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C_4-face-magic toroidal labelings on $C_m \times C_n$

Stephen J. Curran
Department of Mathematics, University of Pittsburgh at Johnstown, 450 Schoolhouse Rd, Johnstown, PA 15904 USA

Richard M. Low
Department of Mathematics, San Jose State University, 1 Washington Sq, San Jose, CA 95192 USA

Stephen C. Locke
Department of Mathematical Sciences, Florida Atlantic University, 777 Glades Rd, Boca Raton, FL 33431 USA

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Abstract

For a graph $G = (V, E)$ naturally embedded in the torus, let $\mathcal{F}(G)$ denote the set of faces of G. Then, G is called a C_n-face-magic toroidal graph if there exists a bijection $f : V(G) \to \{1, 2, \ldots, |V(G)|\}$ such that for every $F \in \mathcal{F}(G)$ with $F \cong C_n$, the sum of all the vertex labels along C_n is a constant S. Let $x_v = f(v)$ for all $v \in V(G)$. We call $\{x_v : v \in V(G)\}$ a C_n-face-magic toroidal labeling on G. We show that, for all $m, n \geq 2$, $C_m \times C_n$ admits a C_4-face-magic toroidal labeling if and only if either $m = 2$, or $n = 2$, or both m and n are even. We say that a C_4-face-magic toroidal labeling $\{x_{i,j} : (i, j) \in V(C_{2m} \times C_{2n})\}$ on $C_{2m} \times C_{2n}$ is antipodal balanced if $x_{i,j} + x_{i+m,j+n} = \frac{1}{2}S$, for all $(i, j) \in V(C_{2m} \times C_{2n})$. We show that there exists an antipodal balanced C_4-face-magic toroidal labeling on $C_{2m} \times C_{2n}$ if and only if the parity of m and n are the same. Furthermore, when both m and n are even, an antipodal balanced C_4-face-magic toroidal labeling on $C_{2m} \times C_{2n}$ is both row-sum balanced and column-sum balanced. In addition, when $m = n$ is even, an antipodal balanced C_4-face-magic toroidal labeling on $C_{2n} \times C_{2n}$ is diagonal-sum balanced.

Keywords: C_4-face-magic graphs, polyomino, toroidal graphs, Cartesian products of cycles.

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E-mail addresses: sjcurran@pitt.edu (Stephen J. Curran), richard.low@sjsu.edu (Richard M. Low), lockes@fau.edu (Stephen C. Locke)

1 Introduction

Kotzig and Rosa [11] formally introduced graph labelings in the 1970s. There are applications of graph labelings to graph decomposition problems, radar pulse code designs, X-ray crystallography and communication network models. The interested reader should read J.A. Gallian's comprehensive dynamic survey on graph labelings [8] for further investigation.

We refer the reader to Bondy and Murty [5] for concepts and notation not explicitly defined in this paper. All graphs in this paper are simple and connected. For a planar graph $G = (V, E)$ embedded in $\mathbb{R}^2$, let $\mathcal{F}(G)$ denote the set of faces of G. Then, G is called a C_n*-face-magic* graph if there exists a bijection $f : V(G) \to \{1, 2, \ldots, |V(G)|\}$ such that for every $F \in \mathcal{F}(G)$ with $F \cong C_n$, the sum of all the vertex labels along C_n is a constant S. Here, the constant S is called a C_n*-face-magic value* of G. A C_n*-face-magic toroidal* (or *cylindrical*) graph G is defined similarly, where G is embedded in the torus (or cylinder), respectively. C_n-face-magic graph labelings are a special case of the more general (a, b, c)-magic labeling introduced by Lih [12]. For assorted values of a, b and c, Baca and others [1, 2, 3, 4, 9, 10, 12] have analyzed the problem for various classes of graphs. Wang [13] showed that the toroidal grid graphs $C_m \times C_n$ are antimagic for all integers $m, n \geq 3$. Butt et al. [6] investigated face antimagic labelings on toroidal and Klein bottle grid graphs.

