pmod{\mathcal{C}_3} \\
&= \gamma_p^{-\hat{\tau}+\hat{\tau}^2-\cdots-\hat{\tau}^{q-2}+\hat{\tau}^{q-1}}.
\end{aligned}$$

Since $\hat{\tau} \not\equiv -1 \pmod{p}$, then

$$\begin{aligned}
\mathbb{V}(C_1) &= \gamma_p^{-\hat{\tau}+\hat{\tau}^2-\cdots-\hat{\tau}^{q-2}+\hat{\tau}^{q-1}} \\
&= \gamma_p^{(\hat{\tau}^q+1)/(\hat{\tau}+1)-1}.
\end{aligned}$$

We may assume this does not generate $\mathcal{C}_p$, for otherwise Factor Group Lemma 2.6 applies. Therefore, since $\hat{\tau}^q \equiv 1 \pmod{p}$, then

$$\begin{aligned}
0 &\equiv (\hat{\tau}^q + 1)/(\hat{\tau} + 1) - 1 \pmod{p} \\
&= 2/(\hat{\tau} + 1) - 1.
\end{aligned}$$

This implies that $\hat{\tau} \equiv 1 \pmod{p}$, which is impossible.

Subcase 2.2. Assume $j = 0$. Then $i \neq 0$ and $c = a_2\gamma_p$. Consider $\overline{G} = \mathcal{C}_2 \times \mathcal{C}_q$. Then we have $\overline{a} = a_2 a_q$ and $\overline{b} = \overline{c} = a_2$. This implies that $|\overline{a}| = 2q$ and $|\overline{b}| = |\overline{c}| = 2$. We have

$$C = (\overline{c}, \overline{a}^{q-1}, \overline{b}, \overline{a}^{-(q-1)})$$

as a Hamiltonian cycle in $\mathrm{Cay}(\overline{G}; \overline{S})$. Since there is one occurrence of b in C, and it is the only generator of G that contains a_3, then by Lemma 2.8 we conclude that the subgroup generated by $\mathbb{V}(C)$ contains $\mathcal{C}_3$. Similarly, since there is one occurrence of c in C, and it is the only generator of G that contains γ_p, then by Lemma 2.8 we conclude that the subgroup generated by $\mathbb{V}(C)$ contains $\mathcal{C}_p$. Therefore, the subgroup generated by $\mathbb{V}(C)$ is G'. So, Factor Group Lemma 2.6 applies.

Subcase 2.3. Assume $i \neq 0$ and $j \neq 0$. Then $c = a_2 a_q^j \gamma_p$. Consider $\overline{G} = \mathcal{C}_2 \times \mathcal{C}_q$. Then we have $\overline{a} = a_2 a_q$, $\overline{b} = a_2$ and $\overline{c} = a_2 a_q^j$. This implies that $|\overline{a}| = |\overline{c}| = 2q$ and $|\overline{b}| = 2$. We may assume j is even by replacing c with its inverse and j with $q - j$ if necessary.

Suppose, for the moment, that $j = q-1$, then $c = a_2 a_q^{-1}\gamma_p$ and $\overline{c} = \overline{a}^{-1}$. We have

$$C_1 = (\overline{c}, \overline{b}, (\overline{a}^{-1}, \overline{b})^{q-1})$$

as a Hamiltonian cycle in $\mathrm{Cay}(\overline{G}; \overline{S})$. Since there is one occurrence of c in C, and it is the only generator of G that contains γ_p, then by Lemma 2.8 we conclude that the subgroup generated by $\mathbb{V}(C_1)$ contains $\mathcal{C}_p$. Also,

$$\begin{aligned}\mathbb{V}(C_1) &= cb(a^{-1}b)^{q-1}\\ &\equiv a_2 \cdot a_2a_3 \cdot (a_2 \cdot a_2a_3)^{q-1} \pmod{\mathcal{C}_q \ltimes \mathcal{C}_p}\\ &= a_3^q\end{aligned}$$

which generates $\mathcal{C}_3$. Therefore, the subgroup generated by $\mathbb{V}(C_1)$ contains G'. Thus, Factor Group Lemma 2.6 applies.

