On deformation of polygonal dendrites preserving the intersection graph

Dmitry Drozdov

Novosibirsk State University, Novosibirsk, Russia

Mary Samuel

Indian Institute of Information Technology Lucknow, Lucknow, India

Andrei Tetenov*

*Novosibirsk State University, Novosibirsk, Russia, and
Gorno-Altai State University, Gorno-Altai, Russia*

Received 6 August 2020, accepted 29 September 2020, published online 15 March 2021

Abstract

Let $\mathcal{S} = S_1, \dots, S_m$ be a system of contracting similarities of $\mathbb{R}^2$. The attractor $K(\mathcal{S})$ of the system $\mathcal{S}$ is a non-empty compact set satisfying $K = S_1(K) \cup \dots \cup S_m(K)$. We consider contractible polygonal systems $\mathcal{S}$ which are defined by a finite family of polygons whose intersection graph is a tree and therefore the attractor $K(\mathcal{S})$ is a dendrite. We find conditions under which a deformation $\mathcal{S}'$ of a contractible polygonal system $\mathcal{S}$ has the same intersection graph and therefore the attractor $K(\mathcal{S}')$ is a self-similar dendrite which is isomorphic to the attractor K of the system $\mathcal{S}$.

Keywords: Self-similar dendrite, generalized polygonal system, attractor, intersection graph, index diagram.

Math. Subj. Class.: 28A80

Introduction

Our work is part of a series of works aimed at studying self-similar dendrites in $\mathbb{R}^d$. A dendrite K is called self-similar, if it can be represented as an union $K = S_1(K) \cup \dots \cup$

*Supported by Russian Foundation of Basic Research project 18-01-00420

E-mail addresses: dimalek97@yandex.ru (Dmitry Drozdov), marysamuel@iiitl.ac.in (Mary Samuel), atet@mail.ru (Andrei Tetenov)

$S_m(K)$ of its images under a finite system $\mathcal{S} = \{S_1, \dots, S_m\}$ of contracting similarities of $\mathbb{R}^d$. This representation defines a self-similar structure on K .

The history of fractal geometry contains remarkable examples of dendrites such as Hata tree [8], Vicsek set, Pentadendrite and others, as well as certain important theorems obtained by various authors.

In 1985, M. Hata [8] obtained a criterion for the connectedness of self-similar sets and proved that if a dendrite K is the attractor of a system of weak contractions in a complete metric space, then the set of its endpoints is infinite. In 1990, C. Bandt showed in [3] that the Jordan arcs connecting pairs of points of a postcritically finite self-similar dendrite are self-similar, and the set of possible dimensions of such arcs is finite. J. Kigami in his work [11] considered finite topological subgraphs of a dendrite generated by a given set of points and studied shortest path metrics on self-similar dendrites.

Many examples of dendrites are defined as fractal squares [14]. A special case of such squares, fractal labyrinths, were studied in [5, 6, 7].

A systematic approach to the study of self-similar dendrites required finding answers to the following questions: What topological constraints characterize the class of dendrites generated by systems of similarities in $\mathbb{R}^d$? What are the explicit algorithms for constructing self-similar dendrites? What are the metric and analytical properties of morphisms of self-similar structures on dendrites?

Starting from the simplest and most obvious settings that have been used implicitly by many authors [3, 19], we considered systems $\mathcal{S} = \{S_1, \dots, S_m\}$ of contracting similarities in $\mathbb{R}^2$, defined by some polygon $P \in \mathbb{R}^2$ and a family of polygons $P_i = S_i(P) \subset P$, which (i) intersect each other only by their vertices; (ii) whose union $\tilde{P} = \bigcup_{i=1}^n P_i$ is a contractible set; and (iii) whose set of vertices contains all the vertices of P . We called such systems $\mathcal{S}$ contractible P -polygonal systems [15, 16, 17, 18].

The attractor K of each such system $\mathcal{S}$ is a dendrite. The properties of such dendrites directly follow from the geometry of the system of polygons P_i and from the properties of the intersection graph $\Gamma(\mathcal{S})$ of this system.

For example, the upper bound for the order of ramification points of K depends only on the values of angles and on the number of vertices of P . The addresses of ramification points of K and their order can be derived from the intersection graph $\Gamma(\mathcal{S})$ of the system $\mathcal{S}$.

Moreover, by [18, Theorem 27], if two contractible polygonal systems $\mathcal{S}, \mathcal{S}'$ have isomorphic intersection graphs $\Gamma(\mathcal{S}), \Gamma(\mathcal{S}')$, then there is a homeomorphism $\varphi : K \rightarrow K'$ of their attractors, which defines the isomorphism of self-similar structures $(K, \mathcal{S})$ and $(K', \mathcal{S}')$.

As we define in Section 2, the system $\mathcal{S}$ is called a generalised P -polygonal system if the system of polygons $P_i = S_i(P)$ satisfies the conditions (ii)–(iii), but the requirement $P_i \subset P$ is omitted.

If $\mathcal{S}$ is a contractible P -polygonal system and $\mathcal{S}'$ is a generalized P' -polygonal system, the intersection graphs $\Gamma(\mathcal{S}), \Gamma(\mathcal{S}')$ are isomorphic and for any i, j , $P'_i \cap P'_j = S'_i(K') \cap S'_j(K')$, then there is a homeomorphism $\varphi : K \rightarrow K'$ which defines the isomorphism of $(K, \mathcal{S})$ and $(K', \mathcal{S}')$.

In the current paper we find the conditions under which a small deformation $\mathcal{S}'$ of a contractible polygonal system $\mathcal{S}$ preserves the intersection graph of the system $\mathcal{S}$.

In Section 1 we expound basic definitions of the theory of self-similar sets and the

definition of contractible polygonal systems. In Section 2 we make a reference to some of our results from [20] on self-similar sets possessing one-point intersection property. Then we proceed to generalized polygonal systems and prove the Theorem 2.6 providing the condition **D0** under which the attractor of a generalized polygonal system is a dendrite. Then we give the Definition 2.7 of δ -deformations of a contractible P -polygonal system $\mathcal{S}$ and prove the Theorem 2.9 which shows that if a δ -deformation $\mathcal{S}'$ of the polygonal system $\mathcal{S}$ satisfies the condition **D0** then the self-similar structures $(K, \mathcal{S})$ and $(K', \mathcal{S}')$ are isomorphic.

So the question arises, how to ensure that the system $\mathcal{S}'$ satisfies Condition **D0** which guarantees that the deformation preserves the intersection graph of the system $\mathcal{S}$. The answer is given by two following statements.

First is the Parameter Matching Theorem 3.12 which gives necessary condition for one-point intersections at common vertices $P'_i \cap P'_j$.

The second is the Small Deformations Theorem 4.6 which states that there is a number $\nu > 0$, depending on the system $\mathcal{S}$ such that if δ is not greater than ν and if the Parameter Matching condition holds, then the system $\mathcal{S}'$ has the same intersection graph as $\mathcal{S}$.

In Section 3 we introduce the Index Diagram $\mathcal{G}(\mathcal{S})$ of a P -polygonal system $\mathcal{S}$. This allows us to understand the role of cyclic vertices of the polygon P and to show that each vertex A of P is subordinate to some cyclic vertex B . In subsection 3.2 we define the standard neighbourhood U_A of a non-cyclic vertex A and therefore of any of its images $S_j(A)$. Proceeding to generalized polygonal systems we prove the existence of invariant arcs at cyclic points of these systems and prove the Parameter Matching Theorem. Section 4 is rather technical. We find the estimates for δ -deformations, and in the end we prove the Small Deformations Theorem 4.6.

1 Preliminaries

1.1 Self-similar sets

Definition 1.1. Let $\mathcal{S} = \{S_1, S_2, \dots, S_m\}$ be a system of (injective) contraction maps on the complete metric space (X, d) . A nonempty compact set $K \subset X$ is called the attractor of the system $\mathcal{S}$, if $K = \bigcup_{i=1}^m S_i(K)$.

Throughout the whole paper, the maps $S_i \in \mathcal{S}$ are supposed to be similarities and the set X to be $\mathbb{R}^2$. We will use complex notation for the point on the plane, so each similarity will be written as $S_j(z) = q_j e^{i\alpha_j}(z - z_j) + z_j$, where $q_j = \text{Lip } S_j$ and $z_j = \text{fix}(S_j)$. For a system $\mathcal{S}$, we denote $q_{\min} = \min\{q_j, j = 1, \dots, m\}$ and $q_{\max} = \max\{q_j, j = 1, \dots, m\}$.

The system $\mathcal{S}$ defines its *Hutchinson operator* T by the equation $T(A) = \bigcup_{i=1}^m S_i(A)$. By Hutchinson's Theorem [9], the attractor K is unique for $\mathcal{S}$ and for any compact set $A \subset X$ the sequence $T^n(A)$ converges to K . We also call the set K *self-similar* with respect to $\mathcal{S}$.

