ISSN 2590-9770
The Art of Discrete and Applied Mathematics 4 (2021) #P1.07
https://doi.org/10.26493/2590-9770.1387.a84
(Also available at http://adam-journal.eu)

A simple construction of exponentially many nonisomorphic orientable triangular embeddings of K_{12s}

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Received 27 August 2019, accepted 24 July 2020, published online 26 January 2021

Abstract

Using an index one current graph with the cyclic current group we give a simple construction of 2^{2s-7} nonisomorphic orientable triangular embeddings of the complete graph K_{12s}, $s \geq 4$. These embeddings have no nontrivial automorphisms.

Keywords: Topological embedding, complete graph, nonisomorphic embeddings, triangular embedding.

Math. Subj. Class.: 05C10, 05C15

1 Introduction

In the present paper, by an embedding of a graph we mean a cellular embedding of the graph in an orientable surface. An embedding of a graph is *triangular* if all faces are 3-gonal. Euler's formula allows the possibility for a complete graph K_n to have a triangular embedding if $n \equiv 0, 3, 4$ or $7 \pmod{12}$. Constructing triangular embeddings of complete graphs was a major step in proving the Map Color Theorem [11].

Let K be a graph without loops and multiple edges. An m-gonal face of an embedding of K will be designated as a cyclic sequence $(v_1, v_2, \ldots, v_m)$ of vertices obtained by listing the incident vertices when traversing the boundary walk of the face in some chosen direction. The sequences $(v_1, v_2, \ldots, v_m)$ and $(v_m, \ldots, v_2, v_1)$ designate the same face.

One can differentiate embeddings of graphs as labeled objects (in this case we speak about different labeled embeddings and they have different face sets) and as unlabeled objects (in this case we speak about nonisomorphic embeddings). Two triangular embeddings f_1 and f_2 of K_n are *isomorphic* if there is a bijection ψ between the vertices of K_n such

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that (w_1, w_2, w_3) is a face of f_1 if and only if $(\psi(w_1), \psi(w_2), \psi(w_3))$ is a face of f_2. The bijection ψ is called an *isomorphism* from the embedding f_1 onto the embedding f_2.

During the proof of the Map Color Theorem, one triangular embedding was constructed for every complete graph K_n, $n \equiv 0, 3, 4$ or $7 \pmod{12}$. In this paper we consider the natural question on the rate of growth of the number of nonisomorphic triangular embeddings of complete graphs. At present there are two approaches to construct many such embeddings.

The first approach uses recursive constructions that generate a face 2-colorable triangular embedding of a complete graph from a face 2-colorable triangular embedding of a complete graph of lesser order. First it was shown [1, 3] that there are at least $2^{an^2-o(n^2)}$ (where a is a positive constant) nonisomorphic face 2-colorable triangular embeddings of K_n for some families of values of n such that $n \equiv 3$ or $7 \pmod{12}$, namely, for $n \equiv 7$ or $19 \pmod{36}$, $n \equiv 15 \pmod{60}$, $n \equiv 15$ or $43 \pmod{84}$, etc. Later it was shown [2, 4, 5] that there are at least $n^{bn^2-o(n^2)}$ nonisomorphic face 2-colorable triangular embeddings of K_n for an infinite, but rather sparse set of values of n (where $n \equiv 3$ or $7 \pmod{12}$). This approach having to do with face 2-colorable triangular embeddings does not work in the case of complete graphs of even order.

The second approach [7, 9, 10] uses the current graph technique. Within the limits of the approach, it was shown that there are constants $M, c > 0$, $b \geq 1/12$ such that for every $n \geq M$, $n \equiv 0, 3, 4$ or $7 \pmod{12}$, there are at least $c2^{bn}$ nonisomorphic triangular embeddings of K_n. In the case $n \equiv 0 \pmod{12}$, this approach (see [9]) gives 2^{s-6} nonisomorphic triangular embeddings of K_{12s}, $s \geq 6$, and, up to the present time, this result was the only known result on the number of nonisomorphic triangular embeddings of K_{12s}. This result was obtained by using index four current graphs with the cyclic current group $\mathbb{Z}_{12s}$, and the constructions involved are rather complicated.

In the present paper we give a simple construction of 2^{2s-7} nonisomorphic triangular embeddings of K_{12s}, $s \geq 4$. We use an index one current graph with current group $\mathbb{Z}_{12s-4}$ that was constructed by T. Sun [12] and which generates an embedding of K_{12s-4}, $s \geq 4$, that can be modified into a triangular embedding of K_{12s} (thereby providing a simple construction of a a triangular embedding of K_{12s}, $s \geq 3$). In the present paper, following the approach used in [7, 9, 10], changing rotations of some vertices of the current graph, we obtain 2^{2s-7} different current graphs generating 2^{2s-7} different embeddings of K_{12s-4} that can be modified into 2^{2s-7} different triangular embeddings of K_{12s}. Analyzing faces of the embeddings, we show (Theorem 3.1) that all these 2^{2s-7} different triangular embeddings of K_{12s}, $s \geq 4$, are nonisomorphic, thereby providing a much simpler construction of exponentially many nonisomorphic orientable triangular embeddings of K_{12s}.

2 Index one current graphs

In this section we describe index one current graphs which generate embeddings of K_{12s-4} that can be modified into triangular embeddings of K_{12s}.