So we may assume $j \neq q-1$. Then we have

$$C_2 = (\overline{c}, \overline{a}^{q-j-1}, \overline{b}, \overline{a}^{-q+j+1}, (\overline{a}^{-1}, \overline{b})^j)$$

and

$$C_3 = (\overline{c}, \overline{a}^{q-j-2}, \overline{b}, \overline{a}^{-q+j+2}, (\overline{a}^{-1}, \overline{b})^{j-1}, \overline{a}^{-2}, \overline{b}, \overline{a})$$

as Hamiltonian cycles in $\mathrm{Cay}(\overline{G}; \overline{S})$. Since there is one occurrence of c in C_2, and it is the only generator of G that contains γ_p, then by Lemma 2.8 we conclude that the subgroup generated by $\mathbb{V}(C_2)$ contains $\mathcal{C}_p$. Also,

$$\begin{aligned}\mathbb{V}(C_2) &= ca^{q-j-1}ba^{-q+j+1}(a^{-1}b)^j\\ &\equiv a_2 \cdot a_2^{q-j-1} \cdot a_2a_3 \cdot a_2^{-q+j+1} \cdot a_3^j \pmod{\mathcal{C}_q \ltimes \mathcal{C}_p}\\ &= a_3^{j+1}.\end{aligned}$$

We may assume this does not generate $\mathcal{C}_3$, for otherwise Factor Group Lemma 2.6 applies. Then $j \equiv -1 \pmod 3$.

Since there is one occurrence of c in C_3, and it is the only generator of G that contains γ_p, then by Lemma 2.8 we conclude that the subgroup generated by $\mathbb{V}(C_3)$ contains $\mathcal{C}_p$. Also,

$$\begin{aligned}\mathbb{V}(C_3) &= ca^{q-j-2}ba^{-q+j+2}(a^{-1}b)^{j-1}a^{-2}ba\\ &\equiv a_2 \cdot a_2^{q-j-2} \cdot a_2a_3 \cdot a_2^{-q+j+2} \cdot a_3^{j-1} \cdot a_2^{-2} \cdot a_2a_3 \cdot a_2 \pmod{\mathcal{C}_q \ltimes \mathcal{C}_p}\\ &= a_2a_3a_2a_3^{j-1}a_2a_3a_2\\ &= a_3^{j-3}\\ &= a_3^j\end{aligned}$$

Since $j \equiv -1 \pmod 3$, this generates $\mathcal{C}_3$. So, Factor Group Lemma 2.6 applies.

Case 3. Assume $a = a_2a_q$ and $b = a_q^m a_3$. If $k \neq 0$, then by Lemma 2.29(1) $\langle a, c\rangle = G$ which contradicts the minimality of S. So we can assume $k = 0$. Now if $i \neq 0$, then by Lemma 2.29(3) $\langle b, c\rangle = G$ which contradicts the minimality of S. Therefore, we may assume $i = 0$. Then $j \neq 0$ and $c = a_q^j \gamma_p$. Consider $\overline{G} = \mathcal{C}_2 \times \mathcal{C}_q$. Then we have $\overline{a} = a_2 a_q$, $\overline{b} = a_q^m$ and $\overline{c} = a_q^j$.

Suppose, for the moment, that $m = j$. Then $\overline{b} = \overline{c}$. We have

$$C_1 = (\overline{c}^{-1}, \overline{b}^{-(q-2)}, \overline{a}^{-1}, \overline{b}^{q-1}, \overline{a})$$

as a Hamiltonian cycle in $\mathrm{Cay}(\overline{G}; \overline{S})$. Since there is one occurrence of c in C_1, and it is the only generator of G that contains γ_p, then by Lemma 2.8 we conclude that the subgroup generated by $\mathbb{V}(C_1)$ contains $\mathcal{C}_p$. Also,

$$\begin{aligned}
\mathbb{V}(C_1) &= c^{-1} b^{-(q-2)} a^{-1} b^{q-1} a \\
&\equiv a_3^{-(q-2)} \cdot a_2 \cdot a_3^{q-1} \cdot a_2 \pmod{\mathcal{C}_q \ltimes \mathcal{C}_p} \\
&= a_3^{-2q+3} \\
&= a_3^{-2q}
\end{aligned}$$

which generates $\mathcal{C}_3$, because $\gcd(2q, 3) = 1$. So, the subgroup generated by $\mathbb{V}(C_1)$ is G'. Therefore, Factor Group Lemma 2.6 applies.

