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Semi-perimeter and inner site-perimeter of k-ary words and bargraphs

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Abstract

Given a bargraph B, a *border cell* of B is a cell of B that shares at least one common edge with an outside cell of B. Clearly, the inner site-perimeter of B is the number of border cells of B. A *tangent cell* of B is a cell of B which is not a border cell of B and shares at least one vertex with an outside cell of B. In this paper, we study the generating function for the number of k-ary words, represented as bargraphs, according to the number of horizontal steps, up steps, border cells and tangent cells. This allows us to express some cases via Chebyshev polynomials of the second kind. Moreover, we find an explicit formula for the number of bargraphs according to the number of horizontal steps, up steps, and tangent cells/inner site-perimeter. We also derive asymptotic estimates for the mean number of tangent cells/inner site-perimeter.

Keywords: Bargraphs, Chebyshev polynomials, k-ary words, semi-perimeter, inner site-perimeter.

Math. Subj. Class.: 05A15, 05A16, 60C05

1 Introduction

The solid–on–solid (SOS) model has received a lot of attention. The SOS model arose from the consideration of the boundary between oppositely magnetized phases in the Ising model [9, 23]. The linear SOS model with a magnetic field and wall interaction was solved in [19]. Later, in [20], the restricted SOS (RSOS) has been considered, where the interface takes on a restricted subset of configurations and the interactions of field and single wall interaction in the half-plane. Then, in [21] is presented the solution of the linear RSOS model confined to a slit. Each configuration of the RSOS presented by a *k-bounded bargraph* (see below) with allowing horizontal steps in the x-axis, where the exact solution in [21] is presented by studying the generating function for the number of k-bounded bargraphs according to the *semi-perimeter*.

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A *bargraph* is a column-convex polyomino where all the columns are bottom justified. Throughout this paper, we represent a bargraph as a lattice path in $\mathbb{Z}^2$, starting at the origin and after ending upon first return to the x-axis, the progression ends at the origin $(0,0)$. The allowed steps are the up step $u = (0,1)$, the down step $d = (0,-1)$, the horizontal step $h = (1,0)$, and the left horizontal step $(-1,0)$. The first step has to be an up step, the horizontal steps must all lie above the x-axis, the left horizontal steps must all lie on x-axis, and an up step cannot follow a down step and vice versa. Alternatively, we identify a bargraph by the 1×1 squares (called *cells*) that lie inside its lattice path. For instance, Figure 1 represents the bargraph $uhuhhuhuhdhhddd$ (as lattice path) and the word $\sigma_1\sigma_2\cdots\sigma_7 = 1223433$ where the ith column contains exactly σ_i cells. For a given bargraph B, we define the *semi-perimeter* of B to be the number of up steps and horizontal steps in the lattice path, the *site-perimeter* of B to be the number of nearest-neighbour cells outside the boundary of B, and the *inner site-perimeter* to be the number of cells inside B that have at least one edge in common with an outside cell. Figure 1 represents a

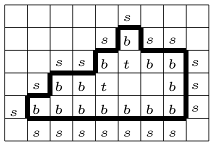

Figure 1: The bargraph of the word 1223433

corresponding bargraph B of the word 1223433 with semi-perimeter 11, site-perimeter 18 (the sum of the cells marked by s), and inner site-perimeter 10 (the sum of the cells marked by b).

In the last decades, the enumeration of bargraphs according to statistics has received a lot of attention (following the interest in Statistical physics as we said at the beginning of the introduction). Earlier authors, such as Prellberg and Brak [22] and Feretić [10] (see also [6, 12]), have found that the generating function that counts all bargraphs (including the empty bargraph) is given by

$$B(x,y) = \frac{1+x-y-xy-\sqrt{(1-x-y-xy)^2-4x^2y}}{2x}, \tag{1.1}$$

where x counts number of horizontal steps and y counts the number of up steps. Note that the generating function for bargraphs according to the semi-perimeter, often called *the isotropic generating function*, is given by $B(x,x)$. To find the asymptotics of the coefficient of x^n in $B(x,x)$, one computes the dominant singularity ρ which is the positive root of $1-4x+2x^2+x^4=0$. Thus, by singularity analysis (for example, see [11]) we have