First we briefly review some material about index one current graphs in the form used in the paper. The reader is referred to [6, 11] for a more detailed development of the material sketched herein. We assume the reader is familiar with current graphs and embeddings generated by current graphs.

Let G be a connected graph (multiple edges and loops are allowed) with the vertex set $V(G)$ whose edges have been given plus and minus direction. Hence each edge e gives rise to two reverse arcs e^+ and e^- of G. The involutary permutation θ of the arc set $A(G)$ of the graph G that permutes reverse arcs is called the *involution* of G. By a *current assignment* on G we mean a function λ from $A(G)$ into the set of nonzero elements of a group $\mathbb{Z}_n$ such that $\lambda(e^-) = -\lambda(e^+)$ for every edge e. The values of λ are called *currents* and the group $\mathbb{Z}_n$ is called the *current group*. If an edge e is incident with a onevalent vertex w and $\lambda(e^-) = \lambda(e^+)$ (that is, $\lambda(e^+)$ is of order 2 in $\mathbb{Z}_n$), then the arcs e^+ and e^- are identified and this arc is called an *end arc* (and in this case we do not consider w to be a vertex of G).

A *rotation* D of G is a permutation of $A(G)$ whose orbits cyclically permute the arcs directed outwards from each vertex. The rotation D can be represented as $D = \{D_w : w \in V(G)\}$, where D_w, called a rotation of the vertex v, is a cyclic permutation of the arcs directed outwards from v. Consider the permutation $D\theta$ of $A(G)$. It is easy to see that the terminal vertex of an arc a is the initial vertex of the arc $D\theta a$, hence a cycle $(a_1, a_2, \ldots, a_m)$ of $D\theta$ can be considered as an oriented path in G called a *circuit* induced by the rotation D of G. By a *one-rotation* of G we mean a rotation of G inducing exactly one circuit.

A triple $\langle G, \lambda, D\rangle$ is called a *current graph*. The *index* of the current graph is the number of circuits induced by D. By the *log* of a circuit $(a_1, a_2, \ldots, a_m)$ of the current graph we mean the cyclic sequence $(\lambda(a_1), \lambda(a_2), \ldots, \lambda(a_m))$.

A current graph $\langle G, \lambda, D\rangle$ can be represented as a figure of G where the rotations of vertices are indicated. The black vertices denote a clockwise rotation and the white vertices a counterclockwise rotation. Each pair of reverse arcs is represented by one of the arcs with the current indicated. The end arc, as is customary, is depicted as a straight line without an arrow, with a vertex at one end and without a vertex at the other end.

If $(a_1, a_2, \ldots, a_t)$ is the rotation of a vertex of a current graph $\langle G, \lambda, D\rangle$, where $\lambda(a_i) = \varepsilon_i$ for $i = 1, 2, \ldots, t$, then the cyclic sequence $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_t)$ is called the *current rotation* of the vertex and the element $\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_t$ is the *excess* of the vertex. If the excess of a vertex equals zero, we say that the vertex satisfies Kirchhoff's Current Law (KCL).

Figure 1(a) shows (for now ignore the labels x, y, z, and w, and the boxes connected by lines with edges of the graph) an index one current graph $\langle G, \lambda, D\rangle$ with the current group $\mathbb{Z}_{12s-4}$, $s \geq 4$, having the following properties (A1)-(A6):

(A1) G has two onevalent vertices, one twovalent vertex, and all other vertices are trivalent.

(A2) The log of the circuit contains every nonzero element of $\mathbb{Z}_{12s-4}$ exactly once.

(A3) G has exactly one end arc which has current $6s - 2$.

(A4) Every trivalent vertex satisfies KCL.

(A5) The two onevalent vertices have excess -1 and $6s + 1$, respectively (each of the two excesses has order $12s - 4$ in $\mathbb{Z}_{12s-4}$).

(A6) The twovalent vertex has current rotation $(1, -3)$.

The fragment of the current graph lying inside the dashed box is shown in Figure 1(b). The current graph is slightly different from the current graph given in [12]: we changed the rotations of some vertices for present purposes.

The current graph generates an embedding $f(D)$ of the graph K_{12s-4} whose vertex set is the set $V(s) = \{0, 1, \ldots, 12s - 5\}$ of all elements of $\mathbb{Z}_{12s-4}$. There is a mapping from the face set onto the vertex set of the current graph. Given a vertex of the current graph,

