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Bracing frameworks consisting of parallelograms*

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Abstract

A rectangle in the plane can be continuously deformed preserving its edge lengths, but adding a diagonal brace prevents such a deformation. Bolker and Crapo characterized combinatorially which choices of braces make a grid of squares infinitesimally rigid using a *bracing graph*: a bipartite graph whose vertices are the columns and rows of the grid, and a row and column are adjacent if and only if they meet at a braced square. Duarte and Francis generalized the notion of the bracing graph to rhombic carpets, proved that the connectivity of the bracing graph implies rigidity and stated the other implication without proof. Nagy Kem gives the equivalence in the infinitesimal setting. We consider continuous deformations of braced frameworks consisting of a graph from a more general class and its placement in the plane such that every 4-cycle forms a parallelogram. We show that rigidity of such a braced framework is equivalent to the non-existence of a special edge coloring, which is in turn equivalent to the corresponding bracing graph being connected.

Keywords: Flexibility, rigidity, bracing, rhombic tiling.

Math. Subj. Class.: 52C25, 51K99, 70B99

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1 Introduction

A planar framework is a graph together with a placement of its vertices in the plane. If there is a non-trivial flex (a deformation of the placement preserving the distances between adjacent vertices that is not induced by a rigid motion), then the framework is said to be flexible, otherwise rigid. Bolker and Crapo [4] studied infinitesimal flexibility of a framework corresponding to a grid of squares with some squares being braced by adding diagonals, see Figure 1. They construct a bipartite graph by taking the columns and rows of the grid to be the two parts of the vertex set; a column and row are connected if and only if their common square is braced. They showed that a braced grid is infinitesimally rigid, i.e., has no non-trivial first order flex, if and only if the bipartite graph is connected.

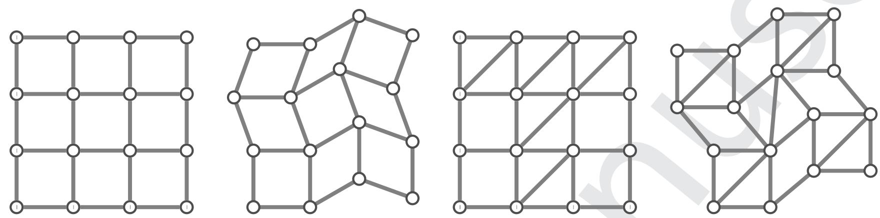

Figure 1: Grid frameworks can be deformed in a way that preserves the edge lengths. By bracing it (i.e. by adding diagonal edges) we can reduce the number of degrees of freedom. The left braced grid is rigid whereas the right one allows a flex.

Generalizations to rectangular grids with holes [14, 24] or placing longer diagonals [13] than those for a single 4-cycle have been studied as well as bracing by cables [20]. Grids with rectilinear boundary are discussed in [17]. Extensions to cubic grids have been studied [3, 21]. The papers [9, 28] describe the number of randomly added braces for which the transition from flexible to rigid occurs. A related problem to the rigidity of a grid is the rigidity of one- and multi-story building [19, 23], also with cables [5, 22, 25]. Simple forms of bracing grids are also known to be suitable as a puzzle, for science communication and for student's exercises (see for instance [26]).

In this work, we focus on parallelograms instead of squares, and we allow a richer combinatorial structure than grids. Flexibility of rhombic/parallelogramic tilings is studied by physicists due to its relation with quasicrystals [29]. The bracing of rhombic carpets, which are 1-skeleta of finite simply connected pieces of rhombic tilings, was investigated by Wester [27]. Duarte and Francis [8] formalized the notions necessary to study the flexibility of rhombic carpets: a natural step from columns and rows of a grid towards a rhombic carpet is to take *ribbons*. These are sequences of rhombi such that every two consecutive ones share an edge and all these edges are parallel. Following the idea of Bolker and Crapo, Duarte and Francis construct a *bracing graph* whose vertices are the ribbons and two ribbons are adjacent if they have a common rhombus that is braced. They prove that if the constructed graph is connected, then the braced rhombic carpet is rigid. Further, they state the other implication without proof. We thank Eliana Duarte for pointing out this statement to us and sharing some hints about a possible proof [7]. Nagy Kem [18] translates the infinitesimal rigidity of a braced rhombic carpet to the rigidity of an auxiliary framework which in turn corresponds to the connectivity of the bracing graph.

We formulate the problem of the flexibility of braced structures in terms of frameworks. In particular, we define ribbons as equivalence classes on edges of the underlying graph us-

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ing its 4-cycles. We consider a special class of graphs, which we call *ribbon-cutting* graphs. A connected graph is ribbon-cutting if every ribbon is an edge cut, i.e., removing the edges of the ribbon makes the graph disconnected. Regarding the placement, we ask all 4-cycles to form parallelograms, see Figure 2. Notice that the frameworks we consider — we call them P-frameworks — form a proper superset of the frameworks corresponding to rhombic carpets and rectangular grids (without holes). The question we address is analogous to the one by Bolker, Crapo, Duarte and others, namely, characterization of choices of braces of parallelograms yielding flexible/rigid P-frameworks. Contrary to Bolker, Crapo and Nagy Kem, we consider continuous flexes, not infinitesimal ones. Furthermore, we use a recently established method of special edge colorings to prove our results.

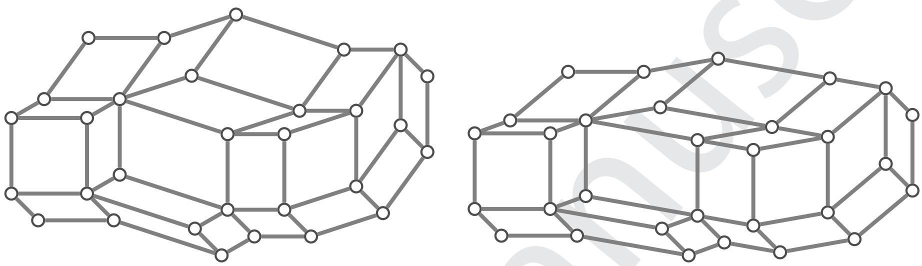

Figure 2: Carpet frameworks can be deformed in a way that preserves the edge lengths.

The notion of *NAC-colorings* was developed in our previous paper [11]. A NAC-coloring is a surjective edge coloring of a graph by red and blue such that for every cycle of the graph, either all edges have the same color or there are at least two edges of each color. We proved that a graph has a flexible framework if and only if it has a NAC-coloring. It appears that the techniques used to prove the theorem fit nicely to the context of bracing P-frameworks if we restrict ourselves to certain NAC-colorings: a NAC-coloring is called *cartesian* if there are no two vertices connected by a red path and blue path simultaneously. The non-existence of a cartesian NAC-coloring serves as a bridge in the proof that a braced P-framework is rigid if and only if the corresponding bracing graph (defined analogously to [8]) is connected. Our results can be summarized as follows.

Theorem 1.1. *For a braced P-framework (G, ρ), the following statements are equivalent:*

(i) (G, ρ) *is rigid,*

(ii) G has no cartesian NAC-coloring, and

(iii) the bracing graph of G is connected.

In particular, the minimum number of braces making a framework rigid is one less than the number of ribbons of its underlying graph.

Theorem 1.1 extends the result by Duarte and Francis [8], which is the implication *(iii)* $\implies$ *(i)* for rhombic carpets. These form a proper subclass of P-frameworks used in this paper since all rhombic carpets are planar embeddings in graph theoretical sense, contrary to P-frameworks.

