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On avoiding 1233

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Abstract

In this paper, we establish a recurrence relation for finding the generating function for the number of k-ary words of length n that avoid 1233 for arbitrary k. Comparable generating function formulas may also be found counting words where a single permutation pattern of length three is avoided in addition to 1233.

Keywords: k-ary words, Kernel method, Avoiding 1233*.*

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1 Introduction

We denote the set of all words of length n over the alphabet $[k] = \{1, \ldots, k\}$ by $[k]^n$ and refer to members of $[k]^n$ as k-ary words. Let $\pi = \pi_1 \cdots \pi_n \in [k]^n$ and $\tau = \tau_1 \cdots \tau_m \in [\ell]^m$ such that each letter from $[\ell]$ appears at least once in τ (possibly with repetitions). We say that π *contains* τ if there exist indices $1 \le i_1 < \cdots < i_m \le n$ such that $\pi_{i_a} \Phi \pi_{i_b}$ if and only if $\tau_a \Phi \tau_b$, for any relation $\Phi \in \{<, =, >\}$ and $a, b \in [m]$. In this context, the word τ is called a *pattern*, and it is said that π *avoids* τ if π fails to contain τ per the preceding definition.

The area of permutation pattern avoidance has received considerable attention in recent decades; see, e.g., [13] and references therein. Alon and Friedgut [2] extended this study to avoidance on k-ary words in obtaining an upper bound on the number of permutations of length n that avoid a given pattern. The question of pattern avoidance on permutations was initiated by Knuth [6], who found that the number of permutations of length n that avoid the pattern τ for any $\tau \in S_3$ is given by the n-th Catalan number $\frac{1}{n+1}\binom{2n}{n}$. Later, Simion and Schmidt [12] extended this result by determining the number of permutations

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of length n that avoid all patterns in any subset T of S_3. Comparable results involving k-ary words were found by Burstein [4] and Albert et al. [1], and later by Burstein and Mansour [5], who allowed patterns to contain repeated letters. See also [11, 14] concerning the avoidance of 123 by words as well as [9] for general enumeration schemes for words avoiding a permutation pattern.

Concerning avoidance of patterns of length four by k-ary words, only the following more general results are known:

- Regev [10] showed that the number of k-ary words of length n that avoid $12\cdots(\ell+1)$ is asymptotic to
$$\frac{n^{\ell(k-\ell)}\ell^n}{\ell^{\ell(k-\ell)}\prod_{i=1}^{\ell}\prod_{j=1}^{k-\ell}(i+j-1)}.$$
This result was re-derived by Brändén and Mansour [3].
- The patterns $11\cdots1$ and $11\cdots121\cdots11$ have been considered in [5].
- Brändén and Mansour [3] (see also [8]) suggested an automaton for the enumeration of k-ary words of length n that avoid a fixed pattern for a given k.

We remark that it is a challenging problem in general to enumerate the k-ary words of length n that avoid a given pattern where k is arbitrary. Even in the case of a pattern of length four, the task at hand is still not a simple one. Here, we consider the problem of enumerating the members of $[k]^n$ that avoid 1233 for arbitrary k. The main purpose of this paper is to provide a recurrence relation on k for finding the number k-ary words of length n that avoid 1233, see Theorem 2.2 below. This recurrence represents an improvement in the case of 1233 over the general procedure described in [3, 8], which was derived using automata theory. Some further results are found involving avoidance of 1233 and a single pattern of length three.

2 k-ary words that avoid 1233

Let $a_{n,k}$ denote the number of k-ary words π of length n that avoid 1233. In order to write recurrences, we must refine $a_{n,k}$ according to the prefix of a word π. Given $s \geq 1$ and $i_1, \ldots, i_s \in [k]$, let $a_{n,k}(i_1, \ldots, i_s)$ denote the number of 1233-avoiding k-ary words π of length n having the form $\pi = i_1 \cdots i_s \pi'$, where π' is possibly empty. Clearly, we have $a_{n,1} = 1$ and $a_{n,2} = 2^n$. Henceforth, we may assume $k \geq 3$. By the definitions, for all $1 \leq i \leq k-2$,
$$a_{n,k}(i) = a_{n,k}(i,k) + \sum_{j=1}^{i} a_{n,k}(i,j) + \sum_{j=i+1}^{k-1} a_{n,k}(i,j).$$
Note that $a_{n,k}(i,k) = a_{n-1,k}(i)$ and $a_{n,k}(i,j) = a_{n-1,k}(j)$ for all $1 \leq j \leq i \leq k-2$. Thus,
$$a_{n,k}(i) = a_{n-1,k}(i) + \sum_{j=1}^{i} a_{n-1,k}(j) + \sum_{j=i+1}^{k-1} a_{n,k}(i,j).$$