$$[x^n]B(x,x) \sim \frac{1}{2}\sqrt{\frac{1-\rho-\rho^3}{\pi\rho n^3}}\rho^{-n} \tag{1.2}$$

with

$$\rho = \frac{1}{3}\left(-1-\frac{2^{8/3}}{(13+3\sqrt{3})^{1/3}}+2^{1/3}(13+3\sqrt{33})^{1/3}\right) \approx 0.295598\cdots. \tag{1.3}$$

Later, Blecher et al. refined the generating function $B(x, y)$ by considering a third statistic such as: levels [1], peaks [3], area [2], descents [2] and height [5]. In [6] is studied the number of bargraphs according to the site-perimeter. In [14] is considered the number of bargraphs according to the inner site-perimeter. In [4] it is derived a functional equation for the generating function that counts the number of k-ary words according to the size of the rightmost letter, the number of letters and the site-perimeter. For more details and motivations related to statistical physics and enumerative combinatorics, we refer the reader to [15] and references therein. The aim of this paper, is to refine some of these results.

For a given bargraph B, we define a *border cell* of B as a cell of B that has at least one edge in common with an outside cell of B. Clearly, the inner site-perimeter of B is the number of border cells of B. Further, we define a *tangent cell* of B to be a cell of B which is not a border cell of B and that has at least one vertex in common with an outside cell of B (see Figure 1). In what follows throughout this paper, whenever we refer to k-ary words, we mean their corresponding bargraph representation.

The paper aims to study the generating function for the number of k-ary words (bounded bargraphs that lie below the line $y = k$), according to the number of horizontal steps, up steps, tangent cells and border cells. More precisely: (1) We find an explicit formula for the generating function for the number of bargraphs according to the number of horizontal steps, the number of up steps, inner site-perimeter, and the number of tangent cells. In particular, we present the average number of tangent cells/inner site-perimeter as the semi-perimeter n of the bargraph tends to infinity. (2) We find an explicit formula for the generating function for the number of k-ary words according to the number of horizontal steps and up steps in terms of Chebyshev polynomials of the second kind, then we rederive (1.1). (3) We find an explicit formula for the generating function for the number of k-ary words according to the number of horizontal steps, up steps and tangent cells in terms of Chebyshev polynomials of the second kind. (4) We also study the generating function for the number of k-ary words according to the number of horizontal steps, up steps and inner site-perimeter.

2 Results

Define $[x]_d = \frac{1-x^d}{1-x} = 1 + x + \cdots + x^{d-1}$ for all $d \geq 0$. Let $C_k = C_k(x, y, p, q)$ be the generating function for the number of k-ary words of length n according to the number of horizontal steps (marked by x), up steps (marked by y), border cells (marked by p) and tangent cells (marked by q). We decompose each k-ary word π as

$$\pi = \pi^{(0)} 1 \pi^{(1)} \cdots 1 \pi^{(s)}, s \geq 0$$

where $\pi^{(j)}$ is a word over alphabet $\{2, 3, \ldots, k\}$, for all $j = 0, 1, \ldots, s$. Then

$$\begin{aligned} C_k &= D_k + \sum_{s \geq 1} (xp)^s (y + D_k - 1)(1 + (D_k - 1)/y)^s \\ &= D_k + \frac{xp(y + D_k - 1)(1 + (D_k - 1)/y)}{1 - px(1 + (D_k - 1)/y)}, \end{aligned} \tag{2.1}$$

where $D_k = D_k(x, y, p, q)$ is the generating function for the number of words of length n over alphabet $\{2, 3, \ldots, k\}$, according to the number of horizontal steps, up steps, border cells and tangent cells.