The set $I = \{1, 2, \dots, m\}$ is called the *set of indices*, while $I^* = \bigcup_{n=1}^{\infty} I^n$ is the set of all finite I -tuples, or *multi-indices* $\mathbf{j} = j_1 j_2 \dots j_n$. The length n of the multi-index $\mathbf{j} = j_1 \dots j_n$ is denoted by $|\mathbf{j}|$ and $\mathbf{i}\mathbf{j}$ denote the concatenation of respective multi-indices. We say $\mathbf{i} \sqsubset \mathbf{j}$, if $\mathbf{j} = \mathbf{i}\mathbf{l}$ for some $\mathbf{l} \in I^*$; if $\mathbf{i} \not\sqsubset \mathbf{j}$ and $\mathbf{j} \not\sqsubset \mathbf{i}$ and $\mathbf{i}$ and $\mathbf{j}$ are *incomparable*.

For any $\mathbf{j} \in I^*$ we write $S_{\mathbf{j}} = S_{j_1 j_2 \dots j_n} = S_{j_1} S_{j_2} \dots S_{j_n}$; given a set A , we denote $S_{\mathbf{j}}(A)$ by $A_{\mathbf{j}}$. The set $G_{\mathcal{S}} = \{S_{\mathbf{j}}, \mathbf{j} \in I^*\}$ is the *semigroup, generated by* $\mathcal{S}$.

The set of all infinite sequences $I^\infty = \{\alpha = \alpha_1\alpha_2\ldots, \alpha_i \in I\}$ is called *the index space* of the system $\mathcal{S}$; the map $\pi : I^\infty \rightarrow K$ called the *index map*, sends each α to the point $\pi(\alpha) = \bigcap_{n=1}^{\infty} K_{\alpha_1\ldots\alpha_n}$. If $\pi(\alpha) = x$, then α is called *an address* of the point x . For each address α of a point $x \in K$, the point $x_n = S_{\alpha_1\ldots\alpha_n}^{-1}(x)$ is called the n -th predecessor of the point x , and the sequence $x_1, x_2, \ldots$ is called the sequence of predecessors of x . Along with the system $\mathcal{S}$ we will consider its n -th refinement $\mathcal{S}^{(n)} = \{S_j, j \in I^n\}$. The Hutchinson operator of the system $\mathcal{S}^{(n)}$ is equal to T^n .

The pair $(K, \mathcal{S})$ is called a self-similar structure. A map $f : K \rightarrow K'$ agrees with the structures $(K, \mathcal{S})$, $(K', \mathcal{S}')$ if for any $x \in K$ and any $i \in I$, $f \circ S_i(x) = S'_i \circ f(x)$. If the map f is a homeomorphism, then it defines the isomorphism of self-similar structures $(K, \mathcal{S})$ and $(K', \mathcal{S}')$.

Definition 1.2. The system $\mathcal{S}$ satisfies the *open set condition* (OSC) if there exists a non-empty open set $O \subset X$ such that the sets $S_i(O)$, $\{1 \leq i \leq m\}$ are pairwise disjoint and are contained in O .

For any $\mathbf{i}, \mathbf{j} \in I^*$, $\mathbf{i} \sqsubset \mathbf{j}$ iff $S_i(O) \supset S_j(O)$ and $\mathbf{i}$ and $\mathbf{j}$ are incomparable, iff $S_i(O) \cap S_j(O) = \emptyset$. The union $\mathcal{C}$ of all intersections $S_i(K) \cap S_j(K)$, $i, j \in I$, $i \neq j$ is called *the critical set* of the system $\mathcal{S}$. The set of all predecessors of the points in $\mathcal{C}$, $\partial K = \{x \in K : \text{for some } \mathbf{j} \in I^*, S_{\mathbf{j}} \in \mathcal{C} \text{ is called the self-similar boundary of the set } K\}$.

The *post-critical set* $\mathcal{P}$ of the system $\mathcal{S}$ is the set of all $\alpha \in I^\infty$ such that $\pi(\alpha) \in \partial K$. A system $\mathcal{S}$ is called *post-critically finite* (PCF) if its post-critical set $\mathcal{P}$ is finite [12]. Thus, if the system $\mathcal{S}$ is post-critically finite then the self-similar boundary ∂K is a finite set $\mathcal{V} = \pi(\mathcal{P})$ such that for any non-comparable $\mathbf{i}, \mathbf{j} \in I^*$, $K_{\mathbf{i}} \cap K_{\mathbf{j}} = S_{\mathbf{i}}(\mathcal{V}) \cap S_{\mathbf{j}}(\mathcal{V})$.

1.2 Dendrites

A *dendrite* is a locally connected continuum containing no simple closed curve [4, 13]. The order $Ord(p, X)$ of the point p with respect to a dendrite X is the number of components of the set $X \setminus \{p\}$. Points of order 1 in a dendrite X are called *end points* of X ; a point $p \in X$ is called a *cut point* of X if $X \setminus \{p\}$ is disconnected; points of order at least 3 are called *ramification points* of X . A continuum X is a dendrite iff X is locally connected and uniquely arcwise connected.

1.3 Contractible polygonal systems

Let $P \subset \mathbb{R}^2$ be a finite polygon homeomorphic to a disk, $\mathcal{V}_P = \{A_1, \ldots, A_{n_P}\}$ be the set of its vertices. Let $\Omega(P, A)$ denote the angle with vertex A in the polygon P . We consider a system of similarities $\mathcal{S} = \{S_1, \ldots, S_m\}$ in $\mathbb{R}^2$ such that:

- (D1) for any $i \in I$ set $P_i = S_i(P) \subset P$;
- (D2) for any $i \neq j$, $i, j \in I$, $P_i \cap P_j = \mathcal{V}_{P_i} \cap \mathcal{V}_{P_j}$ and $\#(\mathcal{V}_{P_i} \cap \mathcal{V}_{P_j}) < 2$;
- (D3) $\mathcal{V}_P \subset \bigcup_{i \in I} S_i(\mathcal{V}_P)$;
- (D4) the set $\tilde{P} = \bigcup_{i=1}^m P_i$ is contractible.

Figure 1: A polygonal system and its intersection graph.

Definition 1.3. The system $\mathcal{S}$ satisfying the conditions **(D1)–(D4)**, is called a *contractible P -polygonal system of similarities*.

Theorem 1.4 ([18, Theorem 12]). *The attractor K of a contractible P -polygonal system of similarities $\mathcal{S}$ is a dendrite.*

It was proved in [16, Theorem 4], that:

1. Every contractible polygonal system satisfies (OSC), where we can take $\dot{P}$ as an open set.
2. $P_j \subset P_i$ iff $j \sqsubset i$, and if $i \sqsubset j$, then $S_i(\mathcal{V}_P) \cap P_j \subset S_j(\mathcal{V}_P)$. If $i, j \in I^*$ are incomparable, then $P_i \cap P_j$ is either empty or is a common vertex of the polygons P_i and P_j .
3. All the vertices of P lie in K , therefore the set $G_{\mathcal{S}}(\mathcal{V}_P)$ of vertices of the polygons P_j is contained in K and dense in K .
4. Every point $x \in K \setminus G_{\mathcal{S}}(\mathcal{V}_P)$ has a unique address.

2 Generalized polygonal systems

2.1 One-point intersection systems and intersection graph

When considering generalized polygonal systems, we will rely on a number of definitions and statements from our paper [20]:

Definition 2.1. Let $\mathcal{A} = \{A_i, i \in I\}$ be a finite system of compact sets such that for any $i \neq j \in I$, $\#A_i \cap A_j \leq 1$. Then $\mathcal{A}$ is a system of sets with one-point intersection (or a FIP1-system of sets).

Let $\mathcal{B}$ be the set of points of pairwise intersection of the sets A_i , and $\mathcal{B}_i = A_i \cap \mathcal{B}$.

Definition 2.2. The intersection graph $\Gamma(\mathcal{A})$ of a FIP1-system of sets $\mathcal{A}$ is a bipartite graph $(\mathcal{A}, \mathcal{B}; E)$ for which $e = \{A_i, B\} \in E$ if and only if $B \in \mathcal{B}_i$.

We call $A_i \in \mathcal{A}$ *white vertices* and $B \in \mathcal{B}$ - *black vertices*. The set $N(A_i)$ of the neighbors of a white vertex A_i is $\mathcal{B}_i$, whereas for a black vertex B , $N(B) = \{A_i : B \in \mathcal{B}_i\}$. Since B is the intersection point of at least two sets A_i , we have $\deg(B) \geq 2$.

Figure 2: A generalized polygonal system and its attractor.

Definitions 2.1 and 2.2 can be applied to the systems of contractions and their attractors. Let $\mathcal{S} = \{S_1, \dots, S_m\}$ be a system of injective contractions in a complete metric space X and K be its attractor. Let $\mathcal{A}(\mathcal{S}) = \{K_1, \dots, K_m\}$ and $\mathcal{A}_n(\mathcal{S}) = \{K_i : i \in I^n\}$.