Next observe that a k-ary word of the form $\pi = ij\pi'$ with $1 \leq i < j \leq k-1$ must have any letters from $[j+1,k] = \{j+1, j+2, \ldots, k\}$ distinct in order to avoid 1233. If we assume that exactly ℓ letters of π belong to $[j+1,k]$, then there are $\binom{k-j}{\ell}$ choices for these letters, $\binom{n-2}{\ell}$ choices for their positions within π and $\ell!$ ways in which to order these letters within their positions. Thus,

$$a_{n,k}(i,j)=\sum_{\ell=0}^{k-j}\ell!\binom{n-2}{\ell}\binom{k-j}{\ell}a_{n-1-\ell,j}(i).$$

Hence, for all $1\leq i\leq k-2$,

$$a_{n,k}(i)=a_{n-1,k}(i)+\sum_{j=1}^{i}a_{n-1,k}(j)+\sum_{j=i+1}^{k-1}\sum_{\ell=0}^{k-j}\ell!\binom{n-2}{\ell}\binom{k-j}{\ell}a_{n-1-\ell,j}(i), \tag{2.1}$$

with $a_{n,k}(k)=a_{n,k}(k-1)=a_{n-1,k}$.

In order to study the sequence determined by recurrence (2.1), we define the distribution polynomial

$$A_{n,k}(v)=\sum_{i=1}^{k}a_{n,k}(i)v^{i-1},\qquad n,k\geq 1,$$

with $A_{0,k}(v)=1$ and the generating function

$$A_k(x,v)=\sum_{n\geq 0}A_{n,k}(v)x^n,\qquad k\geq 1.$$

Multiplying both sides of (2.1) by v^{i-1}, and summing over $i=1,2,\ldots,k-2$, yields for $n,k\geq 3$ the recurrence

$$\begin{aligned}A_{n,k}(v)&-a_{n-1,k}(v^{k-1}+v^{k-2})\\&=A_{n-1,k}(v)-a_{n-2,k}(v^{k-1}+v^{k-2})\\&\quad+\sum_{i=1}^{k-1}a_{n-1,k}(i)\frac{v^{i-1}-v^{k-2}}{1-v}\\&\quad+\sum_{j=2}^{k-1}\sum_{\ell=0}^{k-j}\ell!\binom{n-2}{\ell}\binom{k-j}{\ell}(A_{n-1-\ell,j}(v)-a_{n-2-\ell,j}v^{j-1}),\end{aligned}$$

which, by $a_{n,k}=A_{n,k}(1)$, leads to

$$\begin{aligned}A_{n,k}(v)=&\frac{1}{1-v}(A_{n-1,k}(v)-v^kA_{n-1,k}(1))\\&+\sum_{j=2}^{k}\sum_{\ell=0}^{k-j}\ell!\binom{n-2}{\ell}\binom{k-j}{\ell}(A_{n-1-\ell,j}(v)-v^{j-1}A_{n-2-\ell,j}(1)),\end{aligned}$$

with $A_{1,k}(v)=\alpha_k(v)=\sum_{i=1}^{k}v^{i-1}$ and $A_{2,k}(v)=k\alpha_k(v)$.

Multiplying both sides of the last equation by x^n, and summing over $n \geq 3$, we obtain

$$\begin{aligned} A_k(x,v) &= 1+\alpha_k(v)x(1+kx) \\ &+ \frac{x}{1-v}(A_k(x,v)-\alpha_k(v)x-1-v^kA_k(x,1)+v^k+kv^kx) \\ &+ x(A_k(x,v)-1-\alpha_k(v)x)-v^{k-1}x^2(A_k(x,1)-1) \\ &+ \sum_{j=2}^{k-1}\left(x(A_j(x,v)-1-\alpha_j(v)x)-v^{j-1}x^2(A_j(x,1)-1)\right) \\ &+ \sum_{j=2}^{k-1}\sum_{\ell=1}^{k-j}x^{\ell+2}\binom{k-j}{\ell}\frac{\partial^\ell}{\partial x^\ell}\left(x^{\ell-1}A_j(x,v)-v^{j-1}x^\ell A_j(x,1)\right). \end{aligned} \tag{2.2}$$