Next we write an equation for the generating function D_k with $k \geq 2$. Clearly, $D_2 = 1 + \frac{xy^2p^2}{1-xp^2}$. In order to write a recurrence relation for D_k, we decompose each word π over alphabet $\{2, 3, \dots, k\}$ as

$$\pi = \pi^{(0)}2\pi^{(1)} \cdots 2\pi^{(s)}, s \geq 0$$

where $\pi^{(j)}$ is a word over alphabet $\{3, 4, \dots, k\}$, for all $j = 0, 1, \dots, s$.

The case $s = 0$ contributes

$$F_0 = 1 + x(yp)^3[yp]_{k-2} + yp^2(D_{k-1} - 1 - x(yp)^2[yp]_{k-2}),$$

where first, second and third term counts the empty word, words with one letter, words with at least two letters, respectively.

Similarly, the case $s = 1$ contributes

$$\begin{aligned} F_1 = xp^2(y^2 &+ x(yp)^3[yp]_{k-2} + ypq(D_{k-1} - 1 - x(yp)^2[yp]_{k-2}))\cdot \\ &(1 + xyp^3[yp]_{k-2} + pq/y(D_{k-1} - 1 - x(yp)^2[yp]_{k-2})), \end{aligned}$$

where $xp^2(y^2 + x(yp)^3[yp]_{k-2} + ypq(D_{k-1} - 1 - x(yp)^2[yp]_{k-2}))$ counts the words of the form $\pi^{(0)}2$, and $1 + xyp^3[yp]_{k-2} + pq/y(D_{k-1} - 1 - x(yp)^2[yp]_{k-2})$ counts either the empty word or the nonempty words of the form $\pi^{(1)}$ without first two up steps.

For $s \geq 2$, we have the contribution

$$F_1(xp^2)^{s-1}(1 + xyp^2q[yp]_{k-2} + pq^2/y(D_{k-1} - 1 - x(yp)^2[yp]_{k-2}))^{s-1}.$$

Therefore, by summing all the contributions, we obtain the following result.

Lemma 2.1. *The generating function $D_k = D_k(x, y, p, q)$ satisfies*

$$\begin{aligned} D_k = 1 &+ x(yp)^3[yp]_{k-2} + yp^2(D_{k-1} - 1 - x(yp)^2[yp]_{k-2}) \\ &+ xp^2(y^2 + x(yp)^3[yp]_{k-2} + ypq(D_{k-1} - 1 - x(yp)^2[yp]_{k-2}))\cdot \\ &\quad \cdot (1 + xyp^3[yp]_{k-2} + \frac{pq}{y}(D_{k-1} - 1 - x(yp)^2[yp]_{k-2}))\cdot \\ &\quad \cdot \left(1 - xp^2(1 + xyp^2q[yp]_{k-2} + \frac{pq^2}{y}(D_{k-1} - 1 - x(yp)^2[yp]_{k-2}))\right)^{-1} \end{aligned}$$

with $D_2 = 1 + \frac{xy^2p^2}{1-xp^2}$.

Hence, by (2.1), we have the following formula for the generating function $C_k(x, p, q)$.

Theorem 2.2. *Let $k \geq 1$. Then*

$$C_k(x, y, p, q) = D_k(x, y, p, q) + \frac{xp(y + D_k(x, y, p, q) - 1)(1 + (D_k(x, y, p, q) - 1)/y)}{1 - px(1 + (D_k(x, y, p, q) - 1)/y)},$$

where $D_k(x, y, p, q)$ is given in Lemma 2.1.