We implemented the concepts introduced in the current paper by extending our SageMath package FlexRiLoG [12]. We encourage the reader to experiment online using the Jupyter notebook available at https://jan.legersky.cz/bracingFrameworks.

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The paper is organized as follows: Section 2 recalls the notions from rigidity theory and NAC-colorings. We define ribbons, parallelogram placements and P-frameworks in Section 3 and prove some basic results. We also formalize bracing and the notion of bracing graph in our context. Section 4 provides the proofs yielding Theorem 1.1. We put additional material to the appendix, for instance, we show that rhombic carpets are P-frameworks in Appendix B.

2 Preliminaries

In this section we recall basic notions and definitions commonly used in rigidity theory. The ideas are based on previous work using special edge colorings to find flexes of graphs. We introduce these colorings here and describe what we mean by flexibility.

Definition 2.1. Let $G = (V_G, E_G)$ be a connected graph. A map $\rho : V_G \to \mathbb{R}^2$ such that $\rho(u) \neq \rho(v)$ for all edges $uv \in E_G$ is a *placement*. The pair (G, ρ) is called a *framework*.

Definition 2.2. Two frameworks (G, ρ) and (G, ρ') are *equivalent* if

$$\|\rho(u) - \rho(v)\| = \|\rho'(u) - \rho'(v)\|$$

for all $uv \in E_G$. Two placements ρ and ρ' are *congruent* if there exists a Euclidean isometry M of $\mathbb{R}^2$ such that $M\rho'(v) = \rho(v)$ for all $v \in V_G$.

Definition 2.3. A *flex* of the framework (G, ρ) is a continuous path $t \mapsto \rho_t$, $t \in [0, 1]$, in the space of placements of G such that $\rho_0 = \rho$ and each (G, ρ_t) is equivalent to (G, ρ). The flex is called trivial if ρ_t is congruent to ρ for all $t \in [0, 1]$.

We define a framework to be *(proper) flexible* if there is a non-trivial flex in $\mathbb{R}^2$ (with injective placements). Otherwise it is called *rigid*.

In a previous paper [11], we introduced a special edge coloring, which is called NAC-coloring, in order to classify the graphs that have flexible frameworks.

Definition 2.4. Let G be a graph. A coloring of edges $\delta \colon E_G \to \{\text{blue}, \text{red}\}$ is called a *NAC-coloring*, if it is surjective and for every cycle in G, either all edges have the same color, or there are at least 2 edges in each color (see Figure 3). The NAC-coloring δ gives subgraphs

$$G^\delta_{\text{red}} = (V_G, \{e \in E_G \colon \delta(e) = \text{red}\}) \text{ and } G^\delta_{\text{blue}} = (V_G, \{e \in E_G \colon \delta(e) = \text{blue}\}) .$$

We remark that flexibility in [11] is defined in the following sense: an edge labeling by positive real numbers (interpreted as lengths for edges) is called *flexible* if there are infinitely many non-congruent placements inducing it. Notice that if a framework has a flex, then the induced edge labeling (corresponding to the lengths) is flexible. On the other hand, assuming we have a flexible edge labeling, then placements inducing the edge lengths corresponding to the labeling form an algebraic variety containing an algebraic curve of placements. Considering a nonsingular point in this curve, a local parametrization of the curve around this point gives a flex. Therefore, we can state the result as follows.

Theorem 2.5 ([11])**.** *A connected non-trivial graph allows a flexible framework if and only if it has a NAC-coloring.*

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Two non-adjacent vertices u and v overlap in the flex constructed in the proof of the theorem in [11] if and only if there is a red path from u to v and a blue path from u to v. In order to avoid overlapping vertices, we focus on a special type of NAC-colorings.

Definition 2.6. A NAC-coloring δ of a graph G is called *cartesian* if no two distinct vertices are connected by a red and blue path simultaneously.

We chose the name *cartesian* due to its connection with cartesian products of graphs, see Appendix A.

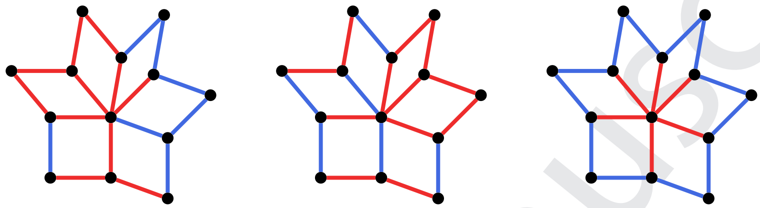

Figure 3: A coloring that is not NAC-coloring (left), cartesian NAC-coloring (middle) and non-cartesian NAC-coloring (right).

Remark 2.7. A NAC-coloring δ of a graph G is cartesian if and only if for every connected component R of G^{δ}_{red} and B of G^{δ}_{blue}, the intersection of the vertex sets of R and B contains at most one vertex.

Notice that in a cartesian NAC-coloring, a 4-cycle subgraph is monochromatic, or the opposite edges have the same color.

3 Ribbons and parallelogram placements

In this section we describe bracings of graphs (Section 3.2). We mainly consider a class of graphs (Section 3.1) which essentially consists of four-cycles which we want to place in the plane, forming parallelograms. Having these 4-cycles in mind we start by defining an equivalence relation on the edges. The equivalence classes, called *ribbons*, generalize the notion of rows and columns in a rectangular grid. Ribbons are a concept that is also used in other places under various names (stripes, worms, de Bruijn lines) and for different purpose (see for instance [2, 6, 10, 29]).

Definition 3.1. Let G be a graph. Consider the relation on the set of edges, where two edges are in relation if they are opposite edges of a 4-cycle subgraph of G. An equivalence class of the reflexive-transitive closure of the relation is called a *ribbon*. Figure 4 shows all ribbons for some small graphs. A ribbon r is *simple* if the subgraph induced by r does not contain any 4-cycle (see Figure 5a for an example of a non-simple ribbon).

In the case of rectangular grids, there is a natural way how to order the edges in a ribbon, i.e., a row or column. In our context, there is no natural order of the edges in a ribbon as Figure 5b indicates.

From now on, given a walk W, the notation $(u, v) \in W$ means that the edge uv belongs to W and u precedes v in W. Similarly, for a ribbon r, the notation $(u, v) \in r \cap W$ means $(u, v) \in W$ and $uv \in r$. If $(u, v) \in r \cap W$ is used to iterate in a sum, the edges of W

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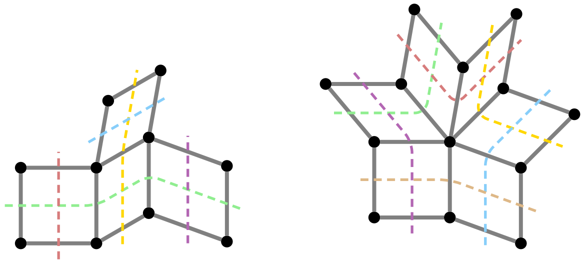

Figure 4: The ribbons of the graphs are indicated by dashed lines. All edges intersecting the line belong to the same ribbon.

(a) The only ribbon is not simple. (b) The yellow ribbon has three “ends”.

