On generalised Petersen graphs of girth 7 that have cop number 4^*

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Abstract

We show that if $n = 7k/i$ with $i \in \{1, 2, 3\}$ then the cop number of the generalised
Petersen graph $GP(n, k)$ is 4, with some small previously-known exceptions. It was pre-
viously proved by Ball et al. (2015) that the cop number of any generalised Petersen graph
is at most 4. The results in this paper explain all of the known generalised Petersen graphs
that actually have cop number 4 but were not previously explained by Morris et al. in a
recent preprint, and places them in the context of infinite families. (More precisely, the
preprint by Morris et al. explains all known generalised Petersen graphs with cop number
4 and girth 8, while this paper explains those that have girth 7.)

Keywords: Cops and robbers, generalised Petersen graphs, girth, cop number.

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1 Introduction

Cops and robbers is a game that can be played on any graph. There are two players:
one playing the cops, with one or more pieces and the other playing the robber, with a
single piece. Both players have perfect information: they can see the graph and the current
locations of all pieces. The two players take turns, with the cop player going first. On

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their first turns, each player chooses a vertex on which to place each of their pieces. On all subsequent turns, they may move any number of their pieces to any neighbouring vertex. The object of the game for the cops is to capture the robber by landing on the same vertex as them with any of the cop pieces. The object for the robber is to avoid being captured forever. The *cop number* of a graph is the minimum number of cops required in order for the cop player to have a winning strategy.

This game first appears in [5] and [6], and has been studied extensively on many families of graphs (see [3] as an excellent reference).

We are interested in studying this game only on the well-known family of generalised Petersen graphs. The *generalised Petersen graph* $GP(n, k)$ is a graph on $2n$ vertices whose vertex set is the union of $A = \{a_0, \ldots, a_{n-1}\}$ and $B = \{b_0, \ldots, b_{n-1}\}$. The edges have one of three possible forms:

- $\{a_i, a_{i+1}\}$;
- $\{a_i, b_i\}$; and
- $\{b_i, b_{i+k}\}$,

where subscripts are calculated modulo n. This family generalises the Petersen graph, which is $GP(5, 2)$. We require $n \geq 5$, and $0 < k < n/2$. This ensures that the graphs are cubic, and avoids some isomorphic copies of graphs.

The girth of a generalised Petersen graph is at most 8, and is well-understood from the parameters n and k (see [2, Theorem 5]). Up to isomorphisms, a generalised Petersen graph has girth 7 if and only if its girth is not 6 or less, and its parameters satisfy one of the following conditions:

- $n = 7k/i$ where $i \in \{1, 2, 3\}$;
- $k = 4$;
- $n = 2k + 3$; or
- $n = 3k \pm 2$.

Generalised Petersen graphs of girth 6 or less are characterised (up to isomorphisms) as those having $k \leq 3$, $n = 2k + 2$, or $n = jk$ where $j \in \{5/2, 3, 4, 5, 6\}$. By carefully checking these with the girth at most 7 conditions for compatibility, we obtain the following precise characterisation. Up to isomorphisms, a generalised Petersen graphs has girth 7 if and only if $k \geq 4$ and its parameters satisfy one of the following conditions:

- $k = 4$ and $n \notin \{10, 12, 16, 20, 24\}$;
- $n = 7k/i$ where $i \in \{1, 2, 3\}$, and if $i = 3$ then $k \neq 6$;
- $n = 2k + 3$ and $k \neq 6$; or
- $n = 3k \pm 2$ and $(n, k) \neq (4, 10)$.

In [1], Ball et al. showed that the cop number of every generalised Petersen graph is at most 4. They also provided a list of all generalised Petersen graphs with $n \leq 40$ that attain this bound. (Minor corrections to this list were included in [4], which also proved that every generalised Petersen graph of girth 8 that appears on this list falls into an infinite class of generalised Petersen graphs of girth 8 that have cop number 4.) Almost all graphs on this list have girth 8, but there are three graphs of girth 7 on the list:
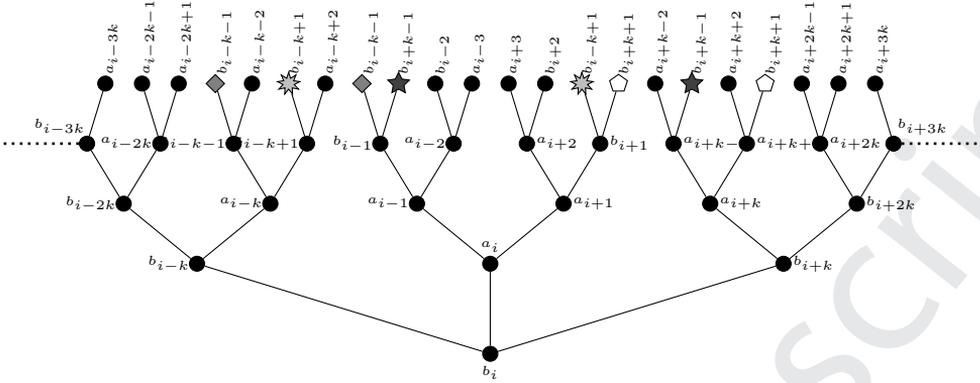
Figure 1: The vertices within distance 4 of a_i, in our families of graphs.