In this paper, we investigate C_4-face-magic toroidal labelings on $C_m \times C_n$ with its natural embedding in the torus. We show that for all $m, n \geq 2$, there exists a C_4-face-magic toroidal labeling on $C_m \times C_n$ if and only if either $m = 2$, or $n = 2$, or both m and n are even. In the case when $m = n$, we say that a C_4-face-magic toroidal labeling on $C_{2n} \times C_{2n}$ is *torus symmetric* if the labeling is row-sum balanced, column-sum balanced and diagonal-sum balanced. Curran and Low [7] show that, up to symmetries on the torus, there are only three torus symmetric C_4-face-magic toroidal labelings on $C_4 \times C_4$. See Theorem 3.5 in Section 3 for details. In this paper, we search for C_4-face-magic toroidal labelings on $C_{2m} \times C_{2n}$ that are row-sum balanced and column-sum balanced. This investigation leads naturally to the concept of an antipodal balanced labeling. We say that a C_4-face-magic toroidal labeling $\{x_{i,j} : (i, j) \in V(C_{2m} \times C_{2n})\}$ on $C_{2m} \times C_{2n}$ is *antipodal balanced* if $x_{i,j} + x_{i+m,j+n} = \frac{1}{2}S$, for all $(i, j) \in V(C_{2m} \times C_{2n})$. We show that there exists an antipodal balanced C_4-face-magic toroidal labeling on $C_{2m} \times C_{2n}$ if and only if the parity of m and n are the same. Furthermore, when $m = n$ is even, we show that any antipodal balanced C_4-face-magic toroidal labeling on $C_{2n} \times C_{2n}$ is torus symmetric.

2 Preliminaries

Theorem 2.1. *Let $m, n \geq 2$. Then, $P_m \times P_n$ is C_4-face-magic.*

Proof. Label the vertex set of $P_m \times P_n$ as

$$V(P_m \times P_n) = \{(i, j) : 1 \leq i \leq m, 1 \leq j \leq n\}$$

and its edge set as

$$E(P_m \times P_n) = \{\{(i, j), (i + 1, j)\} : 1 \leq i < m, 1 \leq j \leq n\} \cup \{\{(i, j), (i, j + 1)\} : 1 \leq i \leq m, 1 \leq j < n\}.$$

We will determine a label $x_{i,j}$ for each vertex $(i, j) \in V(P_m \times P_n)$ and check that this provides a C_4-face-magic labeling.

Case 1. Assume $m-n$ is even. Color the vertex (i,j) white if $i+j$ is even and black if $i+j$ is odd. Note that the vertices $(1,1)$ and (m,n) are white. Let $x_{i,j} = i + m\,(j-1)$ for each white vertex (i,j), and $x_{i,j} = (m-i+1)+m(n-j)$ for each black vertex (i,j). An equivalent definition for $\{x_{i,j} : (i,j) \in V(P_m \times P_n)\}$ would be to write the number $i + m\,(j-1)$ in each cell (i,j), and then rotate the black vertices 180 degrees about the center of the board.

Let $C_{i,j} = \{(i,j), (i+1,j), (i+1,j+1), (i,j+1)\}$. If (i,j) is white, then the two 4-cycles $C_{i,j}$ and $C_{i+1,j}$ have the same face sum, since $x_{i,j} = x_{i+2,j} - 2$ and $x_{i,j+1} = x_{i+2,j+1} + 2$. If (i,j) is black, then the two 4-cycles $C_{i,j}$ and $C_{i+1,j}$ have the same face sum, since $x_{i,j} = x_{i+2,j} + 2$ and $x_{i,j+1} = x_{i+2,j+1} - 2$. A similar proof for $C_{i,j}$ and $C_{i,j+1}$ shows that $\{x_{i,j}\}$ is C_4- face-magic. The sum on each face must be the sum on $C_{1,1}$, which is $1 + (mn-1) + (m+2) + m\,(n-1) = 2mn + 2$. This completes Case 1.

Case 2. Without loss of generality, we may assume that m is even and n is odd. Let $m = 2m_1$ and $n = 2n_1 - 1$ for some positive integers m_1 and n_1. Again, color vertex (i,j) white if $i+j$ is even and black if $i+j$ is odd.