Figure 5: Special cases that might happen for ribbons.

must be considered as a multiset: the summand corresponding to (u, v) is included as many times as uv occurs in W with u preceding v in W. Similarly for the cardinality of $r \cap W$:

$$|r \cap W| = \sum_{(u,v) \in r \cap W} 1 \,.$$

Recall that a set of edges r is an *edge cut* of a connected graph G if the graph $G \setminus r = (V_G, E_G \setminus r)$ is disconnected.

Lemma 3.2. *Let G be a connected graph with a simple ribbon r, which is an edge cut. Then $G \setminus r$ has exactly two connected components. If $w, w' \in V_G$ and W is a walk from w to w', then $|r \cap W|$ is odd if and only if r separates w and w', i.e., w and w' are in the different connected components of $G \setminus r$. In particular, if W is a closed walk in G, then $|r \cap W|$ is even.*

Proof. Let uv be an edge of r. For every edge $u'v' \in r$, there exists a sequence of edges $u_1v_1, \ldots, u_kv_k$ such that $u = u_1, v = v_1, u_k = u', v_k = v'$ and $(u_i, v_i, v_{i+1}, u_{i+1})$ is a 4-cycle in G. Hence, there are walks $(u_1, \ldots, u_k)$ and $(v_1, \ldots, v_k)$ in G. An edge u_iu_{i+1} is in r if and only if v_iv_{i+1} is in r. But if $u_iu_{i+1}, v_iv_{i+1} \in r$, then $(u_i, v_i, v_{i+1}, u_{i+1})$ would be a 4-cycle in the subgraph induced by r, which is not possible since r is simple. Hence, no edge of the two walks is in r. This shows that every vertex of an edge in r is either connected to u, or v in $G \setminus r$, thus, $G \setminus r$ has two connected components.

If W is a walk from w to w', then $|W \cap r|$ is even if and only if w and w' are in the same connected component of $G \setminus r$. □

We want to consider graphs that somehow consist of parallelograms. For interpreting this idea we need to look at frameworks rather than graphs.

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Definition 3.3. Let G be a connected graph. A placement $\rho : V_G \to \mathbb{R}^2$ for G such that ρ is injective and each 4-cycle in G forms a parallelogram[1] in ρ is called a *parallelogram placement*.

Remark 3.4. Let ρ be a parallelogram placement of a connected graph G. Edges of a ribbon of G are parallel line segments of the same length in ρ.

Remark 3.5. By Theorem 3.4, if there was a 4-cycle induced by a ribbon, then two opposite vertices of the 4-cycle would coincide in a parallelogram placement, which contradicts injectivity of the placement. Hence, if a graph allows a parallelogram placement, then all its ribbons are simple.

The following properties of parallelogram placements are needed later on.

Lemma 3.6. *Let G be a connected graph with a parallelogram placement ρ and ribbon r which is an edge cut. If the vertex set of r is $V_1 \cup V_2$, where all vertices of V_i belong to the same connected component of $G \setminus r$, then $\rho(V_2)$ is a translation of $\rho(V_1)$. In particular, the vector $\rho(u_2) - \rho(u_1)$ is the same for all edges $u_1u_2 \in r$, $u_i \in V_i$.*

Proof. The ribbon r is simple by Theorem 3.5. Theorem 3.2 gives the partition $V_1 \cup V_2$ with $|V_1| = |V_2|$. The vector $\rho(u_2) - \rho(u_1)$ is the same for all $u_1u_2 \in r$, $u_i \in V_i$, by Theorem 3.4. □

Lemma 3.7. *Let G be a connected graph with a parallelogram placement ρ. Let r be a ribbon of G which is an edge cut and W be a walk in G. If $|r \cap W|$ is even, then*

$$\sum_{(w_1,w_2)\in r\cap W} (\rho(w_2) - \rho(w_1)) = 0\,.$$

Proof. Let $W = (u_0, u_1, \ldots, u_m)$ be a walk. All ribbons are simple by Theorem 3.5. Let $V_1 \cup V_2$ be as in Theorem 3.6. Let the edges of W that are in r be $u_{j_1}u_{j_1+1}, \ldots, u_{j_k}u_{j_k+1}$ with $j_1 < j_2 < \cdots < j_k$, k is even by assumption. We have that $u_{j_1}, u_{j_2+1}, u_{j_3} \ldots, u_{j_k+1} \in V_1$ and $u_{j_1+1}, u_{j_2}, u_{j_3+1} \ldots, u_{j_k} \in V_2$. By Theorem 3.6,

$$\sum_{i=1}^{k} (\rho(u_{j_i+1}) - \rho(u_{j_i})) = 0\,.$$

□

3.1 Frameworks and graphs consisting of parallelograms

For proving the statements in this paper, we require frameworks with a parallelogram placement where all ribbons of the underlying graph are edge cuts. This yields a so called P-framework. We describe two different subclasses of P-frameworks: an illustrating approach is to start from a set of connected parallelograms with additional properties and form a framework. This will be a carpet framework. The second one is a recursive construction. Furthermore, we describe the relations between these classes. For streamlining the paper, we put some results to Appendix B.

[1] Here we consider only non-degenerate parallelograms, namely, not all vertices are collinear.

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Definition 3.8. A graph G is called *ribbon-cutting graph* if it is connected and every ribbon is an edge cut. If ρ is a parallelogram placement of G, we call the framework (G, ρ) a *P-framework*.

A rectangular lattice graph (grid graph) with its natural placement is a P-framework. as well as the frameworks in Figure 2 and the graphs in Figures 4 and 5b with the placements given by their layouts.

There are ribbon-cutting graphs without any parallelogram placement. Figure 6 shows such a graph, for which the non-existence of a parallelogram placement follows from failing one of the necessary conditions given by Theorem 3.9. On the other hand, the graph in Figure 7 is not ribbon-cutting but has a parallelogram placement.

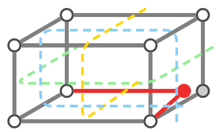

Figure 6: The graph of the framework is ribbon-cutting, but it has no parallelogram placement: if the red vertex and edges were placed forming a parallelogram, the two filled vertices would coincide. Theorem 3.9 also shows this, since these two vertices are not separated by any ribbon.

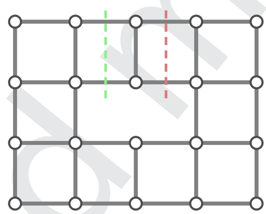

Figure 7: A parallelogram placement of a graph with ribbons that are not edge cuts.

Theorem 3.9. *If (G, ρ) is a P-framework, then there are no odd cycles in G, i.e., the graph is bipartite, and every two vertices are separated by a ribbon.*

Proof. Theorem 3.5 guarantees that all ribbons of G are simple. Every ribbon intersects a cycle in an even number of edges by Theorem 3.2, hence, the cycle is even.

Let u and v be two distinct vertices. Let $W = (u = u_0, u_1, \dots, u_m = v)$ be a walk. Let R be the set of ribbons which contain at least one edge of W. Since u and v are distinct and ρ is injective, we have

$$0 \neq \rho(u_m) - \rho(u_0) = \sum_{i=1}^{m} (\rho(u_i) - \rho(u_{i-1})) = \sum_{r \in R} \sum_{(w_1, w_2) \in r \cap W} (\rho(w_2) - \rho(w_1)) .$$

All ribbons r such that $|r \cap W|$ is even have a zero contribution by Theorem 3.7. Hence, there must be a ribbon r' such that $|r' \cap W|$ is odd. The ribbon r' separates u and v by Theorem 3.2. □

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The following definition is a slight generalization of rhombic carpets used in [8] where we allow parallelograms instead of rhombi.