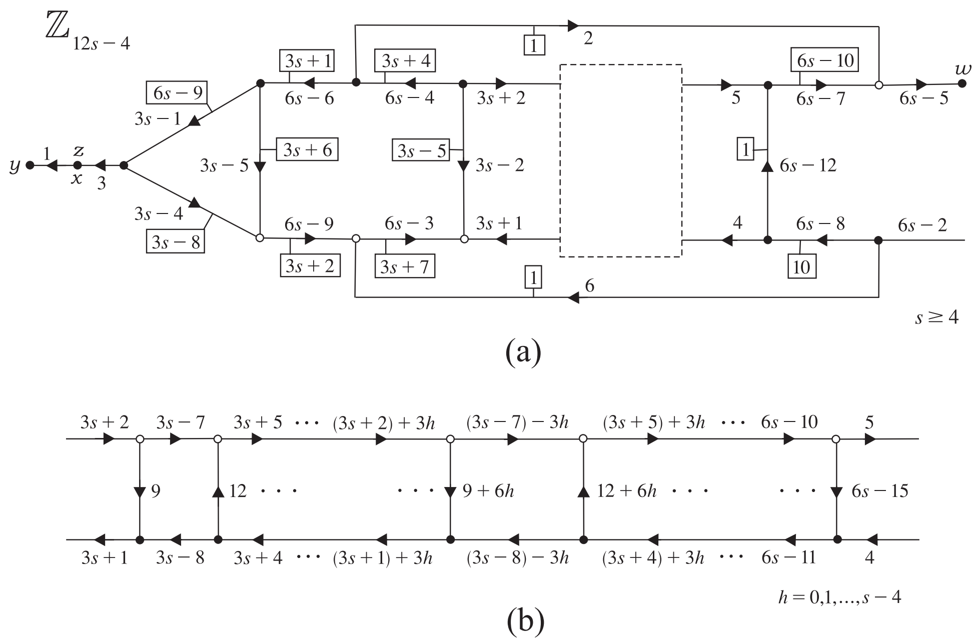

Figure 1: An index one current graph.

the faces mapping onto the vertex are called the faces *induced* by the vertex, and they are determined by Theorem 4.4.1 of [6]. In the case of the current graph $\langle G, \lambda, D\rangle$ satisfying (A1)-(A6) we have the following. A trivalent vertex with current rotation $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ induces $12s-4$ triangular faces $(u, u+\varepsilon_1, u+\varepsilon_1+\varepsilon_2)$, $u \in V(s)$. The onevalent vertex with excess -1 (resp. $6s+1$) induces one $(12s-4)$-gonal face shown in Figure 2(a) (resp. (b)) (now ignore the dashed edges in Figure 2). The twovalent vertex induces two $(12s-4)$-gonal faces shown in Figure 2(c).

The log of the circuit of the current graph $\langle G, \lambda, D\rangle$ (where we ignore the letters x, y, z, and w) determines the cyclic order in which the vertices adjacent to the vertex 0 of G are arranged on the surface around the vertex 0 in $f(D)$.

The fragment of the current graph shown in Figure 1(b) has exactly $2s-7$ vertical edges.

Lemma 2.1 ([8, Lemma 2]). *Let a rotation D of a graph G induce exactly one circuit. Let an edge e of G be incident with distinct trivalent vertices v and w. Then there are two ways to choose rotations of v and w without changing the rotations of other vertices, such that the obtained rotation of G induces exactly one circuit.*