So we may assume $m \neq j$. We may also assume m and j are even, by replacing $\{b, c\}$ with their inverses, m with $q - m$, and j with $q - j$ if necessary. Now suppose, for the moment, $j = 2$. Then we have $c = a_q^2 \gamma_p$. We also have

$$C_2 = (\overline{b}, \overline{c}^{-(m-2)/2}, \overline{a}^{-1}, \overline{c}^{m/2}, \overline{a}^{2q-m-1})$$

as a Hamiltonian cycle in $\mathrm{Cay}(\overline{G}; \overline{S})$. Since there is one occurrence of b in C_2, and it is the only generator of G that contains a_3, then by Lemma 2.8 we conclude that the subgroup generated by $\mathbb{V}(C_2)$ contains $\mathcal{C}_3$. Now by considering the fact that $\mathcal{C}_2$ might centralize $\mathcal{C}_p$ or not, we have

$$\begin{aligned}
\mathbb{V}(C_2) &= bc^{-(m-2)/2} a^{-1} c^{m/2} a^{2q-m-1} \\
&\equiv a_q^m \cdot (a_q^2 \gamma_p)^{-(m-2)/2} \cdot a_q^{-1} a_2 \cdot (a_q^2 \gamma_p)^{m/2} \cdot a_2^{2q-m-1} a_q^{2q-m-1} \pmod{\mathcal{C}_3} \\
&= a_q^m (\gamma_p^{\hat{\tau}^2 + (\hat{\tau}^2)^2 + \cdots + (\hat{\tau}^2)^{(m-2)/2}} a_q^{(m-2)})^{-1} a_q^{-1} a_2 (\gamma_p^{\hat{\tau}^2 + (\hat{\tau}^2)^2 + \cdots + (\hat{\tau}^2)^{m/2}} a_q^m) a_2 a_q^{-m-1} \\
&= a_q^m a_q^{-(m-2)} \gamma_p^{-\hat{\tau}^2(1 + \hat{\tau}^2 + \cdots + (\hat{\tau}^2)^{(m-4)/2})} a_q^{-1} \gamma_p^{\pm\hat{\tau}^2(1 + \hat{\tau}^2 + \cdots + (\hat{\tau}^2)^{(m-2)/2})} a_q^m a_q^{-m-1}.
\end{aligned}$$

Since $\hat{\tau}^2 - 1 \not\equiv 0 \pmod{p}$, then

$$\begin{aligned}
\mathbb{V}(C_2) &= a_q^2 \gamma_p^{-\hat{\tau}^2(\hat{\tau}^{m-2} - 1)/(\hat{\tau}^2 - 1)} a_q^{-1} \gamma_p^{\pm\hat{\tau}^2(\hat{\tau}^m - 1)/(\hat{\tau}^2 - 1)} a_q^{-1} \\
&= \gamma_p^{-\hat{\tau}^4(\hat{\tau}^{m-2} - 1)/(\hat{\tau}^2 - 1) \pm \hat{\tau}^3(\hat{\tau}^m - 1)/(\hat{\tau}^2 - 1)} \\
&= \gamma_p^{\hat{\tau}^3(1 \mp \hat{\tau})(-\hat{\tau}^{m-1} \mp 1)/(\hat{\tau}^2 - 1)}.
\end{aligned}$$

We may assume this does not generate $\mathcal{C}_p$, for otherwise Factor Group Lemma 2.6 applies. Therefore, $\hat{\tau} \equiv \pm 1 \pmod{p}$ or $\hat{\tau}^{m-1} \equiv \pm 1 \pmod{p}$. The first case is impossible. So we may assume $\hat{\tau}^{m-1} \equiv \pm 1 \pmod{p}$. Thus, $\hat{\tau}^{2(m-1)} \equiv 1 \pmod{p}$. We also know that $\hat{\tau}^q \equiv 1 \pmod{p}$. So we have $\hat{\tau}^d \equiv 1 \pmod{p}$, where $d = \gcd(2(m-1), q)$. Since $\gcd(2, q) = 1$ and $2 \leqslant m \leqslant q - 1$, then $d = 1$, which contradicts the fact that $\hat{\tau} \not\equiv 1 \pmod{p}$.