Example 2.1 (Case $k=3$)**.** Note that $A_1(x,v)=1+\frac{x}{1-x}$ and $A_2(x,v)=1+\frac{x(1+v)}{1-2x}$, by the definitions. For $k=3$, we have

$$\begin{aligned} &A_3(x,v)=1+(1+v+v^2)x(1+3x) \\ &+\frac{x}{1-v}(A_3(x,v)-(1+v+v^2)x-1-v^3A_3(x,1)+v^3+3v^3x) \\ &+x(A_3(x,v)-1-(1+v+v^2)x)-v^2x^2(A_3(x,1)-1) \\ &+x(A_2(x,v)-1-(1+v)x)-vx^2(A_2(x,1)-1)+x^3\frac{\partial}{\partial x}\left(A_2(x,v)-vxA_2(x,1)\right). \end{aligned}$$

To solve this functional equation, we make use of the kernel method and take $v=\frac{1-2x}{1-x}$ to obtain

$$\begin{aligned} &1+(1+v+v^2)x(1+3x)+\frac{x}{1-v}(-(1+v+v^2)x-1-v^3A_3(x,1)+v^3+3v^3x) \\ &+x(-1-(1+v+v^2)x)-v^2x^2(A_3(x,1)-1) \\ &+x(A_2(x,v)-1-(1+v)x)-vx^2(A_2(x,1)-1) \\ &+x^3\frac{\partial}{\partial x}\left(A_2(x,v)-vxA_2(x,1)\right)=0. \end{aligned}$$

Hence, $A_3(x,1)=\frac{(1-x)(1-4x+5x^2)}{(1-2x)^4}$. Substituting this expression into the one above for $A_3(x,v)$ then yields

$$A_3(x,v)=\frac{(1-x)(1-4x+5x^2)(1+(v-1)(v+2)x)}{(1-2x)^4}.$$

Following Example 2.1, to solve the functional equation (2.2), we use the kernel method. Taking $v=v_0=\frac{1-2x}{1-x}$ in (2.2) yields

$$\begin{aligned} &A_k(x,1)=\frac{(1-x)^{k-3}}{(1-2x)^{k-1}}\cdot \\ &\quad\cdot\left(1-(k-1)x+\sum_{j=2}^{k-1}\sum_{\ell=0}^{k-j}x^{\ell+2}\binom{k-j}{\ell}\frac{\partial^\ell}{\partial x^\ell}(x^{\ell-1}A_j(x,v)-v^{j-1}x^\ell A_j(x,1))\mid_{v=v_0}\right). \end{aligned}$$

Substituting this expression for $A_k(x,1)$ into (2.2), and observing the identity

$$x\sum_{j=2}^{k}\alpha_{j-1}(v)=(1+kx)\alpha_k(x)-\frac{1+x\alpha_k(v)-v^k(1+kx)}{1-v},\qquad k\geq 2,$$

we obtain our main result.

Theorem 2.2. *The generating function $A_k(x,v)$ for $k\geq 3$ is given by*

$$A_k(x,v)=\frac{(1-v)(1-(k-1)x)}{1-2x-v(1-x)}-\frac{(x+v(1-x))xv^{k-1}}{1-2x-v(1-x)}A_k(x,1)$$
$$+(1-v)\sum_{j=2}^{k-1}\sum_{\ell=0}^{k-j}x^{\ell+2}\binom{k-j}{\ell}\frac{\frac{\partial^\ell}{\partial x^\ell}(x^{\ell-1}A_j(x,v)-v^{j-1}x^\ell A_j(x,1))}{1-2x-v(1-x)},$$

where $A_1(x,v)=\frac{1}{1-x}$, $A_2(x,v)=1+\frac{x(1+v)}{1-2x}$ and

$$A_k(x,1)=\frac{(1-x)^{k-3}}{(1-2x)^{k-1}}\cdot$$
$$\cdot\left(1-(k-1)x+\sum_{j=2}^{k-1}\sum_{\ell=0}^{k-j}x^{\ell+2}\binom{k-j}{\ell}\frac{\partial^\ell}{\partial x^\ell}(x^{\ell-1}A_j(x,v)-v^{j-1}x^\ell A_j(x,1))\mid_{v=v_0}\right),$$

where $v_0=\frac{1-2x}{1-x}$.