2.1 Bargraphs

Clearly, $C(x, y, p, q) = \lim_{k\to\infty} C_k(x, y, p, q)$ is the generating function for the number of bargraphs according to the number of horizontal steps, number of up steps, inner site-perimeter, and the number of tangent cells. Similarly, $D(x, y, p, q) = \lim_{k\to\infty} D_k(x, y, p, q)$

is the generating function for the number of bargraphs such that each nonempty column contains at least two cells according to the number of horizontal steps, number of up steps, inner site-perimeter, and number of tangent cells. By taking $k \to \infty$, Lemma 2.1 gives

$$\begin{aligned} D(x,y,p,q)-1 &= \frac{-\alpha_1+\sqrt{\alpha_1^2-4\alpha_0\alpha_2}}{2\alpha_2} \\ &= p^2xy^2+p^3xy^3+p^4x^2y^2+p^4xy^4+2p^5x^2y^3+p^5xy^5+\cdots, \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} \alpha_0 &= -p^2y^3(p^2y-1)(py-1)x+p^5y^4(py-1)(p-2q+1)x^2 \\ &\quad +p^7y^5(p^2q^2-2pq^2+pq-p+q)x^3, \\ \alpha_1 &= -y(py-1)^2(p^2y-1)+p^2y(py-1)^2(p^2y-2pqy-1)x \\ &\quad +p^4qy^2(py-1)(2p^3qy-3p^2qy+p^2y-pq+1)x^2, \\ \alpha_2 &= -p^3q^2x(1+py-p^2y)(py-1)^2. \end{aligned}$$

On the other hand, by taking $k \to \infty$, Theorem 2.2 and (2.2) imply the following result.

Theorem 2.3. *The generating function for the number of bargraphs according to the number of horizontal steps, number of up steps, inner site-perimeter, and the number of tangent cells is given by*

$$\begin{aligned} &C(x,y,p,q) \\ &= 1+\frac{-\alpha_1+\sqrt{\alpha_1^2-4\alpha_0\alpha_2}}{2\alpha_2}+\frac{xp(2y\alpha_2-\alpha_1+\sqrt{\alpha_1^2-4\alpha_0\alpha_2})^2}{2\alpha_2(2y\alpha_2-px(2y\alpha_2-\alpha_1+\sqrt{\alpha_1^2-4\alpha_0\alpha_2}))} \\ &= 1+pxy+p^2xy^2+p^2x^2y+p^3xy^3+2p^3x^2y^2+p^3x^3y+\cdots. \end{aligned}$$

We illustrate the above theorem through following 2 examples.

Example 2.4. By Theorem 2.3 we have that the generating function $C(1,1,1,p)$ for the number of bargraphs according to inner site-perimeter is given by

$$\frac{2p^6-2p^5+2p^4-2p^2+1-\sqrt{4p^{11}-4p^9+4p^5+4p^4-4p^3-4p^2+1}}{p(-2p^6+2p^5-4p^4+2p^3+4p^2-1+\sqrt{4p^{11}-4p^9+4p^5+4p^4-4p^3-4p^2+1})}.$$

In the case of counting bargraphs according to site-perimeter, we refer the reader to [6].

Next we define a *strong inner site-perimeter* to be the number of border cells and the number of tangent cells.

Example 2.5. By Theorem 2.3 we have that the generating function $C(1,1,p,p)$ for the number of bargraphs according to strong inner site-perimeter is given by

$$\frac{2p^8-3p^7+p^6+p^5-p^4-2p^2+1-\sqrt{\beta})}{p(-2p^8+3p^7-3p^6+p^5+3p^4+2p^2-1+\sqrt{\beta})},$$

where $\beta = p^{14}-2p^{13}+3p^{12}-5p^{10}+6p^9+p^8-2p^7+2p^6-2p^5+2p^4-4p^2+1$.

In particular, the generating function for the bargraphs according to semi-perimeter and the number of tangent cells is given by $C(x,x,1,q)$. Differentiating $C(x,x,1,q)$ with respect to q and evaluating it at $q=1$ gives

$$\frac{\partial}{\partial q}C(x,x,1,q)\mid_{q=1} = \frac{x^8-x^7+7x^6-10x^5+20x^4-25x^3+24x^2-12x+2}{2x^2\sqrt{x^4+2x^2-4x+1}} - \frac{x^7-2x^6+7x^5-13x^4+19x^3-18x^2+10x-2}{2x^2(x-1)}.$$

In what follows ρ is as defined by equation (1.3). By direct calculations and (1.2), we have

$$\lim_{x\mapsto\rho}\frac{\partial}{\partial q}C(x,x,1,q)\mid_{q=1}(1-x/\rho)^{1/2} = 1-\frac{5}{2}\rho-\frac{3}{2}\rho^2.$$

Hence, we have the following result.