We first label the white vertices. Let $x_{i,j} = m(j-1) + i$ if both i and j are odd, and $x_{i,j} = m(j-1) + i - 1$ if both i and j are even. We observe that $x_{2k-1,2\ell-1} = m(2\ell-2) + 2k - 1$ for all $1 \le k \le m_1$ and $1 \le \ell \le n_1$, and $x_{2k,2\ell} = m(2\ell-1) + 2k - 1$ for all $1 \le k \le m_1$ and $1 \le \ell < n_1$. Thus each odd label $1, 3, 5, \ldots, mn-1$ is used exactly once on a white vertex.

Next, we label the black vertices. Let $x_{i,j} = m(n-j+1) - i + 1$ if i is odd and j is even, and $x_{i,j} = m(n-j+1) - i + 2$ if i is even and j is odd. We observe that whenever vertex (i,j) is white, then vertex $(m-i+1, n-j+1)$ is black and $x_{m-i+1,n-j+1} = x_{i,j} + 1$. Thus each even label $2, 4, 6, \ldots, mn$ is used exactly once on a black vertex.

Let $C_{i,j} = \{(i,j), (i+1,j), (i+1,j+1), (i,j+1)\}$. If (i,j) is white, then $x_{i+2,j} = x_{i,j} + 2$ and $x_{i,j+2} = x_{i,j} + 2m$. If (i,j) is black, then $x_{i+2,j} = x_{i,j} - 2$ and $x_{i,j+2} = x_{i,j} - 2m$. An argument similar to that in Case (i) shows that each 4-cycle $C_{i,j}$ has the same face sum as $C_{1,1}$, which is $1 + mn + (m+1) + m(n-1) = 2mn + 2$. This completes Case 2. □

Lemma 2.2. *Let m and n be positive integers. A C_4-face-magic labeling on $P_{2m} \times P_{2n}$ always yields a C_4-face-magic labeling on $C_{2m} \times C_{2n}$ with its natural embedding in the torus. Furthermore, the C_4-face-magic value is $S = 2(4mn+1)$.*

Proof. Let $x_{i,j}$ be the C_4-face-magic labeling on vertex (i,j), for $i = 1, 2, \ldots, 2m$ and $j = 1, 2, \ldots, 2n$. Since S is the C_4-face-magic value on $P_{2m} \times P_{2n}$, we have $x_{i,j} + x_{i+1,j} + x_{i,j+1} + x_{i+1,j+1} = S$, for all $i = 1, 2, \ldots, 2m-1$ and $j = 1, 2, \ldots, 2n-1$. We observe that

$$mnS = \sum_{i=1}^{m} \sum_{j=1}^{n} (x_{2i-1,2j-1} + x_{2i,2j-1} + x_{2i-1,2j} + x_{2i,2j}) = \sum_{i=1}^{4mn} i = \tfrac{1}{2}(4mn)(4mn+1).$$

Thus, $S = 2(4mn+1)$. Since $x_{i,j} + x_{i,j+1} + x_{i+1,j} + x_{i+1,j+1} = S = x_{i+1,j} + x_{i+2,j} + x_{i+1,j+1} + x_{i+2,j+1}$, we have $x_{i,j} + x_{i,j+1} = x_{i+2,j} + x_{i+2,j+1}$. An induction argument shows that $x_{i,j} + x_{i,j+1} = x_{i+2k,j} + x_{i+2k,j+1}$. Since $x_{i+2k,j} + x_{i+2k,j+1} + x_{i+2k+1,j} + x_{i+2k+1,j+1} = S$, we have $x_{i,j} + x_{i,j+1} + x_{i+2k+1,j} + x_{i+2k+1,j+1} = S$.

A similar argument shows that $x_{i,j} + x_{i+1,j} + x_{i,j+2\ell+1} + x_{i+1,j+2\ell+1} = S$. This yields $x_{i,j} + x_{i+2k+1,j} + x_{i,j+2\ell+1} + x_{i+2k+1,j+2\ell+1} = S$. Hence, we have $x_{1,j} + x_{1,j+1} + x_{2m,j} + x_{2m,j+1} = S$, for all $j = 1, 2, \ldots, 2n-1$. Similarly, we have $x_{i,1} + x_{i+1,1} + x_{i,2n} + x_{i+1,2n} = S$, for all $i = 1, 2, \ldots, 2m-1$. Lastly, we have $x_{1,1} + x_{2m,1} + x_{1,2n} + x_{2m,2n} = S$. Therefore, the C_4-face-magic labeling on $P_{2m} \times P_{2n}$ yields a C_4-face-magic labeling on $C_{2m} \times C_{2n}$ with its natural embedding in the torus. □