So we may assume $j \neq 2$. We have

$$C_3 = (\bar{b}, \bar{c}, \bar{a}, \bar{c}^{-1}, \bar{b}^{-1}, \bar{a}^{m-2}, \bar{c}, \bar{a}^{-(j-3)}, \bar{c}, \bar{a}^{2q-m-j-2})$$

as a Hamiltonian cycle in $\mathrm{Cay}(\overline{G}; \overline{S})$. Now we calculate its voltage.

$$\begin{aligned}\mathbb{V}(C_3) &= bcac^{-1}b^{-1}a^{m-2}ca^{-(j-3)}ca^{2q-m-j-2} \\ &\equiv a_3 \cdot a_2 \cdot a_3^{-1} \cdot a_2^{m-2} \cdot a_2^{-j+3} \cdot a_2^{2q-m-j-2} \pmod{\mathcal{C}_q \ltimes \mathcal{C}_p} \\ &= a_3^2\end{aligned}$$

which generates $\mathcal{C}_3$. Also, by considering the fact that $\mathcal{C}_2$ might centralize $\mathcal{C}_p$ or not, we have

$$\begin{aligned}\mathbb{V}(C_3) &= bcac^{-1}b^{-1}a^{m-2}ca^{-(j-3)}ca^{2q-m-j-2} \\ &\equiv a_q^m \cdot a_q^j\gamma_p \cdot a_2a_q \cdot \gamma_p^{-1}a_q^{-j} \cdot a_q^{-m} \cdot a_2^{m-2}a_q^{m-2} \\ &\qquad \cdot a_q^j\gamma_p \cdot a_q^{-j+3}a_2^{-j+3} \cdot a_q^j\gamma_p \cdot a_2^{2q-m-j-2}a_q^{2q-m-j-2} \pmod{\mathcal{C}_3} \\ &= a_q^{m+j}\gamma_p a_2 a_q \gamma_p^{-1} a_q^{-2}\gamma_p a_q^3 a_2 \gamma_p a_q^{-m-j-2} \\ &= a_q^{m+j}\gamma_p a_q \gamma_p^{\mp 1} a_q^{-2}\gamma_p^{\pm 1} a_q^3 \gamma_p a_q^{-m-j-2} \\ &= \gamma_p^{\hat{\tau}^{m+j} \mp \hat{\tau}^{m+j+1} \pm \hat{\tau}^{m+j-1} + \hat{\tau}^{m+j+2}} \\ &= \gamma_p^{\hat{\tau}^{m+j-1}(\hat{\tau}^3 \mp \hat{\tau}^2 + \hat{\tau} \pm 1)}.\end{aligned}$$

We may assume this does not generate $\mathcal{C}_p$, for otherwise Factor Group Lemma 2.6 applies. Therefore,

$$0 \equiv \hat{\tau}^3 \mp \hat{\tau}^2 + \hat{\tau} \pm 1 \pmod{p}.$$

If $\mathcal{C}_2$ centralizes $\mathcal{C}_p$, then

$$0 \equiv \hat{\tau}^3 - \hat{\tau}^2 + \hat{\tau} + 1 \pmod{p}. \tag{3A}$$

We can replace $\hat{\tau}$ with $\hat{\tau}^{-1}$ in the above equation after replacing $\{a, b, c\}$ with their inverses in the Hamiltonian cycle, then

$$0 \equiv \hat{\tau}^{-3} - \hat{\tau}^{-2} + \hat{\tau}^{-1} + 1 \pmod{p}.$$

Multiplying by $\hat{\tau}^3$, we have

$$\begin{aligned}0 &\equiv 1 - \hat{\tau} + \hat{\tau}^2 + \hat{\tau}^3 \pmod{p} \\ &= \hat{\tau}^3 + \hat{\tau}^2 - \hat{\tau} + 1.\end{aligned}$$

Subtracting 3A from the above equation we have

$$\begin{aligned}0 &\equiv 2\hat{\tau}^2 - 2\hat{\tau} \pmod{p} \\ &= 2\hat{\tau}(\hat{\tau} - 1)\end{aligned}$$

which is impossible, because $\hat{\tau} \not\equiv 1 \pmod{p}$.

Now if $\mathcal{C}_2$ inverts $\mathcal{C}_p$, then

$$0 \equiv \hat{\tau}^3 + \hat{\tau}^2 + \hat{\tau} - 1 \pmod{p}. \tag{3B}$$

We can replace $\hat{\tau}$ with $\hat{\tau}^{-1}$ in the above equation after replacing $\{a, b, c\}$ with their inverses. Then

$$0 \equiv \hat{\tau}^{-3} + \hat{\tau}^{-2} + \hat{\tau}^{-1} - 1 \pmod{p}.$$

Multiplying by $\hat{\tau}^3$, then

$$\begin{aligned} 0 &\equiv 1 + \hat{\tau} + \hat{\tau}^2 - \hat{\tau}^3 \pmod{p} \\ &= -\hat{\tau}^3 + \hat{\tau}^2 + \hat{\tau} + 1. \end{aligned}$$

By adding 3B and the above equation, we have

$$\begin{aligned} 0 &\equiv 2(\hat{\tau}^2 + \hat{\tau}) \pmod{p} \\ &= 2\hat{\tau}(\hat{\tau} + 1) \end{aligned}$$

which is also impossible, because $\hat{\tau} \not\equiv -1 \pmod{p}$.