Definition 3.10. Let S be a finite set of arbitrary parallelograms in $\mathbb{R}^2$ (including interiors) such that:

- if a point belongs to two parallelograms, then it is either a vertex of both, or an interior point of an edge of both (in particular, if a point belongs to more than two parallelograms, then it is a vertex),
- the boundary of the union $\bigcup S$ is a simple polygon.

The framework obtained by taking the 1-skeleton of S together with the vertex positions is called a *carpet framework* (see Figure 2 for an example).

Every carpet framework is a P-framework, see Theorem B.6. In order to prove this we introduce a class of graphs $\mathcal{G}_{\text{rec}}$, which contains the underlying graphs of carpet frameworks. The definition is done recursively by adding vertices in a way that a parallelogram placement can be extended. In order to streamline the paper, these discussions can be seen in Appendix B.

3.2 Bracings

A general P-framework is flexible with many degrees of freedom. By adding edges to the graph we can reduce this number. In particular we are interested in adding diagonal edges of 4-cycles. This process is called the bracing of the graph or framework.

Definition 3.11. A *braced ribbon-cutting graph* is a graph $G = (V_G, E_c \cup E_d)$ where E_c and E_d are two non-empty disjoint sets such that the graph (V_G, E_c) is a ribbon-cutting graph and the edges in E_d correspond to diagonals of some 4-cycles of (V_G, E_c). These diagonals are also called *braces*. If r is a ribbon of (V_G, E_c), then

$$r \cup \{u_1u_3 \in E_d \colon \exists \text{ 4-cycle } (u_1, u_2, u_3, u_4) \text{ of } (V_G, E_c) \text{ s.t. } u_1u_2, u_3u_4 \in r\}$$

is a *ribbon* of the braced ribbon-cutting graph G.

The framework (G, ρ) is called *braced P-framework* if G is a braced ribbon-cutting graph and ρ is a parallelogram placement for (V_G, E_c). Figure 8 shows an example.

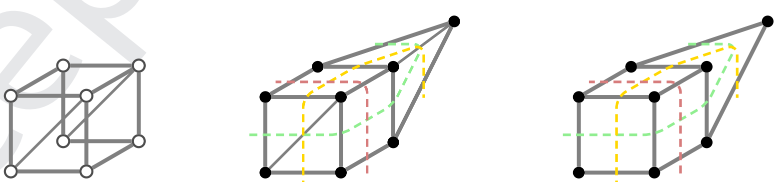

Figure 8: An example of a braced P-framework (left) with the underlying ribbon-cutting graph (right) and its bracing (middle).

Remark 3.12. A ribbon of a braced ribbon-cutting graph $(V, E_c \cup E_d)$ is an edge cut since the corresponding ribbon of (V, E_c) is an edge cut.

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We construct a new graph, which encodes the relations between the ribbons, i.e., we ask whether they share 4-cycles. A subgraph of this graph indicates whether some of the shared 4-cycles is braced.

Definition 3.13. Let G be a braced ribbon-cutting graph. The *ribbon graph* Γ of G is the graph with the set of vertices being the set of ribbons of G and two ribbons r_1, r_2 are adjacent if and only if there is a 4-cycle (u_1, u_2, u_3, u_4) in the underlying unbraced graph of G such that $u_1u_2, u_3u_4 \in r_1$ and $u_1u_4, u_2u_3 \in r_2$. The subgraph (V_Γ, E_b) of Γ, where

$$E_b = \{r_1r_2 \in E_\Gamma : r_1 \cap r_2 \text{ is a subset of braces of } G\} ,$$

is called the *bracing (sub)graph*. See Figure 9 for an example of these definitions.

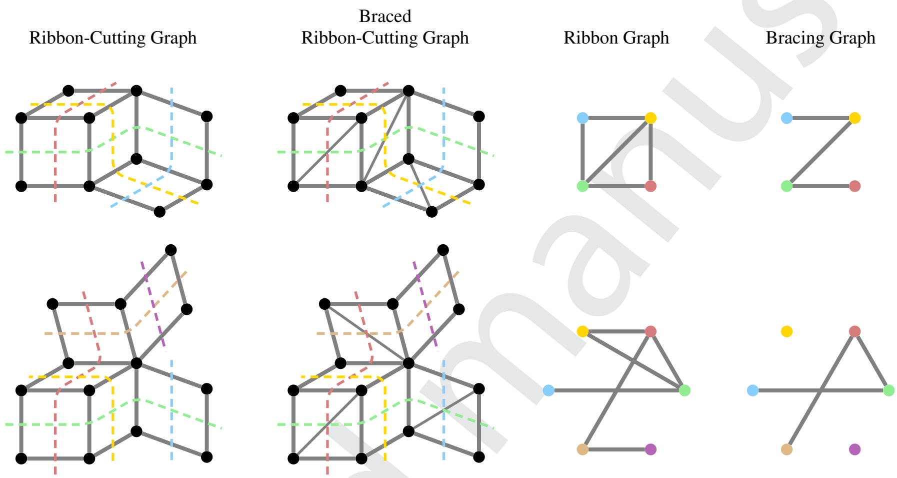

Figure 9: Two ribbon-cutting graphs with an example of a bracing as well as the corresponding ribbon graph and bracing graph. The vertices in the ribbon graph and the bracing graph are colored in correspondence with the indicated ribbons.

We remark that the bracing subgraph according to the definition in [8] does not contain the ribbons which have no brace. In our definition these ribbons are isolated vertices.

There are no loops in ribbon and bracing graphs if all ribbons of the underlying unbraced ribbon-cutting graph are simple. An edge in a bracing graph does not determine uniquely a braced 4-cycle (see the yellow and green ribbon in Figure 8).

Now we have all definitions to recall the main theorem of [8]. In the next section we extend this theorem to P-frameworks and also prove the other direction.

Theorem 3.14 ([8]). *Let (G, ρ) be a braced carpet framework. If the bracing graph of G is connected, then (G, ρ) is rigid.*

4 Flexibility of braced P-frameworks

In this section we determine when a bracing makes the framework rigid and in which cases it remains flexible. We use cartesian NAC-colorings for that. The theory is therefore based on [11]. Indeed, we show that a P-framework is flexible if and only if it has a cartesian NAC-coloring. This finally leads to a proof of the main theorem.

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Cartesian NAC-colorings of a subclass of ribbon-cutting graphs can be characterized using ribbons.