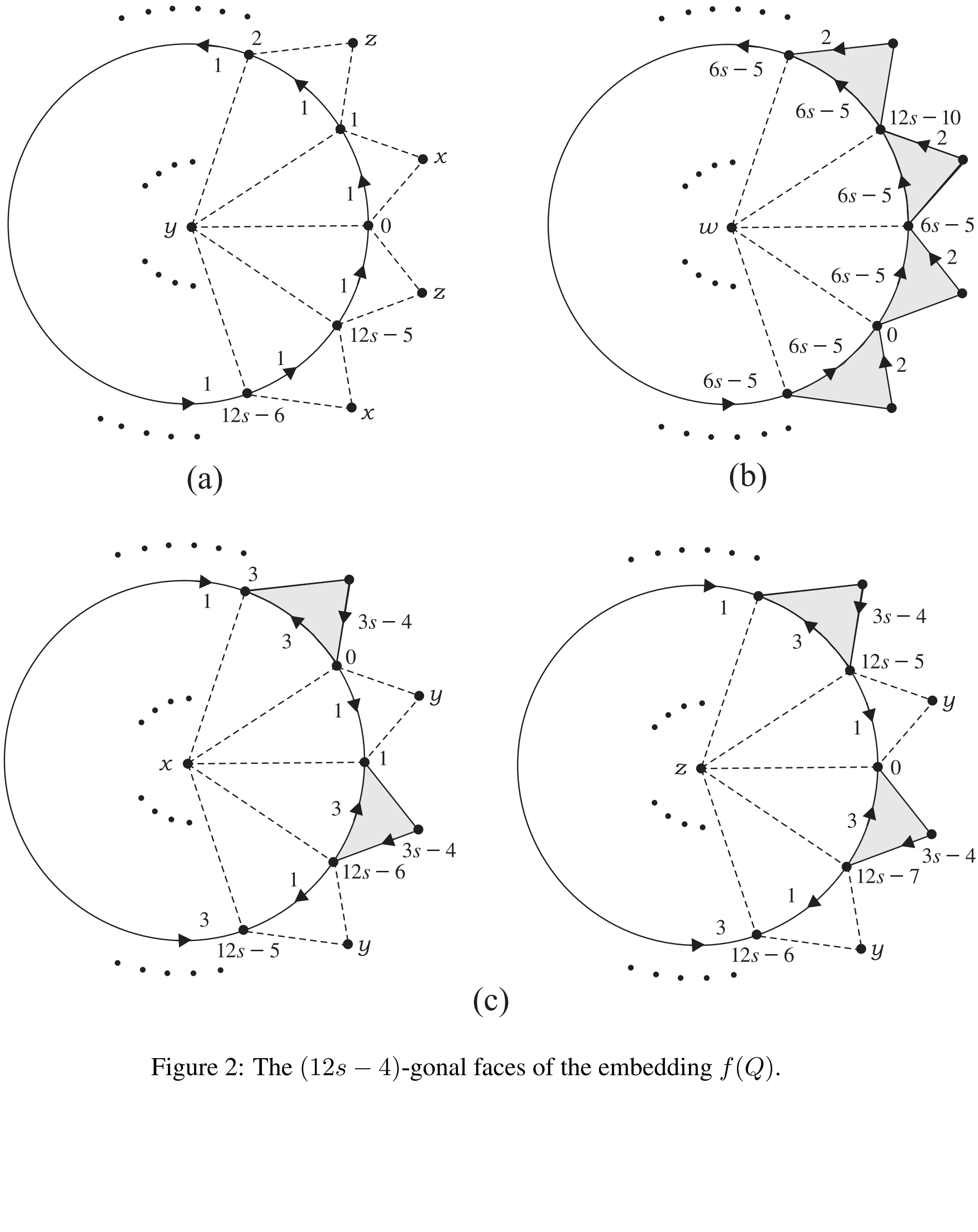

Figure 2: The $(12s-4)$-gonal faces of the embedding $f(Q)$.

Denote by $L(s)$ the set of the vertices of the current graph lying inside the dashed box in Figure 1(a). Now we fix the indicated rotations of the vertices of the current graph in Figure 1(a) that do not lie inside the dashed box, and then, applying Lemma 2.1 to the $2s-7$ vertical edges in Figure 1(b), we can choose the rotations of the vertices of $L(s)$ in 2^{2s-7} different ways such that for the corresponding 2^{2s-7} different one-rotations Q of G, we obtain index one current graphs $\langle G, \lambda, Q\rangle$ satisfying (A1)-(A6). Denote by $\mathcal{D}$ the set of all such 2^{2s-7} different one-rotations Q of G.

The embedding $f(Q)$ of K_{12s-4} generated by $\langle G, \lambda, Q\rangle$, $Q \in \mathcal{D}$, has four $(12s-4)$-gonal faces, and all other faces are triangular. Inserting four new vertices in the four $(12s-4)$-gonal faces, respectively, we obtain a triangular embedding $f'(Q)$ of $K_{12s} - K_4$ which (by attaching one additional handle to gain adjacencies between the new vertices) can be modified into a triangular embedding $\overline{f}(Q)$ of K_{12s}. All embeddings $f'(Q)$ and $\overline{f}(Q)$, $Q \in \mathcal{D}$, have the same vertex set $V(s) \bigcup R$, where $R = \{x, y, z, w\}$ is the set of the four new vertices.

We will show (Theorem 3.1) that all 2^{2s-7} triangular embeddings $\overline{f}(Q)$, $Q \in \mathcal{D}$, are nonisomorphic. Two faces of an embedding are *adjacent* if they share a common edge. To prove Theorem 3.1 we need to know pairs of adjacent faces of the embeddings $\overline{f}(Q)$, $Q \in \mathcal{D}$.

A *link* joining two vertices u and u' of an embedding is every pair (u, u_1, u_2), (u_1, u_2, u') of adjacent triangular faces of the embedding; we say that the vertices u and u' are incident with the link, and that u has the link with u'. By a link $[u, u']$ we mean a link between u and u'.

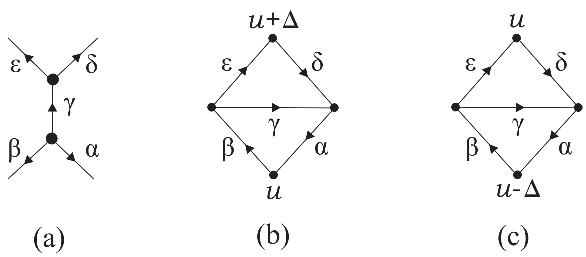

Figure 3: A link of an embedding.

If an edge of $\langle G, \lambda, Q\rangle$, $Q \in \mathcal{D}$, joins two trivalent vertices with current rotations (α, β, γ) and $(\varepsilon, \delta, -\gamma)$, respectively (see Figure 3(a)), then the *type* of the edge is $\Delta = \beta + \varepsilon$. We define the type of an edge up to inversion. Since KCL holds at the vertices, we have $\beta + \varepsilon = -(\alpha + \delta)$, hence the type is well defined. The two adjacent trivalent vertices induce faces of $f(Q)$ that form $12s-4$ links shown in Figure 3(b) where u goes through the values $0, 1, 2, \ldots, 12s-5$; we say that the $12s-4$ links are *induced* by the edge with type $\Delta = \beta + \varepsilon$. Now we have the following.

(B) For any vertex u of $f(Q)$, among the links induced by an edge with type Δ, there are exactly two links incident with u: one of them is a link $[u, u+\Delta]$ shown in Figure 3(b), and another link is a link $[u, u-\Delta]$ shown in Figure 3(c).

Since any two adjacent triangular faces of $f(Q)$ are induced by adjacent trivalent vertices of $\langle G, \lambda, Q\rangle$, every link of $f(Q)$ joining two vertices u and $u+\mu$ is induced by exactly one edge of the current graph and the type of the edge is μ.

3 Links and nonisomorphic embeddings of K_{12s}

To prove Theorem 3.1 we use the fact that in the embeddings $\overline{f}(Q)$, $Q \in \mathcal{D}$, some pairs of vertices have a large number of links joining the vertices, and some pairs of vertices have a small number of links joining the vertices.

Below we describe the modification of $f(Q)$ into $\overline{f}(Q)$, and in so doing we study links of the obtained embeddings.