- $GP(28, 8)$;
- $GP(35, 10)$; and
- $GP(35, 15)$.

Notably, all of these graphs have parameters of the form $n = 7k/2$ or $n = 7k/3$. In this paper, we show that with the exception of the generalised Petersen graph with $n < 42$ that do not appear above, every generalised Petersen graph whose parameters have the form $n = 7k/i$ where $i \in \{1, 2, 3\}$ has cop number 4. Thus, in light of the results of [4], we show that all known generalised Petersen graphs with cop number 4 are included in infinite families that have this property.

For graphs in the family we are studying (generalised Petersen graphs with $n = 7k/i$, $i \in \{1, 2, 3\}$ and $n \geq 42$ or $(n, k) \in \{(28, 8), (35, 10), (35, 15)\}$), the neighbourhood (up to distance 4) of an arbitrary vertex $a_i \in A$ is shown in Figure 1, while that for an arbitrary vertex $b_i \in B$ is shown in Figure 1. Note that at distance 4, some vertices may be the same as others; a pair of nodes that are depicted by the same non-standard shape (e.g. a diamond) represent the same vertex. In our results, we will assume that either $n \geq 42$, or $(n, k) \in \{(28, 8), (35, 10), (35, 15)\}$. Careful calculations (left to the reader) verify that this assumption avoids any additional vertices at distance 4 being the same as each other, so that the graphs are as depicted. This will be of critical importance in our proofs.

2   Main result

We begin by defining what it means for a robber to be trapped.

Definition 2.1. The cops have *trapped* the robber if there is a cop on or adjacent to each v_i ($i \in \{1, 2, 3\}$), where v_1, v_2, and v_3 are the neighbours of the vertex the robber is on (no matter whose turn it is), or if at least one cop is adjacent to the robber on the cops' turn.

Observe that if the cops have trapped the robber, then the robber will be caught on the cops' next move if the robber moves, or within the next two cop moves if the robber passes, and this is the only configuration for which this is true.

Figure 2: The vertices within distance 4 of b_i, in our families of graphs. The dotted lines represent an edge between b_{i+3k} and b_{i-3k}.

We intend to demonstrate that three cops do not have a winning strategy on the graphs we are studying. Together with the upper bound of [1], this is sufficient to show that the cop number for these graphs is 4. In order to prove that three cops do not have a winning strategy, we will first show that if three cops have not already trapped the robber, the robber always has a legal move whereby the cops cannot trap the robber on the cops' next move. After setting up some notation, we will define three cases that together encompass all possible configurations the cops can ever be in relative to the robber's position, and will explain that these cases are exhaustive. Our first lemma will show that one of these three cases can only arise in our graphs if the robber is already trapped. Our second lemma shows that if either of the other cases applies, the robber has a legal move whereby the cops cannot trap the robber on the cops' next turn. These lemmas set up an inductive argument showing that the robber can always avoid becoming trapped unless they begin the game trapped. In the proof of our main theorem, we show that the robber can always find a starting position that is not trapped, thus completing the proof.

We now introduce the labelling system used throughout the rest of this paper and all the proofs within. We assume throughout that the game is being played with 3 cops, C_1, C_2, and C_3. We will use r to denote the vertex the robber starts on. Label the three vertices adjacent to r with v_1, v_2, and v_3. When we mention a "branch" from v_i ($i \in \{1, 2, 3\}$), we are referring to v_i and all of the other 7 vertices along any paths from the robber's vertex to the vertices at distance 4 from the robber, using only a fixed one of v_i's neighbours (excluding r from our count). All of this is illustrated in Figure 2.