Note that Theorem 2.2 provides a recurrence formula for finding the generating function $A_k(x,1)$ (even, more generally, for finding $A_k(x,v)$). For instance, upon making use of software such as Maple or Mathematica, one can obtain from Theorem 2.2 the following explicit formulas for $k=3,4,5,6$:

$$A_3(x,1)=\frac{(1-x)(1-4x+5x^2)}{(1-2x)^4},$$
$$A_4(x,1)=\frac{1-10x+44x^2-104x^3+140x^4-100x^5+31x^6}{(1-2x)^7},$$
$$A_5(x,1)=\frac{1-15x+105x^2-435x^3+1175x^4-2129x^5+2595x^6-2041x^7+946x^8-190x^9}{(1-2x)^{10}},$$
$$A_6(x,1)=\frac{1-20x+192x^2-1136x^3+4604x^4-13380x^5+28599x^6-45154x^7+52338x^8-43320x^9+24401x^{10}-8386x^{11}+1391x^{12}}{(1-2x)^{13}}.$$

Remarks: By Theorem 2.2 and induction on k, one can show that the generating function $A_k(x,v)$ for $k\geq 2$ may be expressed in the form $P_k(x,v)/(1-2x)^{\alpha_k}$, where $\alpha_k\geq 1$ and $P_k(x,v)$ is a polynomial in x and v (and not divisible by $1-2x$). Upon taking $v=1$, it is seen that there exists a constant c_k such that the number of k-ary words of length n that avoid 1233 is asymptotic to $c_k n^{\beta_k}2^n$ for some $1\leq\beta_k<\alpha_k$, which was also shown in [8]. We conjecture that $\alpha_k=3k-5=\beta_k+1$ for all k, the fact of which is demonstrated by programming for $3\leq k\leq 15$. Note that Theorem 2.2 provides a recurrence relation for finding an explicit formula for the generating function $A_k(x,1)$ and is an improvement in the case of 1233 over the more general procedure described in [3, 8].

We close this section with some remarks concerning avoidance of the general pattern 123^m, where $m \geq 2$. Let $a_{n,k}^{(m)}$ denote the number of k-ary words of length n that avoid 123^m, with $a_{n,k}^{(m)}(i_1,\ldots,i_s)$ defined analogously as before. If $1 \leq i \leq k-2$, then

$$a_{n,k}^{(m)}(i) = a_{n-1,k}^{(m)}(i) + \sum_{j=1}^{i} a_{n-1,k}^{(m)}(j) + \sum_{j=i+1}^{k-1} a_{n,k}^{(m)}(i,j).$$

To determine a formula for $a_{n,k}^{(m)}(i,j)$, we consider enumerating a restricted class of finite functions as follows. Given $a,b,c \geq 0$, let $d_{a,b}^{(c)}$ denote the number of functions $f: [b] \rightarrow [a]$ such that $|\{x \in [b] : f(x) = i\}| \leq c$ for all $i \in [a]$. Such functions could be described as being at most c-to-1. Upon considering the number ℓ of letters in a word belonging to $[j+1,k]$, we have

$$a_{n,k}^{(m)}(i,j) = \sum_{\ell=0}^{k-j} d_{k-j,\ell}^{(m-1)} \binom{n-2}{\ell} a_{n-1-\ell,j}^{(m)}(i),$$

if $i < j < k$. Note that the $d_{k-j,\ell}^{(m-1)}$ factor accounts for the number of ways in which to select and arrange the elements of $[j+1,k]$ within ℓ preselected positions such that none of these elements occur m or more times. Hence, for all $1 \leq i \leq k-2$,

$$a_{n,k}^{(m)}(i) = a_{n-1,k}^{(m)}(i) + \sum_{j=1}^{i} a_{n-1,k}^{(m)}(j) + \sum_{j=i+1}^{k-1} \sum_{\ell=0}^{k-j} d_{k-j,\ell}^{(m-1)} \binom{n-2}{\ell} a_{n-1-\ell,j}^{(m)}(i),$$

with $a_{n,k}^{(m)}(k) = a_{n,k}^{(m)}(k-1) = a_{n-1,k}^{(m)}$.