First we describe links in $f(Q)$. In Figure 1(a) there are 13 edges with their types indicated (the type of an edge is given inside a box connected by a line with the edge). A list of the types of the 13 edges is

$$1, 1, 1, 10, 3s-8, 3s-5, 3s+1, 3s+2, 3s+4, 3s+6, 3s+7, 6s-10, 6s-9.$$

It is easy to check that for $s > 6$, $s = 6$, $s = 5$, and $s = 4$, the list contains, respectively, 11, 10, 9, and 10 different types. Hence $\langle G, \lambda, Q\rangle$ contains at least 9 edges having different types. The current graph $\langle G, \lambda, Q\rangle$ has exactly $6s-2$ edges, and exactly $6s-6$ of them join two trivalent vertices, hence at most $(6s-6)-8$ edges of $\langle G, \lambda, Q\rangle$ have the same type, and, by (B), we obtain the following.

(C) In $f(Q)$, $Q \in \mathcal{D}$, every vertex of $V(s)$ has at most $6s-14$ links with any other vertex of $V(s)$.

In what follows an edge joining vertices u and u' is denoted by (u, u').

Now insert new vertices x, y, z and w in the four $(12s-4)$-gonal faces of $f(Q)$ as shown in Figure 2 as dashed lines. (As is customary, in Figure 1(a), if a onevalent or twovalent vertex is labeled by letters, then the letters denote the new vertices that we insert in the faces induced by the vertex.) We obtain a triangular embedding $f'(Q)$ of $K_{12s} - K_4$. Note that the boundary cycle of the two $(12s-4)$-gonal faces in Figure 2(c) contain all edges $(u, u+1)$ and $(u, u-3)$, $u \in V(s)$, and we insert a new vertex x (resp. z) in the face whose boundary cycle contains all edges $(2i, 2i+1)$ and $(2i+1, 2i-2)$ (resp. $(2i+1, 2i+2)$ and $(2i, 2i-3)$, $i = 0, 1, \dots, 6s-3$.

Every link of $f(Q)$ is a link of $f'(Q)$. After we insert a new vertex in a $(12s-4)$-gonal face, every vertex of $V(s)$ lying on the boundary cycle of the face gains a new link with two different vertices of $V(s)$ lying on the cycle. Now, considering Figure 2 where we depict all triangular faces incident with the edges of the boundary cycles of the $(12s-4)$-gonal faces, we obtain the following:

(D) The links of $f'(Q)$ which are not links of $f(Q)$ are as follows: the vertex y has $6s-2$ links with each of x and z; the vertex w has exactly one link with every vertex of $V(s)$; the vertex x (resp. z) has exactly one link with each even (resp. odd) vertex of $V(s)$; every vertex $u \in V(s)$ has three new links $[u, u+2]$, three new links $[u, u-2]$, one new link $[u, u+6]$ and one new link $[u, u-6]$.

The triangular embedding $\overline{f}(Q)$, $Q \in \mathcal{D}$, of K_{12s} is obtained from the embedding $f'(Q)$ of $K_{12s} - K_4$ in the following way. The log of the circuit of $\langle G, \lambda, Q\rangle$ (following [11], the letters x, y, z, w enter the log) determines the cyclic order in which the vertices adjacent to the vertex 0 are arranged on the surface around the vertex 0 in $f'(Q)$. As easily