We now outline the three cases included in [4] as the only possible sets of configurations for the robber to be in, relative to the positions of the three cops. They are as follows:

- **Case 1.** For at least one vertex v_i ($i \in \{1, 2, 3\}$), v_i has no cop on one of its branches, and no cop within distance 2 of the robber on the other branch.

- **Case 2.** For every cop C_i ($i \in \{1, 2, 3\}$), C_i is not within distance 2 of the robber.

- **Case 3.** For at least one vertex v_i ($i \in \{1, 2, 3\}$), v_i has at least one cop on its branches who is at distance 2 or less from the robber. If any vertices v_j ($j \in \{1, 2, 3\}$)
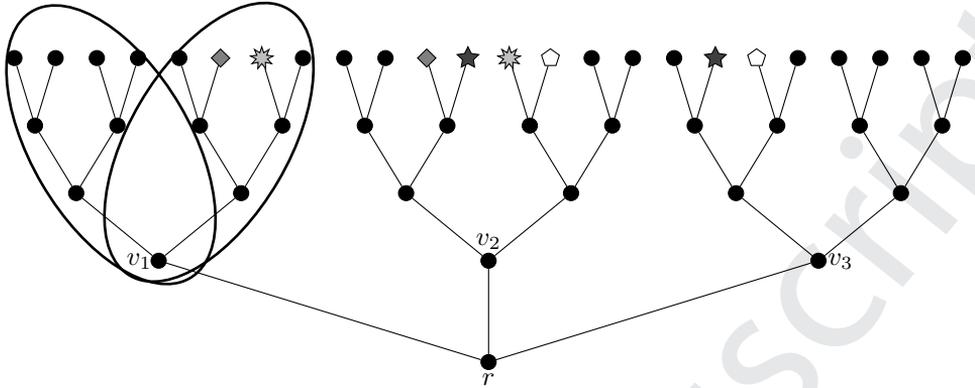
Figure 3: This figure shows our basic notation. In the illustration, r is a vertex in A. The two branches from v_1 are circled.

are not covered by that clause, then for each such v_j, v_j has at least one cop on each of its branches.

Now that we have stated our three cases, we will explain why these cases cover all of the possibilities. Suppose we have a set-up that does not conform to Case 2. This means that there is at least one cop who is at distance 2 or less from the robber. This cop is on v_i ($i \in \{1, 2, 3\}$), or one of its adjacent vertices (other than r). If the situation also does not satisfy the assumptions of Case 3, then there must be at least one vertex (say v_j, where $j \in \{1, 2, 3\}$ and $j \neq i$) that has no cop on one of its branches, and the closest cop through v_j is at distance 3 or more from the robber. This means that Case 1 applies to vertex v_j.

We will now prove our first lemma, which will be used in our main lemma.

Lemma 2.2. *Suppose that $n = 7k/i$ where $i \in \{1, 2, 3\}$, and either $n \geq 42$ or $(n, k) \in \{(28, 8), (35, 10), (35, 15)\}$. If we play cops and robbers with 3 cops on $GP(n, k)$, then the only way for the configuration of the robber with respect to the cops to fall into Case 3 is if the robber is trapped.*

Proof. To prove this lemma, we will use a case analysis on the possible configurations within Case 3, and show why each is impossible unless the robber is trapped. Our proof holds regardless of whether the robber is on a vertex in A or a vertex in B. We assume that both the cop and robber players use their moves optimally.

We now move to our case analysis.

Case A. For exactly one vertex v_i ($i \in \{1, 2, 3\}$), v_i satisfies the conditions of Case 3 by having at least one cop on its branches who is at distance 2 or less from the robber.

In order for our situation to fall under Case A, v_i satisfies the conditions of Case 3 already, but there are two other vertices, v_j and v_k (where $\{i, j, k\} = \{1, 2, 3\}$), that still need to satisfy these conditions. This means that each of v_j and v_k requires both of its branches to be covered by cops. There is no way for two cops to cover all four branches, unless those two cops are directly on v_j and v_k, which does not fall under Case A. This means that it is impossible for a configuration of cops and robber on our graphs to fall under Case A. ∎

Case B. For exactly two vertices v_i and v_j ($i, j \in \{1, 2, 3\}$), v_i and v_j satisfy the conditions of Case 3 by each having at least one cop on its branches who is at distance 2 or less from the robber.

Assuming our scenario meets the parameters of Case B, there is only one vertex v_k (where $\{i, j, k\} = \{1, 2, 3\}$) which does not yet satisfy the conditions of Case 3. This final vertex is required to have at least one cop on each of its branches. Since two of the three cops have already been placed, we only have one cop left. No v_k has a vertex that is a part of both of its branches, other than v_k itself, which cannot have a cop on it because of the conditions of Case B. ∎

Case C. For every vertex v_i ($i \in \{1, 2, 3\}$), v_i satisfies the conditions of Case 3 by having at least one cop on its branches who is at distance 2 or less from the robber.