Lemma 4.1. *Let G be a braced ribbon-cutting graph such that every two vertices are separated by a ribbon. A NAC-coloring of G is cartesian if and only if each ribbon of G is monochromatic.*

Proof. Let δ be a NAC-coloring. If δ is cartesian, then all 4-cycles are either monochromatic or opposite edges have the same color. Since the edges of a braced 4-cycle have the same color, ribbons are monochromatic. On the other hand, if the ribbons are monochromatic, then two vertices cannot be connected by a blue and red path simultaneously since they are separated by a ribbon. □

Theorem 4.2. *If a braced P-framework (G, ρ) is flexible, then G has a cartesian NAC-coloring.*

Proof. A NAC-coloring for G can be constructed as in the proof of [11, Theorem 3.1]. The zero set of the following system of equations for coordinates (x_u, y_u) for $u \in V_G$ describes all placements of G inducing the same edge lengths as ρ:

$$(x_u - x_v)^2 + (y_u - y_v)^2 = \|\rho(u) - \rho(v)\|^2 \quad \text{for all } uv \in E_G\,. \tag{4.1}$$

In order to remove rigid motions, we fix the position of an edge $\bar{u}\bar{v}$ by setting

$$x_{\bar{u}} = 0\,, \quad y_{\bar{u}} = 0\,, \quad x_{\bar{v}} = \|\rho(\bar{u}) - \rho(\bar{v})\|\,, \quad y_{\bar{v}} = 0\,. \tag{4.2}$$

We also impose that each 4-cycle (u_1, u_2, u_3, u_4) in G is a parallelogram:

$$\begin{aligned} x_{u_2} - x_{u_1} &= x_{u_3} - x_{u_4}\,, \quad & x_{u_4} - x_{u_1} &= x_{u_3} - x_{u_2}\,, \\ y_{u_2} - y_{u_1} &= y_{u_3} - y_{u_4}\,, \quad & y_{u_4} - y_{u_1} &= y_{u_3} - y_{u_2}\,. \end{aligned} \tag{4.3}$$

The existence of a flex of (G, ρ) implies that there are infinitely many placements in the zero set of the system consisting of Equations (4.1) to (4.3). Hence, there is an irreducible algebraic curve C in the zero set. For every $u, v \in V_G$ such that $uv \in E_G$, we define $W_{u,v}$ in the complex function field of C by

$$W_{u,v} = (x_v - x_u) + i(y_v - y_u)\,.$$

There exists a valuation ν of the function field of C yielding a NAC-coloring δ of G by taking $\delta(uv) = \text{red}$ if $\nu(W_{u,v}) > 0$ and $\delta(uv) = \text{blue}$ otherwise, see [11, Theorem 3.1] for the details. Since $W_{u_1,u_2} = W_{u_4,u_3}$ for the opposite edges u_1u_2 and u_3u_4 of a 4-cycle (u_1, u_2, u_3, u_4), we have that $\delta(u_1u_2) = \delta(u_3u_4)$. Therefore, ribbons are monochromatic since a 4-cycle with a diagonal is monochromatic. The NAC-coloring δ is cartesian by Theorem 4.1 since every two vertices are separated by a ribbon by Theorem 3.9 applied to the underlying unbraced P-framework and Theorem 3.12. □

Lemma 4.3. *Let (G, ρ) be a P-framework. Let $u, v \in V_G$ and W, W' be walks from u to v in G. If G has a cartesian NAC-coloring δ and $c \in \{red, blue\}$, then*

$$\sum_{\substack{(w_1,w_2)\in W \\ \delta(w_1w_2)=c}} (\rho(w_2) - \rho(w_1)) = \sum_{\substack{(w_1,w_2)\in W' \\ \delta(w_1w_2)=c}} (\rho(w_2) - \rho(w_1))\,.$$

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Proof. Let $\widehat{W}$ be the walk obtained by concatenating W and the inverse of W'. We consider the sum

$$\sum_{\substack{(w_1,w_2)\in\widehat{W}\\ \delta(w_1w_2)=c}} \left(\rho(w_2)-\rho(w_1)\right).$$

Since each ribbon r is monochromatic in a cartesian NAC-coloring and $\widehat{W}$ is closed, the number of edges in $r\cap\widehat{W}$ included in the sum is even by Theorem 3.2 (the ribbons are simple by Theorem 3.5). Hence, the sum is zero by Theorem 3.7. □

Using the lemma we show the reverse direction of Theorem 4.2. The proof is constructive, i.e., it provides a flex. In order to do so, we adapt the "zigzag" grid construction from [11]. A "zigzag" grid is determined by a column of points, whose copies are translated to other positions. Such a grid can flex so that the distances among all vertices in the same column/row remain constant. The idea of the construction of a flexible framework for a given graph with a NAC-coloring is to place the vertices of the graph to the "zigzag" grid using the NAC-coloring so that every blue, resp. red, component is in one column, resp. row of the grid, see Figure 10. Since there are no diagonals, the framework is flexible.

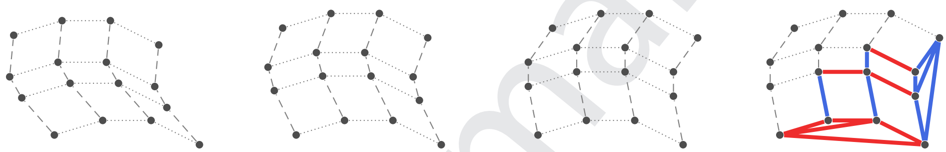

Figure 10: A flex of a "zigzag" grid and an example of a braced ribbon-cutting graph with a NAC-coloring placed to the grid so that there are no diagonals.

In case of P-frameworks (G,ρ), the grid can be chosen so that the flex starts at ρ if the graph allows a cartesian NAC-coloring, see Figure 11. We formalize these observations in the following theorem.

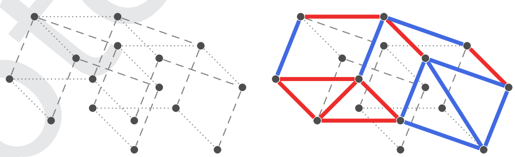

Figure 11: Any parallelogram placement of a braced ribbon cutting graph with a cartesian NAC-coloring can be obtained via some "zigzag" grid.

Theorem 4.4. *If a braced P-framework has a cartesian NAC-coloring, then it is flexible.*

Proof. Let (G',ρ) be a braced P-framework and δ' be a cartesian NAC-coloring of G'. We can assume that $\rho(\bar{u}) = (0,0)$ for a fixed vertex $\bar{u} \in V_{G'}$. Let G be the graph G' with braces removed and δ be the NAC-coloring of G obtained by restricting δ'.

Let $R_1,\dots,R_m$, resp. $B_1,\dots,B_n$, be the vertex sets of the connected components of G^δ_{red}, resp. G^δ_{blue}. We define a map $\rho_{\text{red}} : \{R_1,\dots,R_m\} \to \mathbb{R}^2$ as follows: for R_i, let W

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be any walk in G from $\bar{u}$ to a vertex of G in R_i and

$$\rho_{\text{red}}(R_i) = \sum_{\substack{(w_1,w_2)\in W \\ \delta(w_1w_2)=\text{blue}}} (\rho(w_2) - \rho(w_1)) \,.$$

Theorem 4.3 guarantees that it is well-defined, namely, the sum is independent of the choice of W and the vertex in R_i. We define $\rho_{\text{blue}} : \{B_1, \dots, B_n\} \to \mathbb{R}^2$ analogously by swapping red and blue.