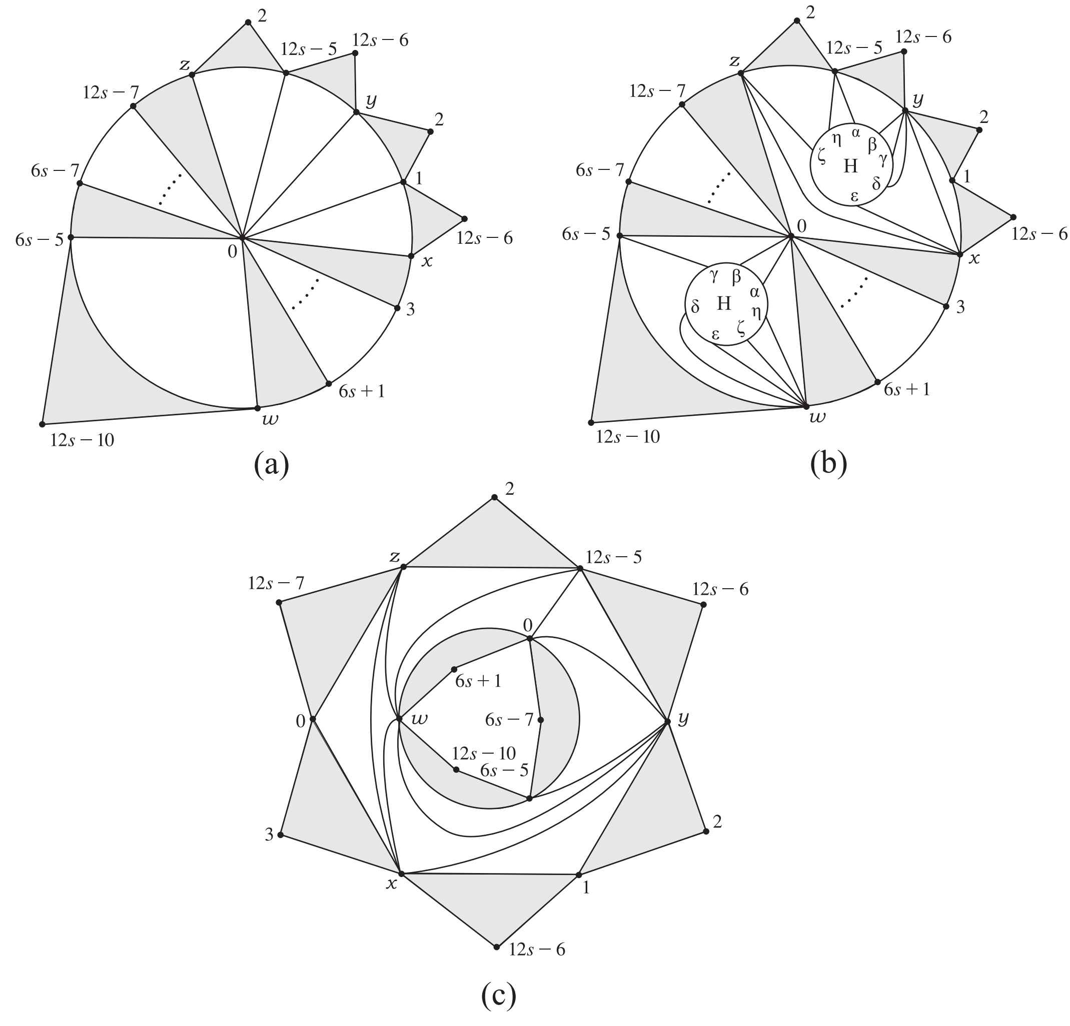

Figure 4: Attaching a handle.

seen, the log is of the form

$$(\ldots, 9s, 3, x, 1, y, 12s-5, z, 12s-7, 9s-3, \ldots, 6s-7, 6s-5, w, 6s+1, 12s-6, \ldots)$$

so that the faces of $f'(Q)$ incident with the vertex 0 are arranged as shown in Figure 4(a). Now, in Figure 4(a), we delete edges $(0, 12s-5), (0, y), (0, 1)$, and then, as shown in Figure 4(b), using a handle (depicted as two blank cycles with the letter H inside; the cycles are to be identified, and the edge ends labeled by the same Greek letter $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta$ are to be identified as well) we gain adjacencies

$$(x, y), (x, w), (x, z), (y, w), (y, 0), (y, 6s-5), (z, w), (0, 12s-5), (w, 12s-5).$$

In Figure 4 (and as in what follows in Figure 5) the shaded faces are faces of $f'(Q)$ that remain unchanged when modifying $f'(Q)$ into $\overline{f}(Q)$. As a result, we obtain a triangular embedding of the graph K_{12s} without two edges (y, z) and $(0, 1)$, but with two extra edges $(w, 12s-5)$ and $(y, 6s-5)$. Note that the faces shown in Figure 4(b) are the same for all $Q \in \mathcal{D}$.

The embedding $f'(Q)$ contains five pairs of adjacent faces shown as non-shaded faces in Figures 5(a) – (e), respectively. For each of the five pairs, we show a fragment of $\langle G, \lambda, Q\rangle$ whose vertices induce the faces of the pair and all the other (shaded) faces adjacent to the faces of the pair (the vertices of the fragment are not vertices of $L(s)$, so that the faces shown in Figures 5(a) – (e) are the same for all $Q \in \mathcal{D}$). The reader can consult Figures 2 and 3 when checking pairs of adjacent faces in Figures 5(a) – (e).

The diagonal flips in the pairs of adjacent non-shaded faces shown in Figures 5(a), (b), and (c), replace the edges $(w, 12s-5)$, $(6s, 6s-6)$, and $(12s-8, 12s-9)$ by the edges $(6s, 6s-6)$, $(12s-8, 12s-9)$, and (y, z), respectively, depicted in dashed line. As a result, we lose an extra edge $(w, 12s-5)$ and gain a missing edge (y, z). The diagonal flips in the pairs of adjacent non-shaded faces shown in Figures 5(d) and (e) replace the edges $(y, 6s-5)$ and $(6s-6, 6s-4)$ by the edges $(6s-6, 6s-4)$ and $(0, 1)$, respectively. As a result, we lose an extra edge $(y, 6s-5)$ and gain a missing edge $(0, 1)$. We obtain the triangular embedding $\overline{f}(Q)$ of K_{12s}. Note that the diagonal flips do not affect the faces shown in Figure 4(b), hence all faces shown in Figure 4(b) are faces of $\overline{f}(Q)$.

Now we need to know what new links we gain and what links incident with vertices of R we lose when modifying $f'(Q)$ into $\overline{f}(Q)$.

In Figure 4(b), a new additional handle is attached to the 6-gonal face $(0, z, 12s-5, y, 1, x)$ and the triangular face $(w, 0, 6s-5)$, and then some new edges are embedded. If we actually identify the two cycles with the letter H inside, then the faces of $\overline{f}(Q)$ incident with the new edges shown in Figure 4(b) can be redrawn as shown in Figure 4(c). Every lost link contains a face that we lose during the modification, hence all lost links are incident with vertices incident with lost faces. Every new link contains a new face, hence all new links are incident with vertices incident with new faces. The reader can consult Figure 2 when checking faces in Figure 4.