Corollary 2.6. *The average number of tangent cells is asymptotic to*

$$\frac{(2-5\rho-3\rho^3)\sqrt{1-\rho-\rho^3}}{2\sqrt{\rho}}n$$

as the semi-perimeter n of the bargraph tends to infinity.

Moreover, by differentiating $C(x,x,p,1)$ with respect to p, evaluating at $p=1$ and using (1.2) gives

$$\lim_{x\mapsto\rho}\frac{\partial}{\partial}C(x,x,p,1)\mid_{p=1}(1-x/\rho)^{1/2} = \frac{25-15\rho-25\rho^2-21\rho^3}{16}.$$

Hence, we have the following result.

Corollary 2.7. *The average inner site-perimeter is asymptotic to*

$$\frac{(25-15\rho-25\rho^2-21\rho^3)\sqrt{1-\rho-\rho^3}}{16\sqrt{\rho}}n$$

as the semi-perimeter n of the bargraph tends to infinity.

2.2 Semi-perimeter and k-ary words

Define $B_k(x,y)=C_k(x,y,1,1)$ and $E_k(x,y)=D_k(x,y,1,1)-1$, for all $k\geq 1$. Then Theorem 2.2 with $p=q=1$ gives

$$B_k(x,y) = \frac{y(1-x+xy)+(xy-x+y)E_k(x,y)}{y(1-x)-xE_k(x,y)}, \tag{2.3}$$

where

$$E_k(x,y) = \frac{xy^3+(1+x)y^2E_{k-1}(x,y)}{y(1-x)-xE_{k-1}(x,y)}.$$

Recall that the Chebyshev polynomials of the second kind $U_m(t)$ satisfy the recurrence relation $U_m(t) = 2tU_{m-1}(t) - U_{m-2}(t)$ with the initial conditions $U_0(t) = 1$ and $U_1(t) = 2t$. By induction ok k, we have

$$E_k(x,y) = \frac{xy\sqrt{y}U_{k-2}\left(\frac{1-x+y+xy}{2\sqrt{y}}\right)}{U_{k-1}\left(\frac{1-x+y+xy}{2\sqrt{y}}\right) - (1+x)\sqrt{y}U_{k-2}\left(\frac{1-x+y+xy}{2\sqrt{y}}\right)}, \tag{2.4}$$

where $U_m(t)$ is the m-th Chebyshev polynomials of the second kind. Substituting into (2.3) gives the following result.

Theorem 2.8. *The generating function for the number of k-ary words, $k \geq 2$, according to the number of horizontal steps and up steps is given by*

$$B_k(x,y) = \frac{(1-x+xy)U_{k-1}\left(\frac{1-x+y+xy}{2\sqrt{y}}\right) - \sqrt{y}U_{k-2}\left(\frac{1-x+y+xy}{2\sqrt{y}}\right)}{(1-x)U_{k-1}\left(\frac{1-x+y+xy}{2\sqrt{y}}\right) - \sqrt{y}U_{k-2}\left(\frac{1-x+y+xy}{2\sqrt{y}}\right)}.$$

Note that $\lim_{k\mapsto\infty} \frac{U_{k-1}(\frac{1}{2\sqrt{t}})}{\sqrt{t}U_k(\frac{1}{2\sqrt{t}})} = C(t)$, where $C(t) = \frac{1-\sqrt{1-4t}}{2t}$ (for example, see [17] and references therein). Thus, Theorem 2.8 shows that