Case 4. Assume $a = a_2$ and $b = a_q a_3$.

Subcase 4.1. Assume $i \neq 0$. Then $c = a_2 a_q^j a_3^k \gamma_p$. By Lemma 2.29(2) $\langle b, c\rangle = G$ which contradicts the minimality of S.

Subcase 4.2. Assume $i = 0$. Then $j \neq 0$ and $c = a_q^j a_3^k \gamma_p$. We may assume j is even by replacing c with its inverse and j with $q - j$ if necessary. Consider $\overline{G} = \mathcal{C}_2 \times \mathcal{C}_q$. Then we have $\overline{a} = a_2$, $\overline{b} = a_q$ and $\overline{c} = a_q^j$. This implies that $|\overline{a}| = 2$ and $|\overline{b}| = |\overline{c}| = q$. We have

$$C_1 = (\overline{c}, \overline{b}^{q-j-1}, \overline{c}, \overline{b}^{-(j-2)}, \overline{a}, \overline{b}^{q-1}, \overline{a})$$

as a Hamiltonian cycle in $\mathrm{Cay}(\overline{G}; \overline{S})$. Now we calculate its voltage.

$$\begin{aligned} \mathbb{V}(C_1) &= cb^{q-j-1}cb^{-(j-2)}ab^{q-1}a \\ &\equiv a_q^j\gamma_p \cdot a_q^{q-j-1} \cdot a_q^j\gamma_p \cdot a_q^{-j+2} \cdot a_2 \cdot a_q^{q-1} \cdot a_2 \pmod{\mathcal{C}_3} \\ &= a_q^j\gamma_p a_q^{-1}\gamma_p a_q^{-j+1} \\ &= \gamma_p^{\hat{\tau}^{j-1}(\hat{\tau}+1)} \end{aligned}$$

which generates $\mathcal{C}_p$. Also

$$\begin{aligned} \mathbb{V}(C_1) &= cb^{q-j-1}cb^{-(j-2)}ab^{q-1}a \\ &\equiv a_3^k \cdot a_3^{q-j-1} \cdot a_3^k \cdot a_3^{-j+2} \cdot a_2 \cdot a_3^{q-1} \cdot a_2 \pmod{\mathcal{C}_q \ltimes \mathcal{C}_p} \\ &= a_3^{k+q-j-1+k-j+2-q+1} \\ &= a_3^{2(k-j+1)}. \end{aligned}$$

We may assume this does not generate $\mathcal{C}_3$, for otherwise Factor Group Lemma 2.6 applies. Then

$$0 \equiv k - j + 1 \pmod{3}. \tag{4.2A}$$

We also have

$$C_2 = (\overline{c}, \overline{a}, (\overline{b}, \overline{a})^{q-j-1}, \overline{b}^j, \overline{a}, \overline{b}^{-(j-1)})$$

as a Hamiltonian cycle in $\mathrm{Cay}(\overline{G}; \overline{S})$. We calculate its voltage. Since there is one occurrence of c in C_2, and it is the only generator of G that contains γ_p, then by Lemma 2.8 we conclude that the subgroup generated by $\mathbb{V}(C_2)$ contains $\mathcal{C}_p$. Also,

$$\begin{aligned}\mathbb{V}(C_2) &= ca(ba)^{q-j-1}b^j ab^{-(j-1)} \\ &\equiv a_3^k \cdot a_2 \cdot (a_3 a_2)^{q-j-1} \cdot a_3^j \cdot a_2 \cdot a_3^{-j+1} \pmod{\mathcal{C}_q \ltimes \mathcal{C}_p} \\ &= a_3^{k-2j+1}.\end{aligned}$$

We may assume this does not generate $\mathcal{C}_3$, for otherwise Factor Group Lemma 2.6 applies. Therefore,

$$0 \equiv k - 2j + 1 \pmod 3.$$

By subtracting the above equation from 4.2A we have $j \equiv 0 \pmod 3$.