In order for the parameters of Case C to be met, there must be a cop on each of v_1, v_2, and v_3, or the adjacent vertices (excluding r, of course). In this case, the robber is trapped. ∎   □

We now prove our main lemma, which will be used inductively in the proof of our theorem.

Lemma 2.3. *Suppose that $n = 7k/i$ where $i \in \{1, 2, 3\}$, and either $n \geq 42$ or $(n, k) \in \{(28, 8), (35, 10), (35, 15)\}$. If we play cops and robbers on $GP(n, k)$ with three cops, then unless the robber starts their turn trapped, they always have a legal move so that they cannot be trapped by the cops on the cops' turn.*

Proof. To prove this lemma, we will use a case analysis of the current positions of the cops compared to that of the robber (on the robber's turn). We will use the techniques from [4] to prove this lemma. We assume that the three cops have not yet trapped the robber. We also assume that both the cop and robber players use their moves optimally. We will go on to prove that unless the robber starts their turn trapped, they always have a legal move so that the robber cannot be trapped at the end of the cops' turn.

In Lemma 2.2, we proved that Case 3 cannot apply to our graphs. This means that we need only prove this lemma under the assumption that the scenario falls into Case 1 or Case 2. These proofs are below.

Case 1. For at least one vertex v_i ($i \in \{1, 2, 3\}$), v_i has no cop on one of its branches, and no cop within distance 2 of the robber on the other branch.

Let us assume that the vertex satisfying the requirements of this case is v_1 (the other cases are similar). Equally, let us assume that the left branch is the one with no cop. This situation is shown in Figure 2. Since there is no cop within distance 2 of the robber on vertex v_1, the robber can go to v_1 safely. No cop can catch the robber on this move, since no cop can be within distance 1 of v_1 at the start of this turn. Since there are no cops on the left branch from v_1, there are no cops within distance 4 of the robber in that direction before the robber's move, so even after the cops' turn there cannot be a cop within distance 2 of the robber in that direction. Therefore, the robber is not trapped at the end of the cops' turn. ∎

Case 2. For every cop C_i ($i \in \{1, 2, 3\}$), C_i is not within distance 2 of the robber.

Figure 2 illustrates this case. In this scenario, the robber can move to any of the three vertices adjacent to their position (v_1, v_2, or v_3). Let's have the robber choose v_1. Since no cops were within distance 2 of the robber at the beginning, once the robber has moved,
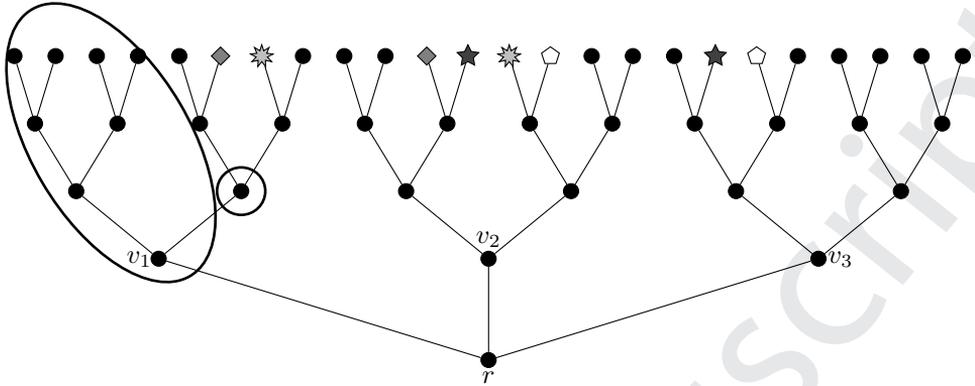
Figure 4: The vertices that we assume do not have cops on them in Case 1 are circled. The illustration shows this when $r \in A$.
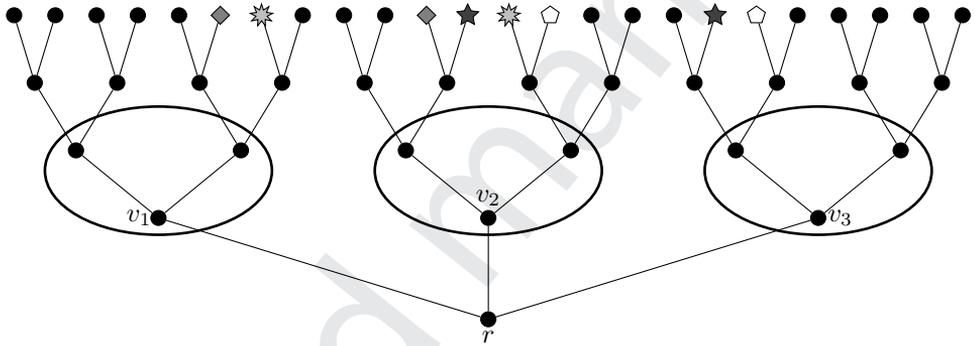