Definition 2.3. A system of injective contractions $\mathcal{S}$ is called a system with one-point intersection property (or FIP1-system of injective contractions), if the system of sets $\mathcal{A}(\mathcal{S}) = \{S_1(K), \dots, S_m(K)\}$ is a FIP1-system of sets.

Theorem 2.4 ([20, Th.1.7]). *Let $\mathcal{S}$ be a system of injective contraction maps in a complete metric space X such that the intersection graph $\Gamma(\mathcal{S})$ is a tree. Then the attractor K of the system $\mathcal{S}$ is a dendrite.*

2.2 Generalized polygonal systems

If we omit the condition **(D1)** in the definition of contractible P -polygonal system $\mathcal{S}$, we get the definition of a *generalized P -polygonal system*:

Definition 2.5. A system $\mathcal{S} = \{S_1, \dots, S_m\}$ satisfying the conditions **D2-D4**, is called a *generalized P -polygonal system of similarities*.

Theorem 2.6. *Let $\mathcal{S}$ be a generalized P -polygonal system. If*

$$\text{for any } i, j \in I \quad S_i(K) \cap S_j(K) = P_i \cap P_j, \quad (\mathbf{D0})$$

then

- (i) *the attractor K of the system $\mathcal{S}$ is a dendrite;*
- (ii) *the system $\mathcal{S}$ satisfies OSC;*
- (iii) *the set of addresses $\pi^{-1}(x)$ of any point $x \in K$ is finite.*

Remark 1. If the generalized polygonal system $\mathcal{S}$ satisfies Condition **D0**, then $\mathcal{S}$ is a system with a connected attractor K which has one-point intersection property and whose self-similar boundary $\partial K = \mathcal{V}_P$ is finite. Such systems were studied in our paper [20]. It was proved there that for any two incomparable multi-indices $\mathbf{i}, \mathbf{j} \in I^*$, the intersection $K_{\mathbf{i}} \cap K_{\mathbf{j}} = S_{\mathbf{i}}(\mathcal{V}_P) \cap S_{\mathbf{j}}(\mathcal{V}_P)$ is either empty set or a singleton.

Figure 3: Example for the Remark 2

Figure 4: Polygonal system $\mathcal{S}$ and its δ -deformation $\mathcal{S}'$

Proof. (i) Indeed, it follows from formula (D0) that the intersections graphs $\Gamma(\{K_i\})$ and $\Gamma(\{P_i\})$ are isomorphic. Also, it follows from the properties (D2)-(D4) that the intersection graph of generalized polygonal system $\Gamma(\{P_i\})$ is a tree. Then, by Theorem 2.4, K is a dendrite. (ii) Since $\mathcal{S}$ is a system of contracting similarities in $\mathbb{R}^2$ which has finite intersection property and has connected attractor K then, by [2, Theorem 2], it satisfies the Open Set Condition. (iii) Finiteness of the set $\pi^{-1}(x)$ follows from (ii) and [20, Proposition 2.3]. $\square$

Remark 2. It is possible for a generalized P -polygonal system $\mathcal{S}$ not to satisfy the Condition D0 and to have the attractor K which is a dendrite. The attractor K of a generalized polygonal system $\mathcal{S}$ in the picture below is a dendrite, but $P_7 \cap P_9 = \emptyset$, whereas $K_7 \cap K_9$ is a line segment.

2.3 δ -deformations of contractible polygonal systems

Definition 2.7. Let $\delta > 0$. A generalized P' -polygonal system $\mathcal{S}' = \{S'_1, \dots, S'_m\}$ is called a δ -deformation of the P -polygonal system $\mathcal{S} = \{S_1, \dots, S_m\}$,

if there is a bijection $f : \bigcup_{k=1}^m \mathcal{V}_{P_k} \rightarrow \bigcup_{k=1}^m \mathcal{V}_{P'_k}$ such that

a) $f|_{\mathcal{V}_P}$ extends to a homeomorphism $\tilde{f} : P \rightarrow P'$;

b) $|f(x) - x| < \delta$ for any $x \in \bigcup_{k=1}^m \mathcal{V}_{P_k}$;

c) $f(S_k(x)) = S'_k(f(x))$ for any $k \in I$ and $x \in \mathcal{V}_P$.

Since $\hat{f}$ is a homeomorphism of polygons mapping vertices to vertices, we can assume that $\hat{f}$ is a simplicial isomorphism of some triangulation of the polygon P whose vertex set is V_P to equivalent triangulation of P' whose vertex set is $V_{P'}$.

By condition c), if $i, j \in I$, $A_1, A_2 \in \mathcal{V}_P$ and $S_i(A_1) = S_j(A_2)$, then $S'_i(f(A_1)) = S'_j(f(A_2))$.

The same relation is fulfilled in the case when $\mathbf{i}, \mathbf{j} \in I^*$ are multi-indices:

Lemma 2.8. *If $A_1, A_2 \in \mathcal{V}_P$, $\mathbf{i}, \mathbf{j} \in I^*$ and $S_{\mathbf{i}}(A_1) = S_{\mathbf{j}}(A_2)$, then $S'_{\mathbf{i}}(f(A_1)) = S'_{\mathbf{j}}(f(A_2))$.*

Proof. Suppose that $S_{\mathbf{i}}(A) = B \in \mathcal{V}_{\bar{P}}$ for some $A \in \mathcal{V}_P$, and let $\mathbf{i} = i_1 i_2 \dots i_n$. We denote $S_{i_{k+1} \dots i_n}(A)$ by A_k .

Then we have a finite sequence of relations between $B \in \mathcal{V}_{\bar{P}}$ and the vertices $A_k \in \mathcal{V}_P$:

$$B = S_{i_1}(A_1), A_1 = S_{i_2}(A_2), \dots, A_{n-1} = S_{i_n}(A). \quad (2.1)$$

By c), the map f transforms these relations into

$$B' = S'_{i_1}(A'_1), A'_1 = S'_{i_2}(A'_2), \dots, A'_{n-1} = S'_{i_n}(A'). \quad (2.2)$$

This implies $S'_{\mathbf{i}}(A') = B'$. Moreover, if $S_{\mathbf{i}}(A_1) = S_{\mathbf{j}}(A_2) \in \mathcal{V}_{\bar{P}}$, then $S'_{\mathbf{i}}(f(A_1)) = S'_{\mathbf{j}}(f(A_2))$.

Now suppose that $S_{\mathbf{i}}(A_1) = S_{\mathbf{j}}(A_2)$ and $\mathbf{i} = \mathbf{l}\mathbf{i}'$, $\mathbf{j} = \mathbf{l}\mathbf{j}'$ and $S_{\mathbf{i}}(A_1) = S_{\mathbf{j}}(A_2) = S_{\mathbf{l}}(B)$ for some $B \in \mathcal{V}_{\bar{P}}$. Then $S_{\mathbf{i}'}(A_1) = S_{\mathbf{j}'}(A_2) = B$, therefore $S'_{\mathbf{i}'}(f(A_1)) = S'_{\mathbf{j}'}(f(A_2)) = f(B)$ and $S'_{\mathbf{i}}(f(A_1)) = S'_{\mathbf{j}}(f(A_2)) = S'_{\mathbf{l}}(f(B))$. $\square$

Theorem 2.9. *Let S' be a δ -deformation of a contractible P -polygonal system $\mathcal{S}$ defined by the map f and let $\pi : I^\infty \rightarrow K, \pi' : I^\infty \rightarrow K'$ be respective address maps.*

- (i) *f has unique continuous extension $\hat{f} : K \rightarrow K'$ such that $\hat{f} \circ \pi = \pi'$;*
- (ii) *if S' satisfies Condition **D0**, then $\hat{f}$ is a homeomorphism.*

Remark 3. The equality $\hat{f} \circ \pi = \pi'$ holds if and only if for any $z \in K$ and any $\mathbf{i} \in I^*$,

$$\hat{f}(S_{\mathbf{i}}(z)) = S'_{\mathbf{i}}(\hat{f}(z)). \quad (2.3)$$

Proof. The proof is similar to (cf. [1, Lemma 1]). First, we define the function $\hat{f}$ which is a surjection of the dense subset $G_{\mathcal{S}}(\mathcal{V}_P) \subset K$ to the dense subset $G_{\mathcal{S}'}(\mathcal{V}_{P'}) \subset K'$. Second, we show that $\hat{f}$ is Hölder continuous on $G_{\mathcal{S}}(\mathcal{V}_P)$ and therefore has unique continuous extension to a surjection from K to K' , which we denote by the same symbol $\hat{f}$. Thirdly, we show that the Condition **D0** implies that $\hat{f}$ is injective and therefore is a homeomorphism.

1. Define a map $\hat{f}(z) : G_{\mathcal{S}}(\mathcal{V}_P) \rightarrow G_{\mathcal{S}'}(\mathcal{V}_{P'})$ by

$$\hat{f}(z) = S'_{\mathbf{i}}(f(S_{\mathbf{i}}^{-1}(z))), \text{ where } z \in S_{\mathbf{i}}(\mathcal{V}_P). \quad (2.4)$$

As it follows from Lemma 2.8, if $S_{\mathbf{i}}(A_1) = S_{\mathbf{j}}(A_2) = z$, then $S'_{\mathbf{i}}(f(S_{\mathbf{i}}^{-1}(z))) = S'_{\mathbf{j}}(f(S_{\mathbf{j}}^{-1}(z)))$, so the map $\hat{f}$ is well defined.