Lemma 2.3. *Let m and n be integers such that $m \geq 3$ and $n \geq 2$. Suppose $P_m \times C_n$ is a C_4-face-magic cylindrical graph with the natural embedding of $P_m \times C_n$ on the cylinder. Then, n is even.*

Proof. For the purpose of contradiction, suppose n is odd, and let $n = 2n_1 + 1$ for some positive integer n_1. Label the vertex set of $P_m \times C_n$ as

$$V(P_m \times C_n) = \{(i,j) : 1 \leq i \leq m, 1 \leq j \leq n\}$$

and its edge set as

$$\begin{aligned} E(P_m \times C_n) = &\{\{(i,j), (i+1,j)\} : 1 \leq i < m, 1 \leq j \leq n\} \\ &\cup \{\{(i,j), (i,j+1)\}, \{(i,n), (i,1)\} : 1 \leq i \leq m, 1 \leq j < n\}. \end{aligned}$$

Let $\{x_{i,j} : (i,j) \in V(P_m \times C_n)\}$ be a C_4-face-magic labeling on $P_m \times C_n$ with C_4-face-magic value S. Let $S_i = x_{i,1} + x_{i+1,1}$, for $i = 1, 2$. Equating the following C_4-face sums to each other:

$$x_{i,j} + x_{i+1,j} + x_{i,j+1} + x_{i+1,j+1} = S = x_{i,j+1} + x_{i+1,j+1} + x_{i,j+2} + x_{i+1,j+2},$$

we obtain $x_{i,j} + x_{i+1,j} = x_{i,j+2} + x_{i+1,j+2}$, where the index j is taken modulo n. Thus,

$$\begin{aligned} &x_{i,1} + x_{i+1,1} = x_{i,2j+1} + x_{i+1,2j+1} \quad \text{and} \\ &x_{i,2} + x_{i+1,2} = x_{i,2j} + x_{i+1,2j}, \text{ for } j = 1, 2, \ldots, n_1. \end{aligned}$$

Also, we have $x_{i,n-1} + x_{i+1,n-1} = x_{i,n+1} + x_{i+1,n+1} = x_{i,1} + x_{i+1,1}$. Hence, $S_i = x_{i,j} + x_{i+1,j}$, for all $i = 1, 2$ and $j = 1, 2, \ldots, n$. From the C_4-face sum

$$S = (x_{i,j} + x_{i+1,j}) + (x_{i,j+1} + x_{i+1,j+1}) = S_i + S_i = 2S_i,$$

we have $S_i = \frac{1}{2}S$. Hence, $x_{1,1} + x_{2,1} = \frac{1}{2}S = x_{2,1} + x_{3,1}$, which in turn, implies that $x_{1,1} = x_{3,1}$. This is a contradiction. Therefore, n is even. □

Proposition 2.4. *Let m be an integer where $m \geq 2$. Then, there is a C_4-face-magic toroidal labeling on $C_m \times C_2$.*

Proof. Let $x_{i,1} = i$ and $x_{i,2} = 2m+1-i$, for $i = 1, 2, \ldots, m$. Then, $x_{i,1} + x_{i,2} = 2m+1$, for $i = 1, 2, \ldots, m$. Thus, $x_{i,1} + x_{i,2} + x_{i+1,1} + x_{i+1,2} = 2(2m+1)$, for $i = 1, 2, \ldots, m$. Hence, $\{x_{i,j} : (i,j) \in V(C_m \times C_2)\}$ is a C_4-face-magic toroidal labeling on $C_m \times C_2$. □

Proposition 2.5. *Let m and n be integers where $m, n \geq 2$. Then, $C_m \times C_n$ has a C_4-face-magic toroidal labeling if and only if either $m = 2$, or $n = 2$, or both m and n are even.*

Proof. ($\Rightarrow$) Suppose $C_m \times C_n$ has a C_4-face-magic toroidal labeling. If either $m = 2$ or $n = 2$, we are done. So assume that $m, n \geq 3$. The C_4-face-magic toroidal labeling on $C_m \times C_n$ is simultaneously a C_4-face-magic cylindrical labeling on both $C_m \times P_n$ and $P_m \times C_n$. By Lemma 2.3, both m and n are even.