$$\begin{aligned}
\lim_{k\mapsto\infty} B_k(x,y) &= \frac{1-x+xy-\sqrt{y}\lim_{k\mapsto\infty}\frac{U_{k-2}\left(\frac{1-x+y+xy}{2\sqrt{y}}\right)}{U_{k-1}\left(\frac{1-x+y+xy}{2\sqrt{y}}\right)}}{1-x-\sqrt{y}\lim_{k\mapsto\infty}\frac{U_{k-2}\left(\frac{1-x+y+xy}{2\sqrt{y}}\right)}{U_{k-1}\left(\frac{1-x+y+xy}{2\sqrt{y}}\right)}} \\
&= \frac{1-x+xy-\frac{y}{1-x+y+xy}C\left(\frac{y}{(1-x+y+xy)^2}\right)}{1-x-\frac{y}{1-x+y+xy}C\left(\frac{y}{(1-x+y+xy)^2}\right)} \\
&= \frac{(1-x+xy)(1-x+y+xy)-yC\left(\frac{y}{(1-x+y+xy)^2}\right)}{(1-x)(1-x+y+xy)-yC\left(\frac{y}{(1-x+y+xy)^2}\right)} \\
&= \frac{1+x-y-xy-\sqrt{(1-x-y-xy)^2-4x^2y}}{2x},
\end{aligned}$$

which agrees with (1.1).

2.3 Tangent cells

Theorem 2.2 with $p = 1$ gives

$$C_k(x,y,1,q) = \frac{y(1-x+xy)+(xy-x+y)E_k(x,y,1,q)}{y(1-x)-xE_k(x,y,1,q)}, \tag{2.5}$$

with

$$xq^2E_k(x,y,1,q) = -y^2(1-x+2xq+q(1-q)x^2y[y]_{k-2}) + e_k,$$

where $e_k = \frac{y^3(1-x+qx)^2}{y(1-x)+y^2(1+qx)(1-x+qx)-e_{k-1}}$ and $e_2 = \frac{y^2(1-x+qx)^2}{1-x}$.

By induction on k and the definition of Chebyshev polynomials of the second kind, we obtain

$$e_k = \frac{y\sqrt{y}(1-x+qx)(\frac{(1-x)^2-qx^2y(1-x+qx)}{\sqrt{y}(1-x+qx)}U_{k-4}(t)-(1-x)U_{k-5}(t))}{\frac{(1-x)^2-qx^2y(1-x+qx)}{\sqrt{y}(1-x+qx)}U_{k-3}(t)-(1-x)U_{k-4}(t)},$$

where $t=\frac{(1-x)(1+y+qxy)+qxy(1+qx)}{2\sqrt{y}(1-x+qx)}$. By substituting into (2.5), we obtain the following result.

Theorem 2.9. *The generating function for the number of k-ary words, $k\geq 2$, according to the number of horizontal steps, up steps and tangent cells is given by*

$$C_k(x,y,1,q)=1+\frac{xy}{1-x}+\frac{y}{1-x}\cdot\frac{E_k(x,y,1,q)}{y(1-x)-xE_k(x,y,1,q)},$$

where

$$\begin{aligned}
&xq^2E_k(x,y,1,q)\\
&\quad=-y^2(1-x+2xq+q(1-q)x^2y[y]_{k-2})\\
&\quad+\frac{y\sqrt{y}(1-x+qx)\left(\frac{(1-x)^2-qx^2y(1-x+qx)}{\sqrt{y}(1-x+qx)}U_{k-4}(t)-(1-x)U_{k-5}(t)\right)}{\frac{(1-x)^2-qx^2y(1-x+qx)}{\sqrt{y}(1-x+qx)}U_{k-3}(t)-(1-x)U_{k-4}(t)}
\end{aligned}$$

and $t=\frac{(1-x)(1+y+qxy)+qxy(1+qx)}{2\sqrt{y}(1-x+qx)}$.

Note that by taking $q=1$ into Theorem 2.9 gives Theorem 2.8, as expected.