For $t \in [0, 2\pi]$ and $v \in V_G = V_{G'}$, where $v \in R_i \cap B_j$, let

$$\rho_t(v) = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix} \cdot \rho_{\text{red}}(R_i) + \rho_{\text{blue}}(B_j) \,.$$

If W is a walk in G from $\bar{u}$ to v, then

$$\begin{aligned} \rho_0(v) &= \rho_{\text{red}}(R_i) + \rho_{\text{blue}}(B_j) \\ &= \sum_{\substack{(w_1,w_2)\in W \\ \delta(w_1w_2)=\text{blue}}} (\rho(w_2) - \rho(w_1)) + \sum_{\substack{(w_1,w_2)\in W \\ \delta(w_1w_2)=\text{red}}} (\rho(w_2) - \rho(w_1)) \\ &= \sum_{(w_1,w_2)\in W} (\rho(w_2) - \rho(w_1)) = \rho(v) - \rho(\bar{u}) = \rho(v) \,. \end{aligned}$$

We follow the argument from [11] that the lengths of the edges in $E_{G'}$ are constant along ρ_t. Notice that the vertex sets of G^{δ}_{red}, resp. G^{δ}_{blue}, and $G'^{\delta'}_{\text{red}}$, resp. $G'^{\delta'}_{\text{blue}}$, are the same since each brace has the same color as the 4-cycle it braces. Let uv be an edge in $E_{G'}$ with $u \in R_i \cap B_j$ and $v \in R_k \cap B_\ell$. If uv is red, then $i = k$ and hence

$$\|\rho_t(v) - \rho_t(u)\| = \|\rho_{\text{blue}}(B_\ell) - \rho_{\text{blue}}(B_j)\| \,.$$

On the other hand, if uv is blue, then $j = \ell$ and

$$\begin{aligned} \|\rho_t(v) - \rho_t(u)\| &= \left\| \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix} \cdot (\rho_{\text{red}}(R_k) - \rho_{\text{red}}(R_i)) \right\| \\ &= \|\rho_{\text{red}}(R_k) - \rho_{\text{red}}(R_i)\| \,. \end{aligned}$$

Since $\rho_0 = \rho$ and ρ is injective, none of the edges has length zero. Therefore, ρ_t is a flex of (G', ρ). □

Finally, we connect the results of flexibility and NAC-colorings with the connectivity of the bracing graph.

Theorem 4.5. *Let G be a braced ribbon-cutting graph such that every two vertices are separated by a ribbon. The bracing graph of G is connected if and only if G does not have a cartesian NAC-coloring.*

Proof. Let B be the bracing graph of G. In a cartesian NAC-coloring of G, ribbons are monochromatic by Theorem 4.1. Hence, if two ribbons are adjacent in B, then the union of their edges is monochromatic. Therefore, if B is connected, all edges of G must have the same color, namely, no cartesian NAC-coloring exists.

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For the opposite implication, assume B is not connected. We color the edges of the ribbons of one connected component by red and the rest by blue. To show that this surjective edge coloring is a NAC-coloring, consider a cycle C. Let uv be an edge of C and r be the ribbon containing uv. Since r separates G, r contains another edge $u'v'$ of C. Since ribbons are monochromatic, either all edges of C have the same color or there are two edges of each color. The obtained NAC-coloring is cartesian by Theorem 4.1. □

This forms the last part of the proof of Theorem 1.1.

Proof of Theorem 1.1. Let (G, ρ) be a braced P-framework. Every two vertices are separated by a ribbon by Theorem 3.9 and Theorem 3.12. Hence, (G, ρ) is rigid if and only if G has no cartesian NAC-coloring (Theorems 4.2 and 4.4) if and only if the bracing graph of G is connected (Theorem 4.5).

Each edge of the bracing graph corresponds to at least one brace. The minimum number of braces making the framework rigid follows from the fact that the number of edges of a spanning tree of the bracing graph is one less than the number of vertices, i.e., ribbons. The result is also illustrated in Figure 12. □

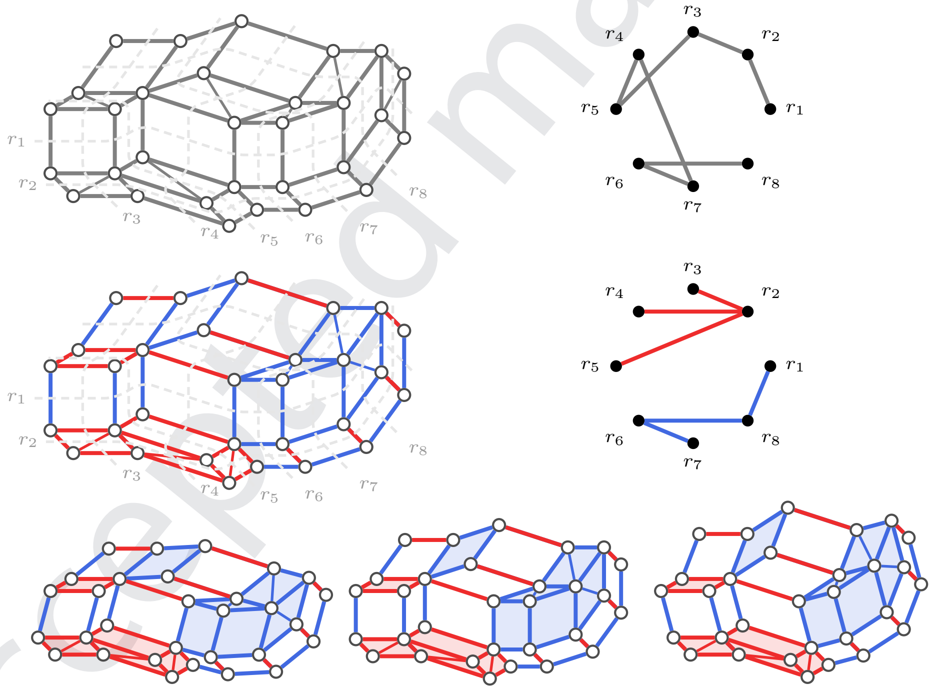

Figure 12: Two bracings of a P-framework where the first one is rigid as visible by the connectivity of the bracing graph. The second bracing yields a flexible framework since the bracing graph is not connected. We show three instances of the flex that is possible with the bracing and the unique resulting cartesian NAC-coloring thereof (shaded parallelograms preserves their shapes).

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A consequence of Theorem 1.1 is that rigidity of a braced P-framework is a combinatorial property, not a geometric one.

Corollary 4.6. *If G is a braced ribbon-cutting graph admitting a parallelogram placement, then either*

(i) (G, ρ) *is rigid for all parallelogram placements* ρ *of* G*, or*

(ii) (G, ρ) *is flexible for all parallelogram placements* ρ *of* G*.*

Conclusion

We have applied the theory of NAC-colorings to P-frameworks generalizing previous results in the area of bracing grids. In fact, we have shown that a P-framework is rigid if and only if it has no cartesian NAC-coloring if and only if the bracing graph is connected. Notice that a consequence of this statement is that a braced rectangular grid/rhombic carpet is rigid if and only if it is infinitesimally rigid. This is not the case for grids with holes as there are instances which are rigid but not infinitesimally rigid (an example can be obtained by bracing all squares besides those with the indicated ribbons in Figure 7).

Similarly as in rectangular grids, there are plenty of interesting questions for further generalizations such as graphs with holes, different types of diagonals or higher dimensions. For P-frameworks these questions are subject to further research.

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Appendix

A Cartesian products of graphs and NAC-colorings

In this section, we show a connection of cartesian NAC-colorings with cartesian products of graphs. Recall that the cartesian product of graphs G and H is given by

$$G \,\square\, H = (V_G \times V_H, \{(u,u')(v,v') \colon (u = v \wedge u'v' \in E_H) \vee (u' = v' \wedge uv \in E_G)\}) .$$

By coloring edges coming from G by red and the rest blue, the following holds.