($\Leftarrow$) Suppose either $m = 2$, or $n = 2$, or both m and n are even. On the one hand, if $m = 2$ or $n = 2$, by Proposition 2.4, $C_m \times C_n$ has a C_4-face-magic toroidal labeling. On the other hand, if both m and n are even, by Theorem 2.1, $P_m \times P_n$ has a C_4-face-magic labeling. By Lemma 2.2, the C_4-face-magic labeling on $P_m \times P_n$ yields a C_4-face-magic toroidal labeling on $C_m \times C_n$. □

Throughout this paper, if $\{x_{i,j} : (i,j) \in V(C_{2m} \times C_{2n})\}$ is a labeling on $C_{2m} \times C_{2n}$, then for convenience we consider the index i modulo $2m$ and the index j modulo $2n$.

Definition 2.6. We say that the C_4-face-magic torus labeling $\{x_{i,j} : i = 1, 2, \ldots, 2m$ and $j = 1, 2, \ldots, 2n\}$ on $C_{2m} \times C_{2n}$ is *antipodal balanced* if $x_{i,j} + x_{i+m,j+n} = 4mn + 1$, for all integers i and j such that $1 \leq i \leq 2m$ and $1 \leq j \leq 2n$.

Remark 2.7. We give a brief explanation for the term *antipodal balanced*. On the n-sphere $S^n \subsetneqq \mathbb{R}^{n+1}$, the antipodal map $p : S^n \to S^n$ is given by $p(\mathbf{x}) = -\mathbf{x}$. Similarly, on the torus $T^2 = S^1 \times S^1 \subsetneqq \mathbb{C}^2 \cong \mathbb{R}^4$, we define the antipodal map $p : T^2 \to T^2$ by $p(e^{i\theta_1}, e^{i\theta_2}) = -(e^{i\theta_1}, e^{i\theta_2}) = (e^{i(\theta_1+\pi)}, e^{i(\theta_2+\pi)})$. Thus an antipodal balanced C_4-face-magic toroidal labeling on $C_{2m} \times C_{2n}$ is one in which the sum of the labels at a vertex and its antipodal vertex is constant for all vertices in $C_{2m} \times C_{2n}$.

Lemma 2.8. *Let m and n be positive integers. Let $\{x_{i,j} : i = 1, 2, \ldots, 2m$ and $j = 1, 2, \ldots, 2n\}$ be an antipodal balanced C_4-face-magic toroidal labeling on $C_{2m} \times C_{2n}$. For all integers i where $1 \leq i \leq m$, we define*

$$d_i = x_{i(m-1)+1,in+1} - x_{(i-1)(m-1)+1,(i-1)n+1}.$$

Then for all integers i and j where $1 \leq i \leq m$ and $1 \leq j \leq 2n$, we have

$$x_{i(m-1)+1,in+j} = x_{(i-1)(m-1)+1,(i-1)n+j} + (-1)^{j+1} d_i.$$

Proof. By the definition of d_i, we have

$$x_{i(m-1)+1,in+1} = x_{(i-1)(m-1)+1,(i-1)n+1} + d_i.$$

We apply an induction argument on j. Thus we assume that

$$x_{i(m-1)+1,in+j-1} = x_{(i-1)(m-1)+1,(i-1)n+j-1} + (-1)^j d_i. \tag{2.1}$$

Since the labeling is antipodal balanced, we have

$$x_{i(m-1)+2,in+j-1} = \tfrac{1}{2}S - x_{(i-1)(m-1)+1,(i-1)n+j-1}, \text{ and} \tag{2.2}$$
$$x_{i(m-1)+2,in+j} = \tfrac{1}{2}S - x_{(i-1)(m-1)+1,(i-1)n+j}. \tag{2.3}$$