Obviously $\hat{f}(G_{\mathcal{S}}(\mathcal{V}_P)) = G_{\mathcal{S}'}(\mathcal{V}_{P'})$, because if $A' \in \mathcal{V}_{P'}$ and $z' = S'_{\mathbf{i}}(A')$, then there is a vertex $A = f^{-1}(A') \in \mathcal{V}_P$, therefore $z' = \hat{f}(S_{\mathbf{i}}(A))$.

Moreover, for any $z \in G_{\mathcal{S}}(\mathcal{V}_P)$ and $\mathbf{i} \in I^*$, $\hat{f}(S_{\mathbf{i}}(z)) = S'_{\mathbf{i}}(\hat{f}(z))$ and if $z_1, z_2 \in G_{\mathcal{S}}(\mathcal{V}_P)$, $\mathbf{i}, \mathbf{j} \in I^*$ and $S_{\mathbf{i}}(z_1) = S_{\mathbf{j}}(z_2)$, then $S'_{\mathbf{i}}(\hat{f}(z_1)) = S'_{\mathbf{j}}(\hat{f}(z_2))$.

Figure 5: An example of a generalized polygonal system and its index diagram. It contains 3 equivalent cyclic vertices.

$$2. \text{ Let } q_k = \text{Lip } S_k, q'_k = \text{Lip } S'_k, \beta = \min_{k \in I} \frac{\log q'_k}{\log q_k}.$$

Then, following the proof [18, Theorem 27, step 4], in which we use K' instead of P' , one can see that for any $z_1, z_2 \in G_S(\mathcal{V}_P)$,

$$|z'_1 - z'_2| \leq \frac{2|K'|}{(\rho_0 \cdot \sin(\alpha_0/2))^\beta} |z_1 - z_2|^\beta.$$

Therefore, the map $\hat{f}$ can be extended to a Hölder continuous surjective mapping of the set K in K' . Since for any $z \in K$ and any $k \in I$, $\hat{f}(S_k(z)) = S'_k(\hat{f}(z))$, then $\hat{f} \circ \pi = \pi'$.

3. Now suppose the system $\mathcal{S}'$ satisfies Condition **D0**. Suppose that for some $\sigma = i_1 i_2 \dots \in I^\infty$ and $\tau = j_1 j_2 \dots \in I^\infty$ $\hat{f} \circ \pi(\sigma) = \hat{f} \circ \pi(\tau)$. Then, if $i_1 \neq j_1$, then, by Condition **D0**, $P'_{i_1} \cap P'_{j_1} \neq \emptyset$, resulting in $P_{i_1} \cap P_{j_1} = \{B\}$ for some $B \in \mathcal{V}_{\bar{P}}$ and $\pi(\sigma) = \pi(\tau) = B$.

Let now $\sigma = \mathbf{l}\sigma'$ and $\tau = \mathbf{l}\tau'$, and $\hat{f} \circ \pi(\sigma) = \hat{f} \circ \pi(\tau)$. Then, by the formula 2.3, $\hat{f} \circ \pi(\sigma') = \hat{f} \circ \pi(\tau')$, so if the first indices in σ' and τ' are different then $\pi(\sigma) = \pi(\tau) = S_1(B)$ for some $B \in \mathcal{V}_{\bar{P}}$.

This implies that the mapping $\hat{f}$ is injective. Thus, $\hat{f}$ is a homeomorphism of the compact sets K and K' . $\square$

3 Parameter matching theorem

3.1 Cyclic vertices and the index diagram

Definition 3.1. Let $\mathcal{S} = \{S_i, i \in I\}$ be a generalized P -polygonal system. The index diagram of the system $\mathcal{S}$ is an edge-labeled directed multigraph $\mathcal{G} = (V_P, E, \mu)$, where the vertex set of $\mathcal{G}$ is the set V_P . Given $A, B \in V_P$, there is an edge $e \in E$ which is directed from A to B and is labeled by an index $i \in I$, iff there is S_i such that $S_i(B) = A$. The labeling map $\mu : E \rightarrow I$ is defined by the equation $\mu(e) = i$.

We use the following notation for edges in directed graphs: if e is an edge in a graph $\mathcal{G}$, directed from A to B , then $\alpha(e) = A$ and $\omega(e) = B$. A walk σ in $\mathcal{G}$ is a sequence of edges $e_1 e_2 \dots e_n \dots$ such that $\alpha(e_k) = \omega(e_{k-1})$ for all $k > 1$. The walk starts at $\alpha(\sigma) = \alpha(e_1)$ and if the walk ends at an edge e_n , then $\omega(\sigma) = \omega(e_n)$. For a walk $\sigma = e_1 e_2 \dots e_n \dots$ we define $\mu(\sigma) = \mu(e_1) \mu(e_2) \dots = i_1 i_2 \dots$.

By condition **D3** for each vertex $A \in \mathcal{V}_P$ there is at least one edge starting from A , so the outdegree of each vertex ≥ 1 , and for any vertex $A \in \mathcal{V}_P$ the set of infinite walks $\sigma = e_1 e_2 \dots$, starting from A is nonempty.

Figure 6: Each walk in the index diagram arrives to some cyclic vertex

Figure 7: An example of disconnected index diagram. The vertex 3 is non-cyclic and is subordinate to the vertex 1 by the maps S_7, S_8 and S_9 .

Consider the sequence $A, A_1, \dots, A_n, \dots$ of vertices of the walk σ . By finiteness of V_P , there are k and l such that $A_l = A_{l+k}$. Then $A_l = S_{i_{l+1} \dots i_{l+k}}(A_l)$. In this case A_l is a *cyclic vertex*.

It follows from **D3** that all the predecessors of a vertex $A \in \mathcal{V}_P$ are also vertices of P . That is, if $A \in S_{i_1 \dots i_n}(P)$, then there is a vertex A_n such that $A = S_{i_1 \dots i_n}(A_n)$. Therefore for any two vertices $A, B \in \mathcal{V}_P$ the equality $A = S_i(B)$ holds iff there is a walk $\sigma_{AB} = e_1 e_2 e_3 \dots e_n$ from A to B such that $\mathbf{i} = \mu(\sigma_{AB})$.

Consider some infinite walk $\sigma = e_1 e_2 e_3 \dots$, and let $A_n = \omega(e_n)$ be its vertices. As $A = S_{i_1 \dots i_n}(A_n)$, the sequence $\{A_n\}$ is a sequence of predecessors of the vertex A . Due to equality $A = \lim_{n \rightarrow \infty} S_{i_1 \dots i_n}(P)$, the infinite sequence $\mu(\sigma) = i_1 \dots i_n \dots \in I^\infty$ is an address of the point A defined by the walk σ . Conversely, each address $i_1 \dots i_n, \dots$ of the point A is equal to $\mu(\sigma)$ for some infinite walk σ starting from A .

Definition 3.2. A vertex $B \in V_P$ is called a *cyclic vertex* if there is a cycle $\sigma_B = e_1 \dots e_k$ such that $\alpha(\sigma_B) = \omega(\sigma_B) = B$. The length k of the cycle σ_B is called *the order* of the vertex B . We call B a *basepoint* of the cycle σ_B .

Remark 4. As we show in Proposition 3.5 (i), each cyclic vertex has outdegree 1 and therefore belongs to only one cycle.

The multi-index $\mathbf{j} = \mu(\sigma_B)$ is the shortest of the multi-indices $\mathbf{k}$ satisfying $S_{\mathbf{k}}(B) = B$. Conversely, if B is a vertex of P and $\mathbf{j} = j_1 \dots j_k$ is the shortest multi-index such that $S_{\mathbf{j}}(B) = B$ then $\mathbf{j} = \mu(\sigma_B)$ for some cycle σ_B in $\mathcal{G}$.

Figure 8: A more complicated example of index diagram. The vertex 5 is subordinate to the vertex 2 by the maps S_3 and S_{56} and to the vertex 1 by the maps S_2 and S_{54} .

Definition 3.3. A finite path η in $\mathcal{G}$ is called a pre-cyclic path if $\omega(\eta)$ is a unique cyclic vertex in η . Let $\mathbf{i} = \mu(\eta)$. Then we say the point $A = \alpha(\eta)$ is *subordinate* to the point $B = \omega(\eta)$ by means of the map $S_{\mathbf{i}}$.

In other words, each pre-cyclic path η starting from a vertex A defines a pair $(B, S_{\mathbf{i}}) = (\omega(\eta), S_{\mu(\eta)})$ such that B is a cyclic vertex and $A = S_{\mathbf{i}}(B)$.

Proposition 3.4. Let $\mathcal{S}$ be a generalised P -polygonal system of similarities. For each vertex $A \in \mathcal{V}_P$ there is a finite number of pairs $(B_k, S_{\mathbf{i}_k})$, such that for each k the vertex A is subordinate to B_k by means of the map $S_{\mathbf{i}_k}$.