Now we have

$$C_3 = (\overline{c}, \overline{a}, \overline{b}^{q-j-1}, \overline{a}, \overline{b}^{-(q-j-2)}, \overline{c}^{-1}, \overline{b}^{j-2}, \overline{a}, \overline{b}^{-(j-1)}, \overline{a})$$

as a Hamiltonian cycle in $\mathrm{Cay}(\overline{G}; \overline{S})$. We calculate its voltage.

$$\begin{aligned}\mathbb{V}(C_3) &= cab^{q-j-1}ab^{-(q-j-2)}c^{-1}b^{j-2}ab^{-(j-1)}a \\ &\equiv a_q^j \gamma_p \cdot a_2 \cdot a_q^{q-j-1} \cdot a_2 \cdot a_q^{-q+j+2} \cdot \gamma_p^{-1} a_q^{-j} \cdot a_q^{j-2} \cdot a_2 \cdot a_q^{-j+1} \cdot a_2 \pmod{\mathcal{C}_3} \\ &= a_q^j \gamma_p a_q \gamma_p^{-1} a_q^{-j-1} \\ &= \gamma_p^{\hat{\tau}^j(1-\hat{\tau})}.\end{aligned}$$

which generates $\mathcal{C}_p$. Also

$$\begin{aligned}\mathbb{V}(C_3) &= cab^{q-j-1}ab^{-(q-j-2)}c^{-1}b^{j-2}ab^{-(j-1)}a \\ &\equiv a_3^k \cdot a_2 \cdot a_3^{q-j-1} \cdot a_2 \cdot a_3^{-q+j+2} \cdot a_3^{-k} \cdot a_3^{j-2} \cdot a_2 \cdot a_3^{-j+1} \cdot a_2 \pmod{\mathcal{C}_q \ltimes \mathcal{C}_p} \\ &= a_3^{k-q+j+1-q+j+2-k+j-2+j-1} \\ &= a_3^{-2q+4j}.\end{aligned}$$

We may assume this does not generate $\mathcal{C}_3$, for otherwise Factor Group Lemma 2.6 applies. Then

$$\begin{aligned}0 &\equiv -2q + 4j \pmod 3 \\ &= q + j\end{aligned}$$

We already know $j \equiv 0 \pmod 3$. By substituting this in the above equation, we have $q \equiv 0 \pmod 3$ which contradicts the fact that $\gcd(q, 3) = 1$. □

3.9 Assume $|S| \geqslant 4$

In this subsection, we prove the following general result that includes the part of Theorem 1.3, where $|S| \geqslant 4$ (see Assumption 3.1). Unlike in the other subsections of this section, we do not assume $|G| = 6pq$.

Proposition 3.10. *Assume $|G|$ is a product of four distinct primes and S is a minimal generating set of G, where $|S| \geqslant 4$. Then $\mathrm{Cay}(G; S)$ contains a Hamiltonian cycle.*

Proof. Suppose $S = \{s_1, s_2, ..., s_k\}$ and let $G_i = \langle s_1, s_2, ..., s_i\rangle$ for $i = 1, 2, ..., k$. Since S is minimal, we know $\{e\} \subset G_1 \subset G_2 \subset ...G_k = G$. Therefore, the number of prime factors of $|G_i|$ is at least i. Since $|G| = p_1p_2p_3q$ is the product of only 4 primes, and $k = |S| \geqslant 4$, we can conclude that $|G_i|$ has exactly i prime factors, for all i. This implies that $|S| = 4$. This also implies every element of S has prime order.

Since $|G|$ is square-free, we know that G' is cyclic (see Proposition 2.15(1)), so $G' \neq G$. We may assume $|G'| \neq 1$, for otherwise G is abelian, so Lemma 2.2 applies. Also, if $|G'|$ is equal to a prime number, then Theorem 2.3 applies. So we may assume $|G'|$ has at least two prime factors. Therefore, the number of prime factors of $|G'|$ is either 2 or 3.