Since $\{x_{i,j}\}$ is a C_4-face-magic toroidal labeling on $C_{2m} \times C_{2n}$, we have

$$x_{i(m-1)+1,in+j-1} + x_{i(m-1)+1,in+j} + x_{i(m-1)+2,in+j-1} + x_{i(m-1)+2,in+j} = S. \tag{2.4}$$

When we substitute the expressions from equations (2.1), (2.2) and (2.3) into equation (2.4), we obtain

$$x_{i(m-1)+1,in+j} = x_{(i-1)(m-1)+1,(i-1)n+j} + (-1)^{j+1} d_i.$$

This completes the proof. □

We next show that if $C_{2m} \times C_{2n}$ has an antipodal balanced C_4-face-magic toroidal labeling, then the parity of m and n are the same.

Lemma 2.9. *Let m and n be positive integers. Let $\{x_{i,j} : i = 1, 2, \ldots, 2m$ and $j = 1, 2, \ldots, 2n\}$ be an antipodal balanced C_4-face-magic toroidal labeling on $C_{2m} \times C_{2n}$. Then, the parity of m and n are the same.*

Proof. We may assume that m is even and n is odd. Let $\{x_{i,j}\}$ be an antipodal balanced C_4-face-magic toroidal labeling on $C_{2m} \times C_{2n}$. We will show that this leads to a contradiction. For all integers i such that $1 \le i \le m$, we define

$$d_i = x_{i(m-1)+1,in+1} - x_{(i-1)(m-1)+1,(i-1)n+1}.$$

By Lemma 2.8, for all integers i and j such that $1 \le i \le m$ and $1 \le j \le 2n$, we have

$$x_{i(m-1)+1,in+j} = x_{(i-1)(m-1)+1,(i-1)n+j} + (-1)^{j+1} d_i.$$

Fix j such that $1 \le j \le 2n$. The equations

$$x_{i(m-1)+1,in+j} = x_{(i-1)(m-1)+1,(i-1)n+j} + (-1)^{j+1} d_i \text{ for } i = 1, 2, \ldots, m,$$

yield

$$x_{m+1,j} = x_{m(m-1)+1,mn+j} = x_{1,j} + (-1)^{j+1}\big(d_1 + d_2 + \cdots + d_m\big). \tag{2.5}$$

Setting $j = 1$, equation (2.5) becomes

$$x_{m+1,1} = x_{1,1} + \big(d_1 + d_2 + \cdots + d_m\big). \tag{2.6}$$

Setting $j = n + 1$, equation (2.5) becomes

$$x_{m+1,n+1} = x_{1,n+1} - \big(d_1 + d_2 + \cdots + d_m\big). \tag{2.7}$$

Because $\{x_{i,j}\}$ is antipodal balanced, we have

$$x_{1,1} + x_{m+1,n+1} = \tfrac{1}{2} S = x_{m+1,1} + x_{1,n+1}. \tag{2.8}$$

Substituting equations (2.6) and (2.7) into equation (2.8) and simplifying yields

$$d_1 + d_2 + \cdots + d_m = 0. \tag{2.9}$$

This implies that $x_{m+1,j} = x_{1,j}$, for all $j = 1, 2, \ldots, 2n$. This is a contradiction. □

We next investigate how the conditions of Lemma 2.8 apply to the case $C_{2m} \times C_{2n}$ where both m and n are even.

Lemma 2.10. *Let m and n be positive even integers. Let $\{x_{i,j} : i = 1, 2, \ldots, 2m$ and $j = 1, 2, \ldots, 2n\}$ be an antipodal balanced C_4-face-magic toroidal labeling on $C_{2m} \times C_{2n}$. For all integers i such that $1 \le i \le m$, we define*

$$d_i = x_{i(m-1)+1,in+1} - x_{(i-1)(m-1)+1,(i-1)n+1}.$$

Then by Lemma 2.8, for all integers i and j such that $1 \le i \le m$ and $1 \le j \le 2n$, we have

$$x_{i(m-1)+1,in+j} = x_{(i-1)(m-1)+1,(i-1)n+j} + (-1)^{j+1} d_i.$$

Also, for all integers i and j such that $m+1 \le i \le 2m$ and $1 \le j \le 2n$, we have