Proof. Let $A \in \mathcal{V}_P$ be a non-cyclic vertex. Consider the set $\{\eta_1, \dots, \eta_m\}$ of all pre-cyclic paths starting from A . Let $(B_k, S_{\mathbf{i}_k})$ be the pair, defined by η_k . Notice that if for some k, l , $\omega(\eta_k) = \omega(\eta_l)$, then $\mu(\eta_k) \neq \mu(\eta_l)$. Therefore all the pairs $(B_k, S_{\mathbf{i}_k})$ are different. $\square$

In the case when $\mathcal{S}$ is a generalized polygonal system, which satisfies condition (1) and particularly in the case when $\mathcal{S}$ is a contractible polygonal system, the vertex set $\mathcal{V}_P$ and the index diagram $\mathcal{G}(\mathcal{S})$ have the following properties:

Proposition 3.5. Let $\mathcal{S}$ be a generalized polygonal system satisfying the condition (1), then:

- (i) All the cyclic vertices of the index diagram $\mathcal{G}(\mathcal{S})$ have the outdegree equal to 1.
- (ii) Each cyclic vertex B is a basepoint of a unique cycle σ_B .
- (iii) There is n such that all cyclic vertices of the system $\mathcal{S}^{(n)} = \{S_{\mathbf{i}}, \mathbf{i} \in I^n\}$ have the order 1.

Proof. Let the outdegree of some cyclic vertex B be greater than 1. This implies that there is a cycle σ_B in $\mathcal{G}$ with a basepoint B and an edge $e_1 : \alpha(e_1) = B$ such that $e_1 \notin \sigma_B$. Consider an infinite walk $\tau = e_1 e_2 e_3 \dots$, starting from B . Then all the walks $\sigma_B^n \tau$ start from B and are pairwise different, therefore B has infinite set of addresses. Since a contractible polygonal system has finite intersection property and satisfies OSC, it follows from [20, Theorem 1.7], that the set of addresses of each vertex B is finite. The contradiction obtained proves (i).

Let B be a cyclic vertex and let τ be the periodic walk generated by the cycle σ_B . By (i) it is the only infinite walk originating from B , which proves (ii). Moreover, $\mu(\tau)$ is the unique address of the point B .

Figure 9: The decomposition (3.1) of the attractor K at the cyclic vertex B (left) and the decomposition (3.2) of the standard neighborhood U_A for a non-cyclic vertex A (right). Notice that the order of the point B is 3, and the order of the point A is 9.

Let $\mathcal{G}$ be the index diagram of the system $\mathcal{S}$. Since $\mathcal{S}^{(n)}$ is a generalized polygonal system satisfying condition (1), the set of vertices of its index diagram $\mathcal{G}^n$ is also $\mathcal{V}_P$, and the edges e' going from A to B correspond to the walks $\eta' = e_1 \dots e_n$ of length n for which $\alpha(\eta') = A$, $\omega(\eta') = B$. Let n be the least common multiple of the orders of all cyclic vertices. Then for every walk η of length n outgoing from a cyclic vertex B , $\omega(\eta) = B$. Therefore, the order of any cyclic vertex in the system $\mathcal{S}^{(n)}$ is equal to 1. $\square$

3.2 The structure of neighborhoods of points of the attractor of a contractible polygonal system

1. Cyclic vertices. Let $B \in \mathcal{V}_P$ be a cyclic vertex of the contractible polygonal system $\mathcal{S}$ and let σ_B be the cycle with the basepoint B . Let $\mathbf{j} = \mu(\sigma_B)$. Then the similarity $S_{\mathbf{j}}$ is a homothety, and the angle Ω_B formed by the sides of P adjacent to B contains P . Moreover, assuming $W = K \setminus S_{\mathbf{j}}(K)$ we obtain a decomposition of the set K to a disjoint union

$$K = \{B\} \cup \bigsqcup_{n=0}^{\infty} S_{\mathbf{j}}^n(W). \quad (3.1)$$

2. Non-cyclic vertices. Let $A \in V_P$ be a non-cyclic vertex. Let $\{\eta_1, \dots, \eta_n\}$ be the set of all pre-cyclic paths starting from A and $\{(B_k, S_{\mathbf{i}_k}), k = 1, \dots, n\}$ be the corresponding set of pairs $(B_k, S_{\mathbf{i}_k})$.

Notice that if $k \neq l$, then $K_{\mathbf{i}_k} \cap K_{\mathbf{i}_l} = \{A\}$. Suppose contrary. Then by Remark 1, one of the multi-indices, say $\mathbf{i}_k$, satisfies $\mathbf{i}_k \sqsubset \mathbf{i}_l$. Then η_k is a subpath of η_l , which is impossible because η_l is pre-cyclic.

The sets $S_{\mathbf{i}_k}(K \setminus B_k)$ are pairwise disjoint and lie inside pairwise disjoint angles $S_{\mathbf{i}_k}(\Omega_{B_k})$ for which A is the common vertex.

The set $U_A = \bigcup_{k=1}^n S_{\mathbf{i}_k}(K)$ is a neighborhood of A in K . It is called the *standard neighborhood* of the vertex A .

Let σ_k be the cycle whose basepoint is B_k and let $\mathbf{j}_k = \mu(\sigma_k)$. Assuming $W_k =$

Figure 10: The decomposition (3.2) of the standard neighborhood U_C of a point $C \in G_S(\mathcal{V}_P)$. The order of the point C is 24.

$K \setminus S_{j_k}(K)$, we get the *canonical decomposition of the standard neighborhood* U_A :

$$U_A = \{A\} \cup \bigcup_{k=1}^n S_{i_k} \bigcup_{l=0}^{\infty} S_{j_k}^l(W_k) \quad (3.2)$$

3. The points of the set $G_S(\mathcal{V}_P)$. Consider the point $A \in G_S(\mathcal{V}_P)$.

Definition 3.6. We say that $A \in G_S(\mathcal{V}_P)$ is subordinate to a cyclic vertex B by means of a map $S_i = S_{i_1 \dots i_k}$ if $A = S_{i_1 \dots i_k}(B)$ and for any $l < k$ the point $S_{i_{l+1} \dots i_k}(B)$ is not a cyclic vertex.

Proposition 3.7. For any point $A \in G_S(\mathcal{V}_P)$ there is a unique finite set of pairs $\mathcal{U}_A = \{(B_k, S_{i_k}), k = 1, \dots, N_A\}$ such that the point A is subordinate to a cyclic vertex B by means of a map S_i if and only if the pair (B, S_i) lies in $\mathcal{U}_A$.

For any non-equal $k, l \leq N_A$, $S_{i_k}(K) \cap S_{i_l}(K) = \{A\}$.

The set $U_A = \bigcup_{k=1}^{N_A} S_{i_k}(K)$ is a neighborhood of A in K ; it admits the decomposition (3.2).

3.3 Parameter matching theorem

Let A be a cyclic vertex of a generalized P -polygonal system $\mathcal{S}$. In this case the map S_i for which $S_i(A) = A$, need not be a homothety and we have to define the rotation parameter for such map. Though the rotation angle α_i of the map S_i is formally defined up to $2n\pi$, in the case of polygonal systems the integer n is uniquely defined by the set $\tilde{P}$ and depends on its geometric configuration.

Proposition 3.8. Let $\mathcal{S}$ be a generalized P -polygonal system satisfying Condition **D0** and let A be a cyclic vertex of the polygon P . Then there is a vertex $B \in V_P$ and a multi-index $\mathbf{i} \in I^*$ such that $S_{\mathbf{i}}(A) = A$ and the Jordan arc $\gamma_{AB} \subset K$ satisfies the inclusion $S_{\mathbf{i}}(\gamma_{AB}) \subset \gamma_{AB}$.

Proof. Let $\mathbf{j}$ be the shortest multi-index for which $A = S_{\mathbf{j}}(A)$. Let $W = K \setminus S_{\mathbf{j}}(K)$, then

$$K = \{A\} \cup \bigcup_{n=0}^{\infty} S_{\mathbf{j}}^n(W).$$

Let $Q = S_{\mathbf{j}}^{-1}(\bar{W} \cap S_{\mathbf{j}}(K))$. From the Remark 1 we see that $Q \subset \mathcal{V}_P \setminus \{A\}$.

The vertex A cannot belong to Q . Otherwise, there would be a piece K_i such that $K_i \cap K_j = \{A\}$, so for any k, l , $S_{\mathbf{j}}^k(K_i) \cap S_{\mathbf{j}}^l(K_i) = \{A\}$. In this case A would be a infinite order ramification point in K , which is impossible by Theorem 2.6.