To write a recurrence for the array $d_{a,b}^{(c)}$, consider the number j of elements in $[a]$ whose pre-image cardinality is exactly c. This implies for $a,b,c \geq 1$,

$$d_{a,b}^{(c)} = \sum_{j=0}^{t} \binom{a}{j} \binom{b}{c,\ldots,c,b-jc} d_{a-j,b-jc}^{(c-1)},$$

where $t = \min\{a, \lfloor b/c \rfloor\}$ and the c index appears exactly j times in the multinomial coefficient of order $j+1$. One may verify the initial conditions $d_{a,0}^{(c)} = 1$ for all $a,c \geq 0$ and $d_{a,b}^{(c)} = 0$ if $ac = 0$ and $b \geq 1$. Note that from the recurrence when $c = 1$, we have $d_{a,b}^{(1)} = 0$ if $b > a$, which is in agreement with the pigeonhole principle, whereas if $b \leq a$, then $d_{a,b}^{(1)} = b!\binom{a}{b} = a(a-1)\cdots(a-b+1)$, as it should. Finding a simple explicit formula for $d_{a,b}^{(c)}$ in general appears not to be an easy task. Note that by induction on c using the recurrence, one has the following multi-sum expression for $d_{a,b}^{(c)}$:

$$d_{a,b}^{(c)} = \sum_{j_c=0}^{\min\{a,\lfloor \frac{b}{c} \rfloor\}} \sum_{j_{c-1}=0}^{\min\{a-j_c,\lfloor \frac{b-cj_c}{c-1} \rfloor\}} \cdots \sum_{j_2=0}^{\min\{a-\sum_{p=3}^{c} j_p,\lfloor \frac{b-\sum_{p=3}^{c} pj_p}{2} \rfloor\}} R_{a,b}(j_2,\ldots,j_c),$$

where

$$R_{a,b}(j_2,\dots,j_c)=\frac{b!}{2!^{j_2}\cdots c!^{j_c}}\binom{a-\sum_{p=2}^{c}j_p}{b-\sum_{p=2}^{c}pj_p}\prod_{i=2}^{c}\binom{a-\sum_{p=i+1}^{c}j_p}{j_i}.$$

3 Further results

As the previous section illustrates, it is challenging in general to ascertain formulas, either explicitly or by a recurrence, for the number of k-ary words for all k that avoid a single fixed pattern of length four (or of arbitrary length). Another possible direction to pursue is that of enumerating words which avoid 1233 and a second pattern τ. Here, we present two cases when τ is of length three demonstrating that even this problem is highly non-trivial. In particular, we consider the cases when $\tau = 132$ or $\tau = 213$ and leave the remaining cases when τ is a permutation pattern of length three as exercises for the interested reader (the patterns 231 and 321 apparently requiring a lengthier analysis than the others).

3.1 Case 132

Let $A_k(x)$ denote the generating function (g.f.) for the number of k-ary words of length n that avoid $\{132, 1233\}$ for each $k \geq 1$ and define $A(x,y) = \sum_{k\geq 0} A_k(x)y^k$, where $A_0(x) = 1$. In order to find a formula for $A(x,y)$, we let $A'(x,y) = \frac{(1-y)A(x,y)-1}{y}$ and $A''(x,y) = \frac{(1-y)A(x,y)-1}{y(1-y)}$, in accordance with [7, Notation 2.2]. Note that $yA''(x,y)$ represents the restriction of the g.f. $A(x,y)$ to nonempty k-ary words, whereas $yA'(x,y)$ is the further restriction to such words that contain 1.

We wish to write an equation for $A(x,y)$. Let π be a nonempty k-ary word that avoids $\{132, 1233\}$. We represent π by $\pi = \pi^{(1)}k\cdots\pi^{(s)}k\pi^{(s+1)}$, where each $\pi^{(j)}$ is $(k-1)$-ary and $s \geq 0$. Proceeding according to [7, Proposition 2.1], we consider the cases $s = 0$, $s = 1$ and $s \geq 2$. This yields the following:

- If $s = 0$, then one has a contribution of $yA(x,y)$.
- If $s = 1$, then $\frac{xy}{1-y} + \frac{xy^2A''(x,y)}{A'(x,y)}((A'(x,y)+1)^2-1)$.
- If $s \geq 2$, then

$$\sum_{s\geq 2}\frac{x^s y}{1-y}+\sum_{s\geq 2}\sum_{d=1}^{s-1}x^s y^2\binom{s-1}{d}B'^{d-1}B''$$
$$+2\sum_{s\geq 2}\sum_{d=0}^{s-1}x^s y^2\binom{s-1}{d}B'^{d}A''(x,y)$$
$$+\sum_{s\geq 2}\sum_{d=0}^{s-1}x^s y^2\binom{s-1}{d}B'^{d}A'(x,y)A''(x,y),$$

  where $B' = \frac{(1-y)B(x,y)-1}{y}$, $B'' = \frac{(1-y)B(x,y)-1}{y(1-y)}$, and $B(x,y) = \frac{1-x}{1-x-y}$ is the g.f. for the number of k-ary words of length n that avoid 12 for all $n, k \geq 0$.