Theorem A.1 ([1, 15])**.** *The cartesian product of any two nontrivial graphs G and H has a cartesian NAC-coloring.*

We remark that the statement of Theorem A.1 has been pointed out by [15] independently of [1]. In [1] a cartesian NAC-coloring is called *good* since applying the grid construction described in [11] yields a proper flexible framework, whereas for a non-cartesian NAC-coloring there are overlapping vertices. Our naming is motivated by the fact that the converse statement can be proved using ideas from [16] about embeddings of graphs into cartesian products.

Theorem A.2. *If a graph G has a cartesian NAC-coloring, then there are graphs Q_1, Q_2 with at least two vertices each and an injective graph morphism $h : G \to Q_1 \,\square\, Q_2$ such that each vertex in $V_{Q_1} \cup V_{Q_2}$ occurs as a coordinate of a vertex in $h(G)$. In particular, G can be viewed as a subgraph of $Q_1 \,\square\, Q_2$.*

Proof. Let δ be a cartesian NAC-coloring of G. Let $R_1, \ldots, R_m$, resp. $B_1, \ldots, B_n$, be the vertex sets of the connected components of G^{δ}_{red}, resp. G^{δ}_{blue}. Since δ is surjective and no blue edge can connect vertices of the same red component [11, Lemma 2.4], $m \geq 2$ and $n \geq 2$. Let $\pi_{\text{red}} : V_G \to \{R_1, \ldots, R_m\}$ and $\pi_{\text{blue}} : V_G \to \{B_1, \ldots, B_m\}$ map a vertex to the vertex set of its red, resp. blue, component, namely, $\pi_{\text{red}}(v) = R_i$ and $\pi_{\text{blue}}(v) = B_j$ if $v \in R_i \cap B_j$. We define the following quotient graphs

$$Q_1 = (\{R_1, \ldots, R_m\}, \{\pi_{\text{red}}(u)\pi_{\text{red}}(v) \colon uv \in E_G \text{ and } \delta(uv) = \text{blue}\}) ,$$
$$Q_2 = (\{B_1, \ldots, B_n\}, \{\pi_{\text{blue}}(u)\pi_{\text{blue}}(v) \colon uv \in E_G \text{ and } \delta(uv) = \text{red}\}) .$$

Let Q be the cartesian product of Q_1 and Q_2, and $h : V_G \to V_Q$ be the graph morphism given by

$$h(v) = (\pi_{\text{red}}(v), \pi_{\text{blue}}(v)) .$$

We check that it is indeed a morphism: if uv is an edge of G, w.l.o.g. red, then $\pi_{\text{red}}(u) = \pi_{\text{red}}(v)$ and $\pi_{\text{blue}}(u) \neq \pi_{\text{blue}}(v)$ from the properties of NAC-colorings. Thus, $h(u)h(v) = (\pi_{\text{red}}(u), \pi_{\text{blue}}(u))(\pi_{\text{red}}(u), \pi_{\text{blue}}(v))$ which is an edge of Q. The morphism h is injective by Theorem 2.7 since δ is cartesian. Each vertex in $V_{Q_1} \cup V_{Q_2}$ occurs as a coordinate of a vertex in $h(G)$, since π_{red} and π_{blue} are surjective. □

B Carpet frameworks are P-frameworks

As indicated in Section 3.1, we prove here that a carpet framework is P-framework. For this, we define the following class of graphs.

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Definition B.1. We define the class of graphs $\mathcal{G}_{\text{rec}}$ recursively. The 4-cycle graph is in $\mathcal{G}_{\text{rec}}$. There are two types of construction (see also Figure 13):

ADD4-CYCLE: If $G \in \mathcal{G}_{\text{rec}}$ with $uv \in E_G$, then the graph $(V_G \cup \{w_1, w_2\}, E_G \cup \{uw_1, w_1w_2, w_2v\})$ is in $\mathcal{G}_{\text{rec}}$, where $w_1, w_2 \notin V_G$.

CLOSE4-CYCLE: If $G \in \mathcal{G}_{\text{rec}}$ with $uv, vw \in E_G$ and the vertex v is separated from any vertex in $V_G \setminus \{u, v, w\}$ by a ribbon which does not contain uv or vw, then the graph $(V_G \cup \{w'\}, E_G \cup \{uw', w'w\})$, where $w' \notin V_G$, is in $\mathcal{G}_{\text{rec}}$.
Note that the separation assumption is needed for avoiding situations as described in Figure 6.

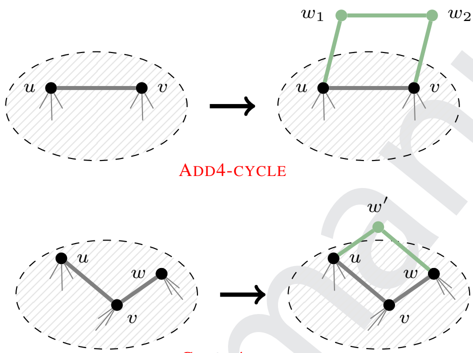

Figure 13: Two recursive $\mathcal{G}_{\text{rec}}$ constructions.

Figure 5b gives an example of a graph in $\mathcal{G}_{\text{rec}}$ that is not the underlying graph of a carpet framework. It is easy to use the construction to show that the class has the ribbon-cutting property.

Proposition B.2. *Every graph in $\mathcal{G}_{rec}$ is ribbon-cutting.*

Proof. By structural induction: the 4-cycle graph is ribbon-cutting. ADD4-CYCLE preserves the property since $\{uw_1, w_2v\}$ is a new ribbon and w_1w_2 belongs to the ribbon of uv. CLOSE4-CYCLE does so as well: the edges uw' and $w'w$ belong to the ribbons of vw and uv respectively. If any ribbon of the extended graph were not an edge cut, than it would not be an edge cut in the original graph. Notice that the separation assumption is not needed for this. □

Recall that for a P-framework (G, ρ), any ribbon r is simple by Theorem 3.5 and $G \setminus r$ has two connected components by Theorem 3.2. This allows us to translate the vertices of one of the components by a constant vector.

Remark B.3. Let (G, ρ) be a P-framework and r be a ribbon of G. Let V_1 and V_2 be the vertex sets of the two connected components of $G \setminus r$. For every vector $t \in \mathbb{R}^2 \setminus \{\rho(u_1) - \rho(u_2) : u_1 \in V_1, u_2 \in V_2\}$, the placement ρ' of G given by $\rho'(v) = \rho(v) + t$ if $v \in V_2$ and $\rho'(v) = \rho(v)$ otherwise is a parallelogram placement.

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We are going to show the relation between P-frameworks, carpet frameworks and the graphs in $\mathcal{G}_{\text{rec}}$. Namely, the underlying graphs of carpet frameworks are in $\mathcal{G}_{\text{rec}}$, which is in turn a subset of the underlying graphs of P-frameworks. For this we need an equivalent condition to the separation assumption in CLOSE4-CYCLE.

Lemma B.4. *For a P-framework (G, ρ) and $uv, vw \in E_G$, the following are equivalent:*

(i) The vertex v is separated from any vertex in $V_G \setminus \{u, v, w\}$ by a ribbon which does not contain uv or vw.

(ii) There exists a parallelogram placement ρ' of the graph $G' = (V_G \cup \{w'\}, E_G \cup \{uw', w'w\})$, where $w' \notin V_G$.