$$x_{i(m-1)+1,in+j} = x_{(i-1)(m-1)+1,(i-1)n+j} + (-1)^{j} d_{i-m}.$$

For all integers j such that $1 \le j \le n$, we define

$$d'_j = x_{jm+1,j(n-1)+1} - x_{(j-1)m+1,(j-1)(n-1)+1}.$$

Then for all integers i and j such that $1 \le i \le 2m$ and $1 \le j \le n$, we have

$$x_{jm+i,j(n-1)+1} = x_{(j-1)m+i,(j-1)(n-1)+1} + (-1)^{i+1} d'_j.$$

Also, for all integers i and j such that $1 \le i \le 2m$ and $n+1 \le j \le 2n$, we have

$$x_{jm+i,j(n-1)+1} = x_{(j-1)m+i,(j-1)(n-1)+1} + (-1)^{i} d'_{j-n}.$$

Proof. By Lemma 2.8, for all integers i and j such that $1 \le i \le m$ and $1 \le j \le 2n$, we have

$$x_{i(m-1)+1,in+j} = x_{(i-1)(m-1)+1,(i-1)n+j} + (-1)^{j+1} d_i. \tag{2.10}$$

Since the indices $in+j$ and $(i-1)n+j$ are reduced modulo $2n$, equation (2.10) holds for all integers j. Let i be an integer such that $m+1 \le i \le 2m$. We replace i with $i-m$ and j with $n+j$ in equation (2.10) to obtain

$$x_{(i-m)(m-1)+1,(i-m)n+n+j} = x_{(i-m-1)(m-1)+1,(i-m-1)n+n+j} + (-1)^{n+j+1} d_{i-m}.$$

This reduces to

$$x_{i(m-1)+m+1,(i+1)n+j} = x_{(i-1)(m-1)+m+1,in+j} + (-1)^{j+1} d_{i-m}. \tag{2.11}$$

Since $\{x_{i,j}\}$ is antipodal balanced, we have

$$x_{i(m-1)+m+1,(i+1)n+j} = \tfrac{1}{2}S - x_{i(m-1)+1,in+j} \text{ and} \tag{2.12}$$

$$x_{(i-1)(m-1)+m+1,in+j} = \tfrac{1}{2}S - x_{(i-1)(m-1)+1,(i-1)n+j}. \tag{2.13}$$

When we substitute the expressions in equations (2.12) and (2.13) into equation (2.11), we have, for all integers i and j such that $m+1 \le i \le 2m$ and $1 \le j \le 2n$,

$$x_{i(m-1)+1,in+j} = x_{(i-1)(m-1)+1,(i-1)n+j} + (-1)^{j} d_{i-m}. \tag{2.14}$$

When we interchange the roles of i and j in the previous argument, when $1 \le i \le 2m$ and $1 \le j \le n$, we have

$$x_{jm+i,j(n-1)+1} = x_{(j-1)m+i,(j-1)(n-1)+1} + (-1)^{i+1} d'_j;$$

and when $1 \le i \le 2m$ and $n+1 \le j \le 2n$, we have

$$x_{jm+i,j(n-1)+1} = x_{(j-1)m+i,(j-1)(n-1)+1} + (-1)^{i} d'_{j-n}.$$

□

We next investigate how the conditions of Lemma 2.8 apply to the case $C_{2m} \times C_{2n}$ where both m and n are odd.

Lemma 2.11. *Let m and n be positive odd integers. Let $\{x_{i,j} : i = 1, 2, \ldots, 2m$ and $j = 1, 2, \ldots, 2n\}$ be an antipodal balanced C_4-face-magic toroidal labeling on $C_{2m} \times C_{2n}$. For all integers i such that $1 \le i \le m$, we define*

$$d_i = x_{i(m-1)+1,in+1} - x_{(i-1)(m-1)+1,(i-1)n+1}.$$

Then, for all integers i and j such that $1 \le i \le m$ and $1 \le j \le 2n$, we have

$$x_{i(m-1)+1,in+j} = x_{(i-1)(m-1)+1,(i-1)n+j} + (-1)^{j+1} d_i \text{ and}$$
$$x_{i(m-1)+m+1,in+j} = x_{(i-1)(m-1)+m+1,(i-1)n+j} + (-1)^{j+1} d_i.$$