To realize the last two cases above, first note that yB'' is seen to enumerate nonempty, weakly decreasing k-ary words of length n, whereas yB' counts such words that contain

1. Observe further that various sections $\pi^{(i)}$ of π are accounted for by B' in the $s \geq 2$ case above, instead of by yB', since one must divide by y to compensate for the fact the minimum letter of one section can coincide with the maximum letter of the subsequent nonempty section. The same also applies when considering the $\pi^{(i)}$ blocks accounted for by $A'(x,y)$.

Combining all of the above contributions, and simplifying, we have that $A(x,y)$ satisfies

$$(1-y)A(x,y) = 1 + \frac{xy}{(1-x)(1-y)} - \frac{x^2y^2}{(1-x)(1-y)} + \frac{x^2y^2}{1-2x-y+xy} + \frac{x((y-1)A(x,y)+1)((y-1)A(x,y)-2y+1)(1-x-y)}{(1-y)(1-2x-y+xy)}.$$

Solving for $A(x,y)$ in the last equation, and simplifying, yields the following result.

Theorem 3.1. *The generating function for the number of k-ary words of length n that avoid both 132 and 1233 for all $n, k \geq 0$ is given by*

$$\frac{1-2x^2-y-xy-\sqrt{\frac{(1-2x-y+xy)((1-x-y-xy)^2-x(1-x)(1-y))}{(1-x)(1-y)}}}{2x(1-x-y)}.$$

For example, extracting the coefficient of y^k in the formula for $A(x,y)$ in Theorem 3.1 yields the following formulas for $A_k(x)$ where $1 \leq k \leq 5$:

$$\begin{aligned}
A_1(x) &= \frac{1}{1-x},\\
A_2(x) &= \frac{1}{1-2x},\\
A_3(x) &= \frac{1-3x+4x^2-x^3}{(1-x)^2(1-2x)^2},\\
A_4(x) &= \frac{1-4x+9x^2-6x^3+2x^4}{(1-x)^2(1-2x)^3},\\
A_5(x) &= \frac{1-6x+21x^2-34x^3+32x^4-16x^5+4x^6}{(1-x)^3(1-2x)^4}.
\end{aligned}$$

3.2 Case 213

By the reverse complement operation, the number of k-ary words of length n that avoid $\{213, 1233\}$ is the same as the number that avoid $\{132, 1123\}$. Here, it is more convenient to enumerate the latter. Let $B_k(x)$ denote the g.f. for the number of k-ary words π of length n that avoid $\{132, 1123\}$ for each $k \geq 1$, with $B_0(x) = 1$. Consider cases based on whether π can be expressed as $\pi = k^\ell\pi'$, where $\ell \geq 0$ and π' is $(k-1)$-ary, or as $\pi = k^\ell\pi''k\pi'''$, where π'' is a word on the alphabet $[i, k-1]$ for some $i \in [k-1]$ such that i occurs at least once and π''' is $(i+1)$-ary on $[i] \cup \{k\}$. Note that π' and π''' both avoid $\{132, 1123\}$, whereas π'' avoids $\{132, 112\}$. This implies

$$B_k(x) = \frac{1}{1-x}B_{k-1}(x) + \frac{1}{1-x}\sum_{j=2}^{k}(M_j(x) - M_{j-1}(x))B_{k+2-j}(x), \qquad k \geq 1,$$

where $M_k(x)$ is the g.f. for the number of k-ary words of the form γk such that γ is $(k-1)$-ary and avoids $\{132, 112\}$. Note that the $M_j(x) - M_{j-1}(x)$ factor accounts for the fact that the letter i must occur at least once in the section π'' of π above.