Next we turn our attention in finding the generating function for the total number of tangent cells over all k-ary words according to number horizontal steps and up steps. Define $Cq_k(x,y)=\frac{\partial}{\partial q}C_k(x,y,1,q)\mid_{q=1}$ and $Eq_k(x,y)=\frac{\partial}{\partial q}E_k(x,y,1,q)\mid_{q=1}$. Differentiating (2.5) with respect to q and evaluating at $q=1$ gives

$$Cq_k(x,y)=\frac{y^2}{(y(1-x)-xE_k(x,y))^2}Eq_k(x,y), \tag{2.6}$$

where

$$\begin{aligned}
&Eq_k(x,y)\\
&=\frac{y^3}{(y(1-x)-xE_{k-1}(x,y))^2}Eq_{k-1}(x,y)\\
&+\frac{xy^2(y+E_{k-1}(x,y))}{(y(1-x)-xE_{k-1}(x,y))^2}\left(xy^2(x-2)[y]_{k-2}+(2+x^2y[y]_{k-2})E_{k-1}(x,y)\right).
\end{aligned}$$

with $Eq_2(x,y)=0$.

By induction on k, we have

$$\begin{aligned}
&Eq_k(x,y)\\
&=\sum_{j=2}^{k-1}\frac{xy^{3(k-j)-1}(y+E_j(x,y))\left(xy^2(x-2)[y]_{j-1}+(2+x^2y[y]_{j-1})E_j(x,y)\right)}{\prod_{i=j}^{k-1}(y(1-x)-xE_i(x,y))^2}.
\end{aligned}$$

Thus, by (2.6), we obtain the following result.

Theorem 2.10. *Let $k \geq 2$. The generating function for the total number of tangent cells over all k-ary words according to the number horizontal steps and up steps is given by*

$$\begin{aligned}&Cq_k(x,y)\\&=\sum_{j=2}^{k-1}\frac{xy^{3(k-j)+1}(y+E_j(x,y))\left(xy^2(x-2)[y]_{j-1}+(2+x^2y[y]_{j-1})E_j(x,y)\right)}{\prod_{i=j}^{k}(y(1-x)-xE_i(x,y))^2}.\end{aligned}$$

where $E_k(x,y)$ is given in (2.4).

For instance, Theorem 2.10 gives $Cq_2(x,y) = 0$ (as expected, since there are no tangent cells in 2-ary words) and

$$\begin{aligned}Cq_3(x,y) &= \frac{x^3y^3(xy-x+3)}{(x^2y-x^2+xy+2x-1)^2(xy-x+1)}\\&= \mathbf{3}x^3y^3+14x^4y^3+4x^4y^4+40x^5y^3+90x^6y^3+34x^5y^4+\cdots.\end{aligned}$$

We emphasize in bold, the three 3-ary words with three horizontal steps and three up steps as bargraphs of 232, 233 and 332 with $1+1+1=3$ tangent cells.

2.4 Border cells

Theorem 2.2 with $q=1$ gives

$$C_k(x,y,p,1)=E_k(x,y,p,1)+1+\frac{xp(y+E_k(x,y,p,1))(1+E_k(x,y,p,1)/y)}{1-px(1+E_k(x,y,p,1)/y)},$$

where

$$\begin{aligned}E_k(x,y,p,1)&=xy^3(p^3-p^4)[yp]_{k-2}+yp^2E_{k-1}(x,y,p,1)\\&\quad+\frac{xp^2(y+pE_{k-1}^2(x,y,p,1))}{1-xp^2(1+xy(p^2-p^3)[yp]_{k-2}+\frac{p}{y}E_{k-1}(x,y,p,1))}\end{aligned}$$

with $E_2(x,y,p,1)=\frac{xy^2p^2}{1-xp^2}$.