When considering the five diagonal flips shown in Figures 5(a) – (e), it is easy to see that if in Figure 5(f) we replace the edge (a, b) by (c, d), then we lose links $[c, d], [a, h], [a, f], [b, g], [b, e]$ and gain new links $[a, b], [c, e], [c, f], [d, g], [d, h]$.

By inspection of Figures 4(c) and 5(a) – (e), the reader can check that during the modification of $f'(Q)$ into $\overline{f}(Q)$: the vertex y lost two links with each of x and z; each of x and z gained at most one link with any vertex of $V(s)$; the vertex w gained at most three new

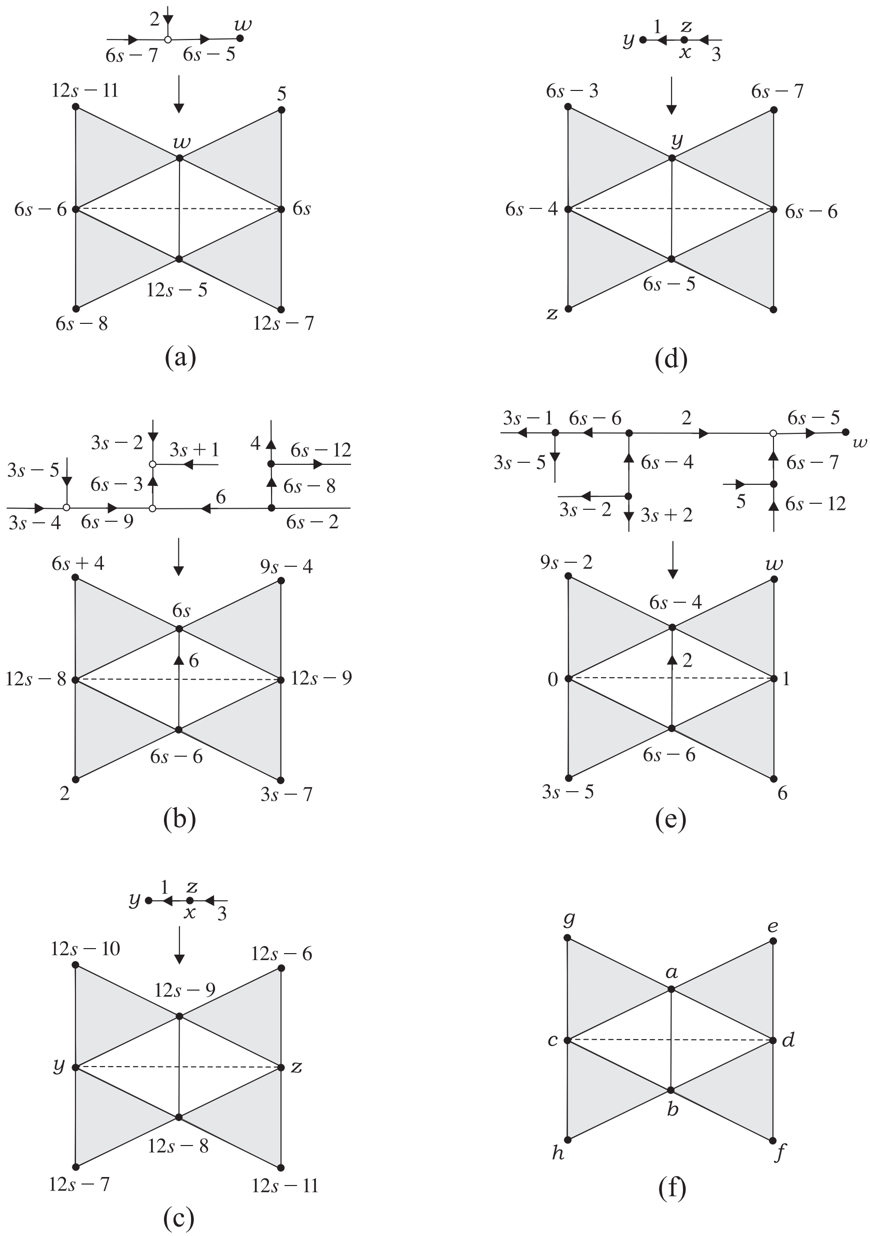

Figure 5: Diagonal flips.

links with any vertex of $V(s)$, one new link with y, and no links with each of x and z; any two vertices of $V(s)$ gained at most one new link; any vertex of $V(s)$ gained at most four new links with vertices of R.

Now, taking into account (C) and (D), we obtain the following.

(E) For any $Q \in \mathcal{D}$, in $\overline{f}(Q)$, we have the following: the vertex y has $6s-4$ links with each of x and z; each of x and z has $6s-4$ links with y only; the vertex w has at most 4 links with any vertex of $V(s) \bigcup R$, and has no links with x and z; every vertex of $V(s)$ has a link either with x or z, and has less than $6s-4$ links with every other vertex of $V(s) \bigcup R$.

By an *automorphism* of $\overline{f}(Q)$ we mean any isomorphism from $\overline{f}(Q)$ onto $\overline{f}(Q)$.