Since K is a dendrite, for any $B \in Q$ there is a unique arc $\gamma_{AB} \subset K$. Let $\gamma_{B'B}$ be the smallest of those subarcs of γ_{AB} for which $B' \in K_j$. Then $B' \in S_{\mathbf{j}}(Q)$. Define a map $\psi : Q \rightarrow Q$ by the formula $\psi(B) = S_{\mathbf{j}}^{-1}(B')$. Then for any $B \in Q$, $S_{\mathbf{j}}(\gamma_{A\psi(B)}) \subset \gamma_{AB}$. Further, for any n , $S_{\mathbf{j}}^n(\gamma_{A\psi^n(B)}) \subset \gamma_{AB}$.

Thus ψ is a mapping of a finite set Q to itself. There is n and $B \in Q$, such that $\psi^n(B) = B$. Then $S_{\mathbf{j}}^n(\gamma_{AB}) \subset \gamma_{AB}$. So we put $S_i = S_{\mathbf{j}}^n$. $\square$

Definition 3.9. The arc γ_{AB} is called an *invariant arc* of the cyclic vertex A .

Let A be a cyclic vertex and γ_{AB} be its invariant arc and $S_i(A) = A$. Let $B' = S_i(B)$. We denote by α the total increase of the argument of $z - A$ as z travels along γ_{AB} from B to B' . This gives a unique representation $S_i(z) = q_i e^{i\alpha}(z - A) + A$.

Remark 5. The following picture shows how the angle α depends on the geometric configuration of the system $\mathcal{S}$, though the similarity which fixes A and sends B to B' is the same.

Definition 3.10. The number $\lambda_A = \frac{\alpha}{\ln q_i}$ is called the parameter of the cyclic vertex A .

Definition 3.11. Generalized P -polygonal system $\mathcal{S}$ of similarities satisfies the *Parameter Matching Condition*, if for any $B \in \bigcup_{i=1}^m \mathcal{V}_{P_i}$ and for any cyclic vertices A, A' such that for some $\mathbf{i}, \mathbf{j} \in I^*$, $S_i(A) = S_j(A') = B$, the equality $\lambda_A = \lambda_{A'}$ holds.

From Propositions 3.4 and 3.8 and V.V.Aseev's Lemma on disjoint periodic arcs [1, Lemma 3.1] we come to the following Parameter Matching Theorem:

Theorem 3.12. Let $\mathcal{S}$ be a generalized P -polygonal system whose attractor K is a dendrite. Then the system $\mathcal{S}$ satisfies the *Parameter Matching Condition*.

Proof. Let $\mathcal{S}$ be a generalized polygonal system whose attractor K is a dendrite. Let $C \in \bigcup_{i=1}^m \mathcal{V}_{P_i}$ and $A, A' \in \mathcal{V}_P$ be such cyclic vertices that for some $i, j \in I$, $S_i(A) = S_j(A') = C$. Denote the images $S_i(K)$ and $S_j(K)$ by K_i, K_j respectively. Without loss of generality we can suppose that the point C has coordinate 0 in $\mathbb{C}$. Since for some

Figure 11: The choice of parameters α_0 , ρ_0 , ρ_1 and ρ_2 for the polygonal system.

$i, j \in I^*$, $S_i(A) = A$ and $S_j(A') = A'$, the maps $S_{b1} = S_i S_i^{-1}$ and $S_{b2} = S_j S_j^{-1}$ have C as their fixed point and $S_{b1}(K_i) \subset K_i$ and $S_{b2}(K_j) \subset K_j$. Let $S_{b1}(z) = q_i e^{i\alpha_i} z$ and $S_{b2}(z) = q_j e^{i\alpha_j} z$. So the parameters of the vertices A and A' will be $\lambda_1 = \frac{\alpha_i}{\log q_i}$ and $\lambda_2 = \frac{\alpha_j}{\log q_j}$. Let $\gamma_{AB} \subset K$ and $\gamma_{A'B'} \subset K$ be invariant arcs for the vertices A and A' . Let also $\gamma_1 = S_i(\gamma_{AB})$ and $\gamma_2 = S_j(\gamma_{A'B'})$. Then $S_{b1}(\gamma_1) \subset \gamma_1$ and $S_{b2}(\gamma_2) \subset \gamma_2$. From [1, Lemma 3.1] it follows that if $\gamma_1 \cap \gamma_2 = \{C\}$, then $\lambda_1 = \lambda_2$. $\square$

4 Small deformation theorem

4.1 Main parameters of a contractible polygonal system

For any set $X \subset \mathbb{R}^2$ or point A by $V_\varepsilon(X)$ (resp. $V_\varepsilon(A)$) we denote ε -neighborhood of the set X (resp. of the point A) in the plane.

Parameter ρ_0 : By $\rho_0 > 0$, we denote the smallest of all distances between the points of non-intersecting polygons P_i, P_j and distances between the vertices $A \in \mathcal{V}_P$ and the points of polygons $P_i \not\ni A$:

- (i) for any vertex $A \in \mathcal{V}_P$, $V_\rho(A) \cap P_k \neq \emptyset \Rightarrow A \in P_k$;
- (ii) for any $x, y \in P$, such that there are $P_k, P_l : x \in P_k, y \in P_l$ and $P_k \cap P_l = \emptyset$, $d(x, y) \geq \rho_0$.

Parameters ρ_1, ρ_2 : Let $C \in \mathcal{V}_{\bar{P}}$ be a non-cyclic vertex. Let U_C be its standard neighborhood and $\tilde{U}_C = \bigcup_{l=1}^k S_{i_l}(W_l)$. Then ρ_1 and ρ_2 are chosen so that for any $C \in \mathcal{V}_{\bar{P}}$, the set $\tilde{U}_C$ is contained in the ring $\rho_1 < |z - C| \leq \rho_2$.

Parameter α_0 : α_0 is the smallest possible angle between those sides of the polygons $P_i, P_j, i, j \in I$, which have a common vertex.

Notation for maps of cyclic vertices. In the case when $\mathcal{S}$ is a contractible P -polygonal system all of whose cyclic vertices are of order 1, we order the indices in I and enumerate the vertices in $\mathcal{V}_P$ in such a way that each cyclic vertex A_l corresponds to a homothety $S_l(z) = q_l(z - A_l) + A_l$. In this case, K is within the angle $\Omega(P, A_l)$ and $K \setminus \{A_l\} = \bigcup_{n=0}^{\infty} S_l^n(W_l)$.

4.2 Estimate of δ and Main Theorem

Initial assumptions. Let $\mathcal{S} = \{S_1, \dots, S_m\}$ be a contractible P -polygonal system, and the map f defines a δ -deformation of the system $\mathcal{S}$ to a generalized P' -polygonal system $\mathcal{S}' = \{S'_1, \dots, S'_m\}$, where $S_k(z) = q_k e^{i\alpha_k}(z - z_k) + z_k$ and $S'_k(z) = q'_k e^{i\alpha'_k}(z - z'_k) + z'_k$. We assume that $\text{diam } P = 1$, and $\delta > 0$ is such that

$$\delta < q_{\min}/8 \quad \text{and} \quad \delta < (1 - q_{\max})/8. \quad (4.1)$$

The estimates for $\Delta q_k = |q'_k - q_k|$ and $\Delta \alpha_k = |\alpha'_k - \alpha_k|$ under the deformation f are given by the following Lemma.

Lemma 4.1. For any $k \in I$,

$$\Delta q_k < 3\delta \quad \text{and} \quad \Delta \alpha_k < C_\alpha \delta, \text{ where } C_\alpha = 2.1(1 + 1/q_{\min}). \quad (4.2)$$

Proof. Choose vertices A, B of the polygon P such that $|B - A| = 1$. For the images of A we use the notation $A_k = S_k(A)$, $A' = f(A)$ and $A'_k = f(A_k) = S'_k(A')$, and similar notation for the vertex B .

We estimate the increments $\Delta q_k = |q'_k - q_k|$ and $\Delta \alpha_k = |\alpha'_k - \alpha_k|$ for

$$\frac{B_k - A_k}{B - A} = q_k e^{i\alpha_k} \quad \text{and} \quad \frac{B'_k - A'_k}{B' - A'} = q'_k e^{i\alpha'_k}.$$

Since f is a δ -deformation, $|(B_k - A_k)| - 2\delta \leq |(B'_k - A'_k)| \leq |(B_k - A_k)| + 2\delta$. This implies

$$3q_{\min}/5 < \frac{q_{\min} - 2\delta}{1 + 2\delta} < \frac{q_k - 2\delta}{1 + 2\delta} \leq q'_k \leq \frac{q_k + 2\delta}{1 - 2\delta} < \frac{q_{\max} + 2\delta}{1 - 2\delta} < \frac{1 + 3q_{\max}}{3 + q_{\max}}. \quad (4.3)$$

Since

$$\alpha'_i - \alpha_i = \arg \frac{B'_k - A'_k}{B_k - A_k} - \arg \frac{B' - A'}{B - A}, \quad (4.4)$$

we get

$$|\alpha'_k - \alpha_k| \leq \arcsin 2\delta + \arcsin \frac{2\delta}{q_k}. \quad (4.5)$$

Substituting the inequalities (4.1) to (4.3) and (4.5) and taking into account the inequality $0 < \arcsin x < 1.05x$, which holds for any $0 < x < 0.5$, we obtain the estimates

$$\Delta q_k = |q'_k - q_k| < \frac{2\delta(1 + q_k)}{1 - 2\delta} < 3\delta \quad \text{and} \quad \Delta \alpha_k = |\alpha'_k - \alpha_k| < C_\alpha \delta, \quad (4.6)$$

where $C_\alpha = 2.1(1 + 1/q_{\min})$. □

Let $V_\delta(P)$ denote the δ -neighborhood of the polygon P .