Proof. $(i) \implies (ii)$ If we want to extend ρ to a parallelogram placement of G', the position $\rho(w')$ of the new vertex w' is uniquely determined by the requirement that $(\rho(u), \rho(v), \rho(w), \rho(w')$ is a parallelogram. We can assume that $\rho(u), \rho(v), \rho(w)$ are not collinear, hence, $\rho(v) \neq \rho(w')$. If it is not so, we replace ρ by a parallelogram placement obtained by Theorem B.3 for the ribbon of uv and a non-zero translation.

If $\rho : V_{G'} \to \mathbb{R}^2$ is injective, we are done. Otherwise, $\rho(w') = \rho(u')$ for a unique vertex $u' \in V_G \setminus \{u, v, w\}$. By assumption, there is a ribbon r separating v from u such that $uv, vw \notin r$. Thus, u, v, w are in the same connected component of $G \setminus r$, whereas u' is in the other one. Using Theorem B.3, there is a parallelogram placement ρ' of G such that $\rho(w') \neq \rho'(u')$. Moreover, the translation vector can be chosen so that the whole image $\rho'(V_G)$ avoids $\rho(w')$. Therefore, ρ' uniquely extends to a parallelogram placement of G' by setting $\rho'(w') = \rho(w')$.

$\neg(i) \implies \neg(ii)$ Assume that $u' \in V_G \setminus \{u, v, w\}$ is a vertex such that it is separated from v only by the ribbon of uv or vw. Let $W = (v = u_0, u_1, \ldots, u_m = u')$ be a walk from v to u'. Let R be the set of ribbons which contains at least one edge of W. All ribbons are simple by Theorem 3.5. By the assumption and Theorem 3.2, $|r \cap W|$ is even for every ribbon r avoiding uv and vw. For any parallelogram placement ρ of G, we have

$$\begin{aligned}\rho(u') - \rho(v) &= \sum_{i=1}^{m} (\rho(u_i) - \rho(u_{i-1})) = \sum_{r \in R} \sum_{(w_1, w_2) \in r \cap W} (\rho(w_2) - \rho(w_1)) \\ &\overset{3.7}{=} \sum_{\substack{r \in R \\ uv \in r \vee vw \in r}} \sum_{(w_1, w_2) \in r \cap W} (\rho(w_2) - \rho(w_1)) \\ &\overset{3.6}{=} \alpha(\rho(w) - \rho(v)) + \beta(\rho(u) - \rho(v)),\end{aligned}$$

where $\alpha, \beta \in \{0, 1\}$. Actually, $\alpha = \beta = 1$, otherwise $\rho(u') = \rho(w)$ or $\rho(u') = \rho(u)$, which violates injectivity. Hence, $\rho(u') = \rho(w) + \rho(u) - \rho(v)$. Assume for contradiction that there is a parallelogram placement ρ' of G'. Since $\rho'|_{V_G}$ is a parallelogram placement of G, we have by the previous $\rho'(u') = \rho'(w) + \rho'(u) - \rho'(v)$. But this is a contradiction since $\rho'(w') = \rho'(w) + \rho'(u) - \rho'(v)$ as well and $w' \neq u'$. □

Corollary B.5. *There exists a P-framework (G, ρ) for every $G \in \mathcal{G}_{rec}$.*

Proof. We proceed by structural induction. The 4-cycle can be placed as a parallelogram. For a graph G' constructed using ADD4-CYCLE from G, a parallelogram placement of G can be extended to a parallelogram placement of G' by placing the two new vertices to

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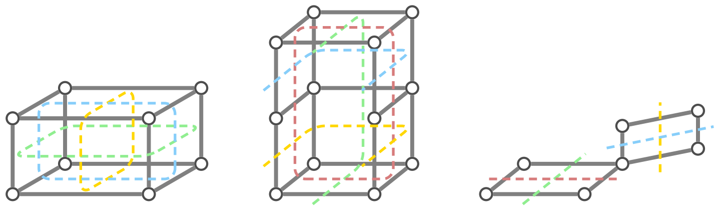

Figure 14: P-frameworks whose underlying graphs are not in $\mathcal{G}_{\text{rec}}$.

form a parallelogram so that the placement is injective. If G' is constructed from G by CLOSE4-CYCLE, then there exists a parallelogram placement of G' by Theorem B.4. □

Note that there are P-frameworks whose underlying graphs are not in $\mathcal{G}_{\text{rec}}$, see Figure 14.

Corollary B.6. *If (G,ρ) is a carpet framework, then $G \in \mathcal{G}_{rec}$. In particular, (G,ρ) is a P-framework.*

Proof. By the definition of carpet framework, ρ is a parallelogram placement. Once we show that $G \in \mathcal{G}_{\text{rec}}$, the fact that (G,ρ) is a P-framework follows from Theorem B.2.

We proceed by induction on the number of parallelograms yielding a carpet framework. Let S be the set of parallelograms in $\mathbb{R}^2$ giving a carpet framework (G',ρ') according to Theorem 3.10. If $|S| = 1$, then (G',ρ') is the 4-cycle with a parallelogram placement, hence, $G' \in \mathcal{G}_{\text{rec}}$. Suppose that $|S| \geq 2$. The boundary of $\bigcup S$ is a simple polygon M with k edges. We divide the parallelograms having an edge in the polygon M into the following categories (see Figure 15):

- K_1 — parallelograms with one edge in M,
- K_2 — parallelograms with two incident edges in M such that the vertex that is not in these two edges is not in M,
- K_2' — parallelograms with two incident edges in M that are not in K_2,
- K_2'' — parallelograms with two opposite edges in M,
- K_3 — parallelograms with three edges in M.

Clearly, $k = |K_1| + 2|K_2| + 2|K_2'| + 2|K_2''| + 3|K_3|$. The sum of the interior angles of the simple polygon M equals $(k-2)\pi$. Considering contributions to the sum for parallelograms in the categories above (see Figure 15), we have

$$|K_1|\pi + |K_2|\pi + 2|K_2'|\pi + 2|K_2''|\pi + 2|K_3|\pi \leq (k-2)\pi$$
$$\iff \quad 2 \leq |K_2| + |K_3|\,.$$

For a parallelogram s in $K_2 \cup K_3$, $S \setminus s$ satisfies the assumptions of Theorem 3.10. Thus, we have a carpet framework (G,ρ) and G is in $\mathcal{G}_{\text{rec}}$ by induction assumption. If $s \in K_3$, then G can be extended to G' by ADD4-CYCLE. If $s \in K_2$, then G can be extended to G' by CLOSE4-CYCLE, since the separation assumption is satisfied by Theorem B.4 and the placement ρ'. □

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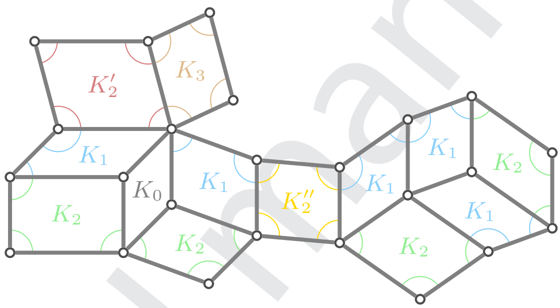

Figure 15: An example of dividing parallelograms into categories according to their intersection with the boundary. The angles whose contribution to the sum of the interior angles is considered are indicated. Note that the parallelogram labeled K_0 belongs to S but is not part of the boundary.

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