Case 1. Assume $|G'|$ has only two prime factors. This implies $|\overline{G}| = p_1p_2$, where p_1 and p_2 are two distinct primes. Suppose $s \in S$, then $\overline{s} \in \overline{S}$. We know that $|\overline{s}| \neq 1$ (see Assumption 3.1(6)). Now since every element of S has prime order, then $|s|$ is either p_1 or p_2. Also, every element of order p_1 must commute with every element of order p_2, because the subgroup H generated by any element of S that has order p_1, together with any element of S that has order p_2 has exactly two prime factors, so $|H| = p_1p_2$, $H' \subseteq G'$, and $|G'| = p_3p_4$. Thus, $|H'| = 1$. Let S_{p_1} be the elements of order p_1 in S, and let S_{p_2} be the elements of order p_2. Also let H_{p_1} and H_{p_2} be the subgroups generated by S_{p_1} and S_{p_2}, respectively. This implies that $\mathrm{Cay}(G; S) \cong \mathrm{Cay}(G_{p_1}; S_{p_1}) \,\square\, \mathrm{Cay}(G_{p_2}; S_{p_2})$. Therefore, $\mathrm{Cay}(G; S)$ contains a Hamiltonian cycle (see Corollary 2.11).

Case 2. Assume $|G'|$ has three prime factors. We may write (see Proposition 2.15(3))

$$G = \mathcal{C}_q \ltimes G' = \mathcal{C}_q \ltimes (\mathcal{C}_{p_1} \times \mathcal{C}_{p_2} \times \mathcal{C}_{p_3}),$$

where p_1, p_2, p_3 and q are distinct primes. Note that $G' \cap Z(G) = \{e\}$ (see Proposition 2.15(2)). Now we may assume $\langle s_4\rangle = \mathcal{C}_q$. Since $|\langle s_i, s_4\rangle|$ has only two prime factors (for $1 \leqslant i \leqslant 3$), we must have $s_i = s_4^{k_i} a_{p_i}$ (after permuting p_1, p_2, p_3), where a_{p_i} is a generator of $\mathcal{C}_{p_i}$. We may also assume $S \cap G' = \varnothing$ (see Lemma 2.12), so $k_i \not\equiv 0 \pmod q$. Now consider

$$G_2 = \langle s_1, s_2\rangle = \langle s_4^{k_1} a_{p_1}, s_4^{k_2} a_{p_2}\rangle.$$

Since $\mathcal{C}_{p_1}$ is a normal subgroup in G, we can consider $\overline{G}_2 = G_2/\mathcal{C}_{p_1}$, then $\{\overline{s}_1, \overline{s}_2\} = \{\overline{s}_4^{k_1}, \overline{s}_4^{k_2}\overline{a}_{p_2}\}$. We have

$$\overline{s}_4^{k_2^{-1}} = (\overline{s}_4^{k_1})^{k_1^{-1}k_2^{-1}} = \overline{s}_1^{k_1^{-1}k_2^{-1}}.$$

Multiplying by $\overline{s}_2$, then

$$\overline{a}_{p_2} = \overline{s}_4^{k_2^{-1}} \cdot \overline{s}_4^{k_2} a_{p_2} = \overline{s}_1^{k_1^{-1}k_2^{-1}}\, \overline{s}_2 \in \overline{G}_2.$$

Since a_{p_2} generates $\mathcal{C}_{p_2}$, this implies $|G_2|$ is divisible by p_2. Similarly, we can show that $|G_2|$ is divisible by p_1. Also, $|s_1| = q$, so $|G_2|$ is divisible by q. Therefore, $|G_2|$ has three prime factors, which is a contradiction. □

References

[1] A. Cayley, The theory of groups: Graphical representation, *Amer. J. Math.* **1** (1878), 174–176, doi:10.2307/2369306.

[2] C. C. Chen and N. Quimpo, Hamiltonian Cayley graphs of order pq, in: *Combinatorial mathematics, X (Adelaide, 1982)*, Springer, Berlin, volume 1036 of *Lecture Notes in Math.*, pp. 1–5, 1983, doi:10.1007/BFb0071505.

[3] C. C. Chen and N. F. Quimpo, On some classes of Hamiltonian graphs, *Southeast Asian Bull. Math.* (1979), 252–258.

[4] C. C. Chen and N. F. Quimpo, On strongly Hamiltonian abelian group graphs, in: *Combinatorial mathematics, VIII (Geelong, 1980)*, Springer, Berlin-New York, volume 884 of *Lecture Notes in Math.*, pp. 23–34, 1981, doi:10.1007/BFb0091805.

[5] S. J. Curran and J. A. Gallian, Hamiltonian cycles and paths in Cayley graphs and digraphs—a survey, *Discrete Math.* **156** (1996), 1–18, doi:10.1016/0012-365X(95)00072-5.