Lemma 4.2. Let $\delta_1 = \frac{4\delta}{1 - q_{\max}}$, and $U = V_{\delta_1}(P)$. Then

(1) For any $k \in I$, $S_k(U) \subset U$ and $S'_k(U) \subset U$.

(2) For any $z \in U$, $|S'_k(z) - S_k(z)| < C_\Delta \delta$, where $C_\Delta = 6.5 + 1.5C_\alpha$.

Proof. (1) By Definition 2.7, $V_\delta(P_k) \supset P'_k$, $V_\delta(P'_k) \supset P_k$, $V_\delta(P) \supset P'$ and $V_\delta(P') \supset P$.

Thus, $S'_k(P') \subset V_\delta(P_k) \subset V_\delta(P)$. Therefore $S'_k(P) \subset V_{2\delta}(P_k) \subset V_{2\delta}(P)$.

Since S'_k is a similarity, $S'_k(V_\rho(P)) \subset V_{2\delta+q'_k\rho}(P)$ for any positive ρ .

If $\rho \geq \frac{2\delta}{1-q'_k}$, then $2\delta + q'_k\rho \leq \rho$. Therefore $S'_k(V_\rho(P)) \subset V_\rho(P)$.

From the inequality $q'_k < \frac{q_{max} + 2\delta}{1 - 2\delta}$ it follows that $\frac{2\delta}{1 - q'_k} \leq \frac{2\delta(1 - 2\delta)}{1 - q_{max} - 4\delta} < \frac{4\delta}{1 - q_{max}}$.

The latter implies (1). Moreover, due to (4.1), $\delta_1 < 1/2$.

(2) Take $z \in U$ and consider the difference $S'_k(z) - S_k(z)$. It can be represented in the form $S'_k(A) - S_k(A) + (q'_k e^{i\alpha'_k} - q_k e^{i\alpha_k})(z - A)$. Therefore

$$|S'_k(z) - S_k(z)| < |S'_k(A) - S_k(A)| + (|q'_k - q_k| + q_k |e^{i\alpha'_k} - e^{i\alpha_k}|) |z - A|. \quad (4.7)$$

Since $|z - A| < 1 + \delta_1 < 1.5$, $|S'_k(A) - S_k(A)| < 2\delta$, and $|e^{i\alpha'_k} - e^{i\alpha_k}| \leq |\alpha'_k - \alpha_k|$, the right hand side of (4.7) is no greater than $2\delta + 1.5(3\delta + C_\alpha\delta)$. $\square$

Applying the Displacement Theorem [10, Theorem 17] to $\mathcal{S}, \mathcal{S}'$ and U we obtain the following statement.

Proposition 4.3. *Let π, π' be the address maps for the systems $\mathcal{S}$ and $\mathcal{S}'$ respectively.*

1. *For any $\sigma \in I^\infty$,*

$$|\pi'(\sigma) - \pi(\sigma)| < C_K \delta \text{ where } C_K = \frac{2C_\Delta}{1 - q_{max}}. \quad (4.8)$$

2. *If the system $\mathcal{S}'^{(n)}$ satisfies D2–D4, then it is a $(C_K\delta)$ -deformation of the system $\mathcal{S}^{(n)}$.*

Remark 6. Let $\mathcal{S}' = \{S'_1, \dots, S'_m\}$ be a δ -deformation of the contractible P -polygonal system $\mathcal{S}$. Let $A \in S_j(\mathcal{V}_P)$ for some $j \in I$. Let $g(z) = z - A + A'$ and $S''_k = g \circ S'_k \circ g^{-1}$. Then $\mathcal{S}'' = \{S''_1, \dots, S''_m\}$ is a 2δ -deformation of the system $\mathcal{S}$, for which $A'' = A$, $K'' = g(K')$, $P''_j = g(P_j)$. Since g is a translation, the estimates (4.1) and (4.2) for $\mathcal{S}''$ remain the same with the same δ , while $|\pi''(\sigma) - \pi(\sigma)| < (C_K + 1)\delta$. Thus we will denote $\delta_2 = (C_K + 1)\delta$.

Taking into account the Propositions 3.4 and 4.3, it is sufficient to prove the Theorem for the case when all cyclic vertices of the system $\mathcal{S}$ have order 1.

Proposition 4.4. *Let the initial assumptions be fulfilled, and let $S_k(z) = q_k(z - A_k) + A_k$ be the homothety fixing $A_k \in \mathcal{V}_P$. Then the parameter λ_k of the similarity S'_k satisfies the inequality*

$$|\lambda_k| < C_\lambda \delta, \text{ where } C_\lambda = \frac{2.1(1 + 1/q_{max})}{\log(3 + q_{max}) - \log(3q_{max} + 1)}. \quad (4.9)$$

Proof. From Lemma 4.1 we have

$$|\lambda_k| \leq \frac{\arcsin 2\delta + \arcsin \frac{2\delta}{q_k}}{|\log(q_k + 2\delta) - \log(1 - 2\delta)|}. \quad (4.10)$$

Given the inequalities (4.3) and (4.5), we get (4.9). $\square$

Lemma 4.5. *Let $\mathcal{S}$ be a contractible P -polygonal system whose cyclic vertices have order 1 and $\mathcal{S}'$ be its δ -deformation. Then if*

$$2.1 \frac{\delta_2}{\rho_1} + \lambda \log \frac{\rho_2 + \delta_2}{\rho_1 - \delta_2} < \alpha_0 \text{ and } 2\delta_2 < \rho_0, \quad (4.11)$$

then the system $\mathcal{S}'$ satisfies condition (D0).

Remark 7. On the assumption that $\delta_2 < \rho_1/4$, and $\delta_2 < (1 - \rho_2)/4$, the inequality (4.11) holds if

$$2.1 \frac{\delta_2}{\rho_1} + \lambda \log \frac{1 + 3\rho_2}{3\rho_1} < \alpha_0.$$

Proof. Take a vertex $B \in V_{\bar{P}}$. Taking into account Remark 6, we can assume that $B = 0$ and $f(0) = 0$, so $B' = B = 0$.

The decomposition of a standard neighborhood U_B of the point B has the form

$$U_B = \{B\} \cup \bigsqcup_{l=1}^k S_{j_l} \bigsqcup_{n=0}^{\infty} S_l^n(W_l). \quad (4.12)$$

The maps $\bar{S}_l = S_{j_l} S_{i_l} S_{j_l}^{-1}$ are homotheties with a fixed point B such that

$$K_{j_l} \setminus \{B\} = \bigsqcup_{n=0}^{\infty} \bar{S}_l^n(W_l). \quad (4.13)$$

Similarly, let $W'_l = \hat{f}(W_l)$ and $\bar{S}'_l = S'_{j_l} S'_{i_l} S'_{j_l}{}^{-1}$. Then

$$K'_{j_l} \setminus \{B\} = \bigsqcup_{n=0}^{\infty} \bar{S}'_l{}^n(W'_l). \quad (4.14)$$

Notice that for any l , $\bar{S}_l(z) = q_{i_l} z$ and $\bar{S}'_l(z) = q'_{i_l} e^{i\alpha_{i_l}} z$, and due to Parameter Matching Condition, there is such λ , that for any l , $\alpha_{i_l} = \lambda \log q'_{i_l}$.

Consider the map $z = \exp(w)$ of the plane ($w = \varrho + i\varphi$) as a universal cover of the punctured plane $\mathbb{C} \setminus \{0\}$.

Consider the polygons P_{j_l} and choose their liftings in the plane ($w = \varrho + i\varphi$). We may suppose that these liftings lie in respective horizontal strips $\theta_l^- \leq \varphi \leq \theta_l^+$, where $0 < \theta_l^- < \theta_l^+ < 2\pi$ and $\theta_l^+ + \alpha_0 < \theta_{l+1}^-$ for any $l < k$ and $\theta_k^+ + \alpha_0 < \theta_1^- + 2\pi$. We also consider liftings of K_{j_l} , W_l , K'_{j_l} and W'_l . We denote these liftings by $\mathcal{K}_{j_l}$, $\mathcal{W}_l$, $\mathcal{K}'_{j_l}$ and $\mathcal{W}'_l$.

It follows from the equations (4.13) and (4.14) that

$$\mathcal{K}_{j_l} = \bigsqcup_{n=0}^{\infty} \bar{T}_l^n(W_l) \quad \text{and} \quad \mathcal{K}'_{j_l} = \bigsqcup_{n=0}^{\infty} \bar{T}'_l{}^n(W'_l), \quad (4.15)$$

where $T_l(w) = w + \log q_l$ and $T'_l(w) = w + (1 + i\lambda) \log q'_l$ are parallel translations for which $T_l(\mathcal{K}_l) \subset \mathcal{K}_l$ and $T'_l(\mathcal{K}'_l) \subset \mathcal{K}'_l$.