Theorem 3.1. *All 2^{2s-7} embeddings $\overline{f}(Q)$, $Q \in \mathcal{D}$, of K_{12s}, $s \geq 4$, are nonisomorphic and each of them has no nontrivial automorphisms.*

Proof. Suppose there is an isomorphism ψ of $\overline{f}(Q_1)$ onto $\overline{f}(Q_2)$, where $Q_1, Q_2 \in \mathcal{D}$. If two adjacent faces (u_1, u_2, u_3) and (u_2, u_3, u_4) are a link in $\overline{f}(Q_1)$, then the two adjacent faces $(\psi(u_1), \psi(u_2), \psi(u_3))$ and $(\psi(u_2), \psi(u_3), \psi(u_4))$ are a link in $\overline{f}(Q_2)$. Since $\overline{f}(Q_1)$ and $\overline{f}(Q_2)$ have the same number of links, namely, the number of edges of K_{12s}, it follows that the number of links between any two vertices u and u' in $\overline{f}(Q_1)$ equals the number of links between any two vertices $\psi(u)$ and $\psi(u')$ in $\overline{f}(Q_2)$.

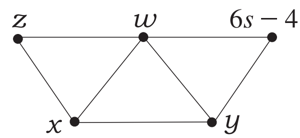

Figure 6: Common faces of all $\overline{f}(Q)$, $Q \in \mathcal{D}$.

By (E), the vertex y is the only vertex in each of $\overline{f}(Q_1)$ and $\overline{f}(Q_2)$ that has $6s-4$ links with each of two vertices, hence $\psi(y) = y$. Since x and z are the only vertices in each of $\overline{f}(Q_1)$ and $\overline{f}(Q_2)$ such that each of the vertices has $6s-4$ links with exactly one other vertex, we have $\{\psi(x), \psi(z)\} = \{x, z\}$. Since w is the only vertex of $V(s) \bigcup \{w\}$ in each of $\overline{f}(Q_1)$ and $\overline{f}(Q_2)$ that has no links with x and z, we have $\psi(w) = w$. Considering Figure 4(c), we see that $\overline{f}(Q_1)$ and $\overline{f}(Q_2)$ have the same faces shown in Figure 6. Since $(\psi(w), \psi(x), \psi(y)) = (w, \psi(x), y)$ is a face of $\overline{f}(Q_2)$, and $\psi(x) \in \{x, z\}$, we obtain (see Figure 6) that $\psi(x) = x$, and then $\psi(z) = z$.

If $\overline{f}(Q_1)$ and $\overline{f}(Q_2)$ have common adjacent faces $(\psi(u_1), \psi(u_2), \psi(u_3))$ and $(\psi(u_2), \psi(u_3), \psi(u_4))$, where $\psi(u_j) = u_j$ for $j = 1, 2, 3$, then $\psi(u_4) = u_4$. The faces incident with w (the faces are the same for all $\overline{f}(Q)$, $Q \in \mathcal{D}$), form a sequence $F_1, F_2, \ldots, F_{12s-1}$ where:

(i) $F_1 = (w, x, y)$ and $\psi(w) = w$, $\psi(x) = x$, $\psi(y) = y$;

(ii) for $j = 1, 2, \ldots, 12s-1$, the faces F_j and F_{j+1} (here $F_{12s} = F_1$) share a common edge (w, b_j), where $\{b_1, b_2, \ldots, b_{12s-1}\} = (V(s) \bigcup R) \setminus \{w\}$.

It follows that $\psi(u) = u$ for every $u \in V(s) \bigcup R$, hence $\overline{f}(Q_1)$ and $\overline{f}(Q_2)$ have the same faces. If $Q_1 = Q_2$, then we obtain that ψ is a trivial automorphism, hence $\overline{f}(Q_1)$ does not have nontrivial automorphisms.

Suppose, for a contradiction, that $Q_1 \neq Q_2$. Since $\overline{f}(Q_1)$ and $\overline{f}(Q_2)$ have the same faces, considering the modification of $f(Q)$ into $\overline{f}(Q)$, we see that $f(Q_1)$ and $f(Q_2)$ have the same faces as well, hence we have:

(a) The cyclic order in which the vertices adjacent to the vertex 0 are arranged on the surface around the vertex 0 in $f(Q_1)$ is (up to reversal) the cyclic order in which the vertices adjacent to the vertex 0 are arranged on the surface around the vertex 0 in $f(Q_2)$.

The embeddings $f(Q_1)$ and $f(Q_2)$ are generated by the current graphs $\langle G, \lambda, Q_1 \rangle$ and $\langle G, \lambda, Q_2 \rangle$, respectively. Since $Q_1 \neq Q_2$, a trivalent vertex v (resp. w) of G has the same rotation (resp. different rotations) in Q_1 and Q_2. Then the circuit of $\langle G, \lambda, Q_1 \rangle$ (resp. $\langle G, \lambda, Q_2 \rangle$) is of the form $(a_1, a_2, \ldots, b_1, b_2, \ldots)$ (resp. $(a_1, a_2, \ldots, b_1, b_3, \ldots)$) where a_1 and a_2 are arcs incident with v, and b_1, b_2, b_3 are arcs incident with w, where $b_2 \neq b_3$. Hence the two cascades have different logs of their circuits, namely, $(\lambda(a_1), \lambda(a_2), \ldots, \lambda(b_1), \lambda(b_2), \ldots)$ and $(\lambda(a_1), \lambda(a_2), \ldots, \lambda(b_1), \lambda(b_3), \ldots)$ where $\lambda(b_2) \neq \lambda(b_3)$, contrary to (a) (note that in the cascades, $\lambda(a) \neq \lambda(a')$ for different arcs a and a'). □

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