[6] S. J. Curran, D. W. Morris and J. Morris, Cayley graphs of order $16p$ are Hamiltonian, *Ars Math. Contemp.* **5** (2012), 185–211, doi:10.26493/1855-3974.207.8e0.

[7] E. Durnberger, Connected Cayley graphs of semidirect products of cyclic groups of prime order by abelian groups are Hamiltonian, *Discrete Math.* **46** (1983), 55–68, doi:10.1016/0012-365X(83)90270-4.

[8] E. Durnberger, Every connected Cayley graph of a group with prime order commutator group has a Hamilton cycle, in: *Cycles in graphs (Burnaby, B.C., 1982)*, North-Holland, Amsterdam, volume 115 of *North-Holland Math. Stud.*, pp. 75–80, 1985, doi:10.1016/S0304-0208(08)72997-9.

[9] E. Ghaderpour and D. W. Morris, Cayley graphs of order $27p$ are Hamiltonian, *Int. J. Comb.* (2011), Art. ID 206930, 16, doi:10.1155/2011/206930.

[10] E. Ghaderpour and D. W. Morris, Cayley graphs of order $30p$ are Hamiltonian, *Discrete Math.* **312** (2012), 3614–3625, doi:10.1016/j.disc.2012.08.017.

[11] C. Godsil and G. Royle, *Algebraic Graph Theory*, Springer New York., 2001, doi:10.1007/9781461301639.

[12] M. Hall, *The Theory of Groups*, American Mathematical Soc., 1959, https://books.google.com/books?hl=sl&lr=&id=K8hEDwAAQBAJ&oi=fnd&pg=PP1&ots=BE6c1Bzt3u&sig=W90_d6E83hb0OBijkVUNNbHfxJU.

[13] D. Jungreis and E. Friedman, Cayley graphs on groups of low order are Hamiltonian (unpublished).

[14] K. Keating and D. W. Morris, On Hamilton cycles in Cayley graphs in groups with cyclic commutator subgroup, in: *Cycles in graphs (Burnaby, B.C., 1982)*, North-Holland, Amsterdam, volume 115 of *North-Holland Math. Stud.*, pp. 89–102, 1985, doi:10.1016/S0304-0208(08)72999-2.

[15] E. Konstantinova, Some problems on Cayley graphs, *Linear Algebra Appl.* **429** (2008), 2754–2769, doi:10.1016/j.laa.2008.05.010.

[16] K. Kutnar, D. Marušič, D. W. Morris, J. Morris and P. Šparl, Hamiltonian cycles in Cayley graphs whose order has few prime factors, *Ars Math. Contemp.* **5** (2012), 27–71, doi:10.26493/1855-3974.177.341.

[17] D. Li, Cayley graphs of order pqr are Hamiltonian (chinese)., *Acta Math. Sinica* **44** (2001), 351–358.

[18] F. Maghsoudi, Cayley graphs of order $6pq$ are Hamiltonian, Master's thesis, University of Lethbridge, 2020, https://hdl.handle.net/10133/5771.

[19] D. Marušič, Hamiltonian circuits in Cayley graphs, *Discrete Math.* **46** (1983), 49–54, doi: 10.1016/0012-365X(83)90269-8.

[20] D. W. Morris and K. Wilk, Cayley graphs of order kp are hamiltonian for $k < 48$, *Art Discrete Appl. Math.* **17** (2020), https://arxiv:1805.00149.

[21] I. Pak and R. Radoičić, Hamiltonian paths in Cayley graphs, *Discrete Math.* **309** (2009), 5501–5508, doi:10.1016/j.disc.2009.02.018.

[22] J. S. Robinson, *A Course in the Theory of Groups*, Springer New York., 1996, doi:10.1007/978-1-4419-8594-1.

[23] D. Witte, Cayley digraphs of prime-power order are Hamiltonian, *J. Combin. Theory Ser. B* **40** (1986), 107–112, doi:10.1016/0095-8956(86)90068-7.

[24] D. Witte and J. A. Gallian, A survey: Hamiltonian cycles in Cayley graphs, *Discrete Math.* **51** (1984), 293–304, doi:10.1016/0012-365X(84)90010-4.

[25] D. S. Witte, On Hamiltonian circuits in Cayley diagrams, *Discrete Math.* **38** (1982), 99–108, doi:10.1016/0012-365X(82)90174-1.