The sets $\mathcal{K}_l$ lie in the half-strips $\varrho \leq \log \rho_2$, $\theta_l^- \leq \varphi \leq \theta_l^+$, while the sets $\mathcal{W}_l$ are contained in the rectangles $R_l = \{\log \rho_1 \leq \varrho \leq \log \rho_2, \theta_l^- \leq \varphi \leq \theta_l^+\}$.

Figure 12: The images of the set K' under the map $w = \log(z - O)$ and the map $w = \log(z - B)$

Then the sets W'_l lie in the rectangle

$$R'_l = \left\{ \log(\rho_1 - \delta_2) \leq \varrho \leq \log(\rho_2 + \delta_2), \theta_l^- - 1.05 \frac{\delta_2}{\rho_1} \leq \varphi \leq \theta_l^+ + 1.05 \frac{\delta_2}{\rho_1} \right\}. \quad (4.16)$$

Each union $\bigcup_{n=0}^{\infty} T_l'^n(R'_l)$ lies in a half-strip

$$\begin{cases} \varrho \leq \log(\rho_2 + \delta_2) \\ \theta_l^- - 1.05 \frac{\delta_2}{\rho_1} - \lambda \log(\rho_2 + \delta_2) \leq \varphi - \lambda \varrho \leq \theta_l^+ + 1.05 \frac{\delta_2}{\rho_1} - \lambda \log(\rho_1 - \delta_2). \end{cases} \quad (4.17)$$

Therefore, the set $\mathcal{K}'_{j_l}$ also lies in this half-strip. So if

$$\theta_{l-1}^+ + 1.05 \frac{\delta_2}{\rho_1} - \lambda \log(\rho_1 - \delta_2) < \theta_l^- - 1.05 \frac{\delta_2}{\rho_1} - \lambda \log(\rho_2 + \delta_2), \quad (4.18)$$

then $\mathcal{K}'_{j_{l-1}} \cap \mathcal{K}'_{j_l} = \emptyset$.

The inequality (4.18) holds for any l if $2.1 \frac{\delta_2}{\rho_1} + \lambda \log \frac{\rho_2 + \delta_2}{\rho_1 - \delta_2} < \alpha_0$.

If, moreover, $2\delta_2 < \rho_0$, then for any $i_1, i_2 \in I$ such that $P_{i_1} \cap P_{i_2} = \emptyset$, $P'_{i_1} \cap P'_{i_2} = \emptyset$ and $K'_{i_1} \cap K'_{i_2} = \emptyset$. This implies that condition (D0) is fulfilled. $\square$

Theorem 4.6. *Let $\mathcal{S}$ be a contractible P -polygonal system. There is $\delta > 0$ such that for any δ -deformation $\mathcal{S}'$ of the system $\mathcal{S}$ satisfying parameter matching condition the attractor $K(\mathcal{S}')$ is a dendrite, homeomorphic to $K(\mathcal{S})$.*

Proof. Let all the cyclic vertices of the P -polygonal system $\mathcal{S}$ have order 1. If we combine inequalities 4.2, 4.8, 4.9, 4.11 and take into account Remark 7, we see that if the following inequality holds:

$$\delta < \min \left(\frac{\min(q_{\min}, 1 - q_{\max})}{8}, \frac{\min(\rho_0, \rho_1, 1 - \rho_2)}{2(C_K + 1)}, \frac{\alpha_0}{\frac{2.1(C_K + 1)}{\rho_1} + C_\lambda \log \frac{1 + 3\rho_2}{3\rho_1}} \right), \quad (4.19)$$

then the attractor K' of a δ -deformation S' of the system S satisfies condition **(D0)**. Therefore K' is a dendrite. By Theorem 2.9, the map $\hat{f} : K \rightarrow K'$ is a bijection and therefore it is a homeomorphism.

Suppose now that S has cyclic vertices of order greater than 1. There is such n , that all the cyclic vertices of the system $S^{(n)}$ have order 1. Suppose any δ -deformation of the system $S^{(n)}$ generates a dendrite. Then for any δ/C_K -deformation S' of the system S , the system $S'^{(n)}$ is a δ -deformation of the system $S^{(n)}$. $\square$

ORCID iDs

Dmitry Drozdov <https://orcid.org/0000-0002-8099-808X>

Mary Samuel <https://orcid.org/0000-0001-9645-1006>

Andrei Tetenov <https://orcid.org/0000-0001-5419-5710>

References

- [1] V. V. Aseev, A. V. Tetenov and A. S. Kravchenko, Self-similar Jordan curves on the plane, *Sibirsk. Mat. Zh.* **44** (2003), 481–492, doi:10.1023/a:1023848327898.
- [2] C. Bandt and H. Rao, Topology and separation of self-similar fractals in the plane, *Nonlinearity* **20** (2007), 1463–1474, doi:10.1088/0951-7715/20/6/008.
- [3] C. Bandt and J. Stahnke, Self-similar sets 6. interior distance in deterministic fractals, 1990, preprint, Greifswald.
- [4] J. J. Charatonik and W. J. Charatonik, Dendrites, in: *XXX National Congress of the Mexican Mathematical Society (Spanish) (Aguascalientes, 1997)*, Soc. Mat. Mexicana, México, volume 22 of *Aportaciones Mat. Comun.*, pp. 227–253, 1998, https://www.researchgate.net/publication/285354319_Dendrites.
- [5] L. L. Cristea and B. Steinsky, Curves of infinite length in 4×4 -labyrinth fractals, *Geom. Dedicata* **141** (2009), 1–17, doi:10.1007/s10711-008-9340-3.
- [6] L. L. Cristea and B. Steinsky, Curves of infinite length in labyrinth fractals, *Proc. Edinb. Math. Soc. (2)* **54** (2011), 329–344, doi:10.1017/s0013091509000169.
- [7] L. L. Cristea and B. Steinsky, Mixed labyrinth fractals, *Topology Appl.* **229** (2017), 112–125, doi:10.1016/j.topol.2017.06.022.
- [8] M. Hata, On the structure of self-similar sets, *Japan J. Appl. Math.* **2** (1985), 381–414, doi:10.1007/bf03167083.
- [9] J. E. Hutchinson, Fractals and self-similarity, *Indiana Univ. Math. J.* **30** (1981), 713–747, doi:10.1512/iumj.1981.30.30055.
- [10] K. G. Kamalutdinov and A. V. Tetenov, Twofold Cantor sets in $\mathbb{R}$, *Sib. Èlektron. Mat. Izv.* **15** (2018), 801–814, doi:10.17377/semi.2018.15.066.
- [11] J. Kigami, Harmonic calculus on limits of networks and its application to dendrites, *J. Funct. Anal.* **128** (1995), 48–86, doi:10.1006/jfan.1995.1023.
- [12] J. Kigami, *Analysis on fractals*, volume 143 of *Cambridge Tracts in Mathematics*, Cambridge University Press, Cambridge, 2001, doi:10.1017/cbo9780511470943.
- [13] K. Kuratowski, *Topology. Vol. I & 2*, Academic Press, New York-London; Państwowe Wydawnictwo Naukowe, Warsaw, 1966, doi:10.1016/C2013-0-11023-9.
- [14] K.-S. Lau, J. J. Luo and H. Rao, Topological structure of fractal squares, *Math. Proc. Cambridge Philos. Soc.* **155** (2013), 73–86, doi:10.1017/s0305004112000692.

- [15] M. Samuel, A. Tetenov and D. Mekhontsev, On dendrites generated by symmetric polygonal systems: The case of regular polygons, in: *Advances in Algebra and Analysis. Trends in Mathematics*, Springer, volume 1, pp. 17–25, 2019, doi:10.1007/978-3-030-01120-8_4.
- [16] M. Samuel, A. Tetenov and D. Vaulin, On dendrites generated by polyhedral systems and their ramification points, *Proc. Krasovskii Inst. Math. Mech. UB RAS* **23** (2017), 281–291, doi:10.21538/0134-4889-2017-23-4-281-291.
- [17] M. Samuel, A. Tetenov and D. Vaulin, On dendrites generated by polyhedral systems and their ramification points, 2017, <https://arxiv.org/abs/1707.02875v1>.
- [18] M. Samuel, A. Tetenov and D. Vaulin, Self-similar dendrites generated by polygonal systems in the plane, *Sib. Elektron. Mat. Izv.* **14** (2017), 737–751, doi:10.17377/semi.2017.14.063.
- [19] R. S. Strichartz, Isoperimetric estimates on Sierpinski gasket type fractals, *Trans. Amer. Math. Soc.* **351** (1999), 1705–1752, doi:10.1090/s0002-9947-99-01999-6.
- [20] A. Tetenov, Finiteness properties for self-similar sets, 2020, <https://arxiv.org/abs/2003.04202v1>.