Now we find the generating function for the total inner site-perimeter (the number of border cells) over all k-ary words according to the number horizontal steps and up steps. Define $Cp_k(x,y)=\frac{\partial}{\partial p}C_k(x,y,p,1)\mid_{p=1}$ and $Ep_k(x,y)=\frac{\partial}{\partial p}E_k(x,y,p,1)\mid_{p=1}$. Differentiating with respect to p and evaluating at $p=1$ gives

$$Cp_k(x,y)=\frac{y^2}{(y(1-x)-xE_k(x,y))^2}Ep_k(x,y)+\frac{xy(y+E_k(x,y))^2}{(y(1-x)-xE_k(x,y))^2}, \tag{2.7}$$

where

$$Ep_k(x,y)=\frac{y^3}{(y(1-x)-xE_k(x,y))^2}Ep_{k-1}(x,y)+F_k(x,y)$$

with

$$\begin{aligned}F_k(x,y)=&\frac{y^4(3-x-x^2y[y]_{k-2})}{x(y(1-x)-xE_{k-1}(x,y))^2}-\frac{y^3(5-2x^2y[y]_{k-2})}{x(y(1-x)-xE_{k-1}(x,y))}\\&+\frac{y^2(3-x-2x^2y[y]_{k-2})}{x}-\frac{y}{x}(y(1-x)-xE_{k-1}(x,y)).\end{aligned} \tag{2.8}$$

By induction on k with $Ep_2(x,y)=\frac{2xy^2}{(1-x)^2}$, we obtain

$$Ep_k(x,y)=\sum_{j=2}^{k}\frac{y^{3(k-j)}F_j(x,y)}{\prod_{i=j}^{k-1}(y(1-x)-xE_i(x,y))^2}.$$

Hence, by (2.7), we have the following result.

Theorem 2.11. *Let $k\geq 2$. The generating function for the total inner site-perimeter (the number of border cells) over all k-ary words according to the number of horizontal steps and up steps is given by*

$$Cp_k(x,y)=\sum_{j=2}^{k}\frac{y^{3(k-j)+2}F_j(x,y)}{\prod_{i=j}^{k}(y(1-x)-xE_i(x,y))^2}+\frac{xy(y+E_k(x,y))^2}{(y(1-x)-xE_k(x,y))^2},$$

where $E_k(x,y)$ and $F_k(x,y)$ are given in (2.4) and (2.8), respectively.

For instance, Theorem 2.11 gives

$$\begin{aligned}Cp_2(x,y)&=\frac{xy(x^2(y-1)^2+2x(y-1)+2y+1)}{(x^2(y-1)+2x-1)^2}\\&=xy+2x^2y+2xy^2+\mathbf{10}x^2y^2+3x^3y+28x^3y^2+4x^4y+\cdots.\end{aligned}$$

We emphasize in bold, the three 2-ary words with two horizontal steps and two up steps as bargraphs of 12, 21 and 22 with inner site perimeter $3+3+4=10$.

We end this paper by the following comment on the relation between bargraphs and Chebyshev polynomials. We recall that a *Dyck path* of semi-length n is a lattice path that starts at $(0,0)$, ends at $(2n,0)$, remains weakly above the x-axis, and consists of up steps $(1,1)$ and down steps $(1,-1)$. Apparently, for the first time the relation between restricted permutations and Chebyshev polynomials was discovered by Chow and West in [7], then explored in [18], and characterized as Dyck paths in [13]. Chebyshev polynomials of the second kind also occur in the enumeration of height-restricted Dyck paths, and they are much more natural there (for instance, see [13, 18]). On the other hand, Deutsch and Elizalde [8] established a bijection ρ between Dyck paths and bargraphs, where the semi-length of a Dyck path becomes the semi-perimeter minus the number of peaks of the corresponding bargraph (a peak in a bargraph B is an occurrence of uh^jd for some $j\geq 1$). Besides that, as discussed in [15], due to the geometric nature of bargraphs, we tried to study the statistics tangent cells, semi-perimeter and inner-site perimeter directly on bargraphs, and not to transfer our statistics via the bijection ρ. We followed this approach since sometimes the bargraph statistics can not be transferred to nice statistics in Dyck paths, and sometimes the enumeration of the statistics in Dyck paths requires the same amount of work as working directly in bargraphs. It is the main reason that directs us to choose by our techniques rather then bijection ρ. In our present study, transferring the statistics tangent cells, inner-perimeter in bargraphs to statistics in Dyck paths remains a nice point of exploration for the interested readers.

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