Figure 5: The vertices that cannot have cops on them in Case 2 are circled. The illustration shows this when $r \in A$.

they cannot be within distance 1 of the robber, so cannot catch the robber. Furthermore, since no cop was on vertex v_1, v_2, v_3, or any of their neighbours, after the cops' move no cop is on r or any of its neighbours (and r is a neighbour of the robber's new position). This means that the robber is not trapped.   ∎

We have now proven that the robber cannot be trapped at the end of the cops' turn if they did not start their turn trapped.   □

This allows us to prove our main result.

Theorem 2.4. *Suppose that* $n = 7k/i$ *where* $i \in \{1, 2, 3\}$, *and* $n \geq 42$ *or* $(n, k) \in \{(28, 8), (35, 10), (35, 15)\}$. *Then the cop number of the graph* $GP(n, k)$ *is* 4.

Proof. We begin by showing that there is always a vertex the robber can choose for their first move, so that they do not begin the game trapped.

Let w be an arbitrary vertex with neighbours v_1, v_2, and v_3. Recall that for the robber to be trapped, there must be cops on each of its neighbours or their adjacent vertices, or
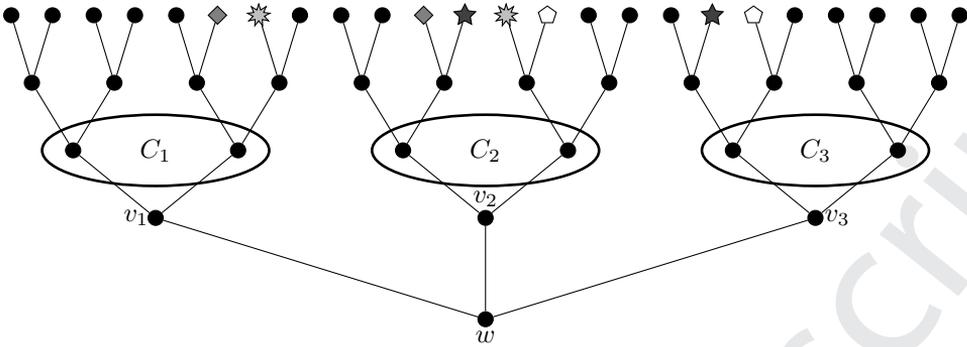
Figure 6: The circles in this figure show the ideal positions for the cops at the beginning of the game, when $w \in A$.

there must be at least one cop directly adjacent to the robber. In order for the cops to be in a position where they could have the robber trapped, a cop should be placed on each of v_1, v_2, and v_3 or their adjacent vertices. Without loss of generality, let us assume that C_i ($i \in \{1, 2, 3\}$) is on v_i or an adjacent vertex.

Suppose momentarily that all three cops choose to go on vertices adjacent to w. Under this scenario, should the robber go on any vertex adjacent to a cop, they will be trapped. This gives us 10 vertices where the robber cannot go without being trapped (including those with cops already on them).

Now, suppose that all three cops choose to go on vertices at distance 2 from w. In this set-up, in addition to the robber being trapped if they are adjacent to a cop, they will also be trapped if they go on w, since there are cops adjacent to v_1, v_2, and v_3. Therefore, in this case, there are 13 different vertices where the robber cannot go without being trapped.

This means that our second set of positions, as shown in Figure 2, are the ideal choices for the cops when there are only three of them, and if they go in the ideal positions, they can still only ensure that the robber will be trapped if placed on one of 13 different vertices. Since $n \geq 28$, our graph has at least 56 vertices, and the robber still has many choices that do not leave them trapped, meaning that the robber will not start trapped.

Combining this with the results of Lemma 2.3, we conclude that the robber never has to become trapped, so $c(G) > 3$.

By [1], if G is a generalised Petersen graph, then $c(G) \leq 4$, so $c(G) = 4$.   □

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