We now must determine $M_k(x)$. Note that $M_1(x) = x$, so assume $k \geq 2$. Then ρ enumerated by $M_k(x)$ is either of the form $\rho = (k-1)^\ell \rho' k$, where $\ell \geq 0$ and ρ' contains no $k-1$, or of the form $\rho = (k-1)^\ell \rho''(k-1)\rho''' k$, where ρ'' is a word on $[i, k-2]$ for some $i \in [k-2]$ containing at least one i and ρ''' is $(i+1)$-ary on $[i] \cup \{k-1\}$. Note that ρ' and ρ''' both avoid $\{132, 112\}$, whereas ρ'' avoids $\{132, 11\}$. Furthermore, observe that within ρ''', any $(k-1)$'s must occur prior to any i's, for otherwise there would be a 1123 in ρ of the form $ii(k-1)k$, where the first i occurs in ρ''. Concerning ρ''', we therefore consider additional cases based on whether ρ''' contains (i) neither i nor $k-1$, (ii) exactly one of $i, k-1$ or (iii) both i and $k-1$. Note that all $(k-1)$'s in ρ''' occur as an initial run in case (iii), for otherwise a 132 would occur. Hence, we get contributions of $M_i(x)$, $2(M_{i+1}(x) - M_i(x))$ and $\frac{x}{1-x}(M_{i+1}(x) - M_i(x))$ for (i), (ii) and (iii), respectively. Considering all possible i, and replacing i with $k-i$, then gives for all $k \geq 2$ the recurrence

$$M_k(x) = \frac{1}{1-x} M_{k-1}(x) + \frac{1}{1-x} \sum_{i=2}^{k-1} (L_i(x) - L_{i-1}(x)) \left(\frac{2-x}{1-x} M_{k+1-i}(x) - \frac{1}{1-x} M_{k-i}(x) \right),$$

where $L_k(x)$ is the g.f. for the number of k-ary words of the form γk that avoid $\{132, 11\}$. Since such words correspond to 132-avoiding permutations whose largest letter is last, we have $L_k(x) = \sum_{j=0}^{k-1} C_j \binom{k-1}{j} x^{j+1}$ for $k \geq 1$.

Define the bivariate g.f.'s by $B(x,y) = \sum_{k \geq 0} B_k(x) y^k$, $M(x,y) = \sum_{k \geq 1} M_k(x) y^k$ and $L(x,y) = \sum_{k \geq 1} L_k(x) y^k$. Then the recurrences above for $B_k(x)$ and $M_k(x)$ imply

$$\left(1 - \frac{y}{1-x}\right) B(x,y) = 1 + \frac{1}{(1-x)y^2} \left((1-y)M(x,y) - xy\right) \left(B(x,y) - 1 - \frac{y}{1-x}\right)$$

and

$$\left(1 - \frac{y}{1-x}\right) M(x,y) = xy + \frac{((1-y)L(x,y) - xy)((2-x-y)M(x,y) - (2-x)xy)}{(1-x)^2 y},$$

where $L(x,y) = \frac{xy}{1-y} C\left(\frac{xy}{1-y}\right)$ and $C(z) = \frac{1-\sqrt{1-4z}}{2z} = \sum_{n \geq 0} C_n z^n$ denotes the g.f. for the Catalan number sequence.

Solving the preceding equations for $B(x,y)$ yields after several algebraic steps the following result.

Theorem 3.2. *The generating function for the number of k-ary words of length n that avoid both* 213 *and* 1233 *(*132 *and* 1123*) for all $n, k \geq 0$ is given by*

$$\frac{4(1-2x)(1-x)^2 - 2(1-x)(4-7x+4x^2)y + (4-7x+4x^2)y^2 + xy^2\sqrt{1 - \frac{4xy}{1-y}}}{2(2(1-2x)(1-x)^2 - (1-x)(2-x)(3-4x)y + 2(1-x)(3-2x)y^2 - (2-x)y^3)}.$$

By Theorem 3.2, we have for example the following formulas for $B_k(x)$ where $1 \leq k \leq 5$:

$$\begin{aligned}
B_1(x) &= \frac{1}{1-x},\\
B_2(x) &= \frac{1}{1-2x},\\
B_3(x) &= \frac{1-3x+4x^2-x^3}{(1-x)^2(1-2x)^2},\\
B_4(x) &= \frac{1-3x+6x^2+2x^4}{(1-x)(1-2x)^3},\\
B_5(x) &= \frac{1-6x+21x^2-34x^3+32x^4-26x^5+13x^6+8x^7-8x^8}{(1-x)^3(1-2x)^4}.
\end{aligned}$$

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