For all integers i such that $1 \le j \le n$, we define

$$d'_j = x_{jm+1,j(n-1)+1} - x_{(j-1)m+1,(j-1)(n-1)+1}.$$

Then, for all integers i and j such that $1 \le i \le 2m$ and $1 \le j \le n$, we have

$$x_{jm+i,j(n-1)+1} = x_{(j-1)m+i,(j-1)(n-1)+1} + (-1)^{i+1} d'_j \text{ and}$$
$$x_{jm+i,j(n-1)+n+1} = x_{(j-1)m+i,(j-1)(n-1)+n+1} + (-1)^{i} d'_j.$$

Proof. By Lemma 2.8, for all integers i and j such that $1 \le i \le m$ and $1 \le j \le 2n$, we have

$$x_{i(m-1)+1,in+j} = x_{(i-1)(m-1)+1,(i-1)n+j} + (-1)^{j+1} d_i. \tag{2.15}$$

Because $in + j$ and $(i-1)n + j$ are reduced modulo $2n$, equation (2.15) holds for all integers j. Because $\{x_{i,j}\}$ is antipodal balanced, we have

$$x_{i(m-1)+1,in+j} = \tfrac{1}{2}S - x_{i(m-1)+m+1,(i+1)n+j} \text{ and} \tag{2.16}$$
$$x_{(i-1)(m-1)+1,(i-1)n+j} = \tfrac{1}{2}S - x_{(i-1)(m-1)+m+1,in+j}. \tag{2.17}$$

When we substitute the expressions from equations (2.16) and (2.17) into equation (2.15), we obtain

$$x_{i(m-1)+m+1,(i+1)n+j} = x_{(i-1)(m-1)+m+1,in+j} + (-1)^{j} d_i. \tag{2.18}$$

When we replace j with $j+n$ in equation (2.18), we have, for all integers i and j such that $1 \le i \le m$ and $1 \le j \le 2n$,

$$x_{i(m-1)+m+1,in+j} = x_{(i-1)(m-1)+m+1,(i-1)n+j} + (-1)^{j+1} d_i.$$

When we interchange the roles of i and j in the above argument, we have for all integers i and j such that $1 \le i \le 2m$ and $1 \le j \le n$,

$$x_{jm+i,j(n-1)+1} = x_{(j-1)m+i,(j-1)(n-1)+1} + (-1)^{i+1} d'_j \text{ and}$$
$$x_{jm+i,j(n-1)+n+1} = x_{(j-1)m+i,(j-1)(n-1)+n+1} + (-1)^{i} d'_j.$$

□

3 Results on $C_4 \times C_4$

Curran and Low [7] determine all antipodal balanced C_4-face-magic toroidal labelings on $C_4 \times C_4$ (up to symmetries on a torus). In order to state this result precisely, we must introduce some definitions. This result, as stated in Theorem 3.5, is the basis for the investigation of antipodal balanced C_4-face-magic toroidal labelings on $C_{2m} \times C_{2n}$ in this paper.

Definition 3.1. Let n be a positive integer Let $\{x_{i,j} : i, j = 1, 2, \ldots, 2n\}$ be a C_4-face-magic torus labeling on $C_{2n} \times C_{2n}$. We say that the C_4-face-magic labeling $\{x_{i,j}\}$ on $C_{2n} \times C_{2n}$ is *torus symmetric* if all row sums, column sums, and diagonal sums have a constant value S'. In other words, the sums

$$R_i = \sum_{j=1}^{2n} x_{i,j} = S' \qquad \text{for all } i = 1, 2, \ldots, 2n,$$

$$C_j = \sum_{i=1}^{2n} x_{i,j} = S' \qquad \text{for all } j = 1, 2, \ldots, 2n,$$

$$D_j = \sum_{i=1}^{2n} x_{i,i+j} = S' \qquad \text{for all } j = 1, 2, \ldots, 2n \text{ and}$$

$$D'_j = \sum_{i=1}^{2n} x_{i,j-i} = S' \qquad \text{for all } j = 1, 2, \ldots, 2n,$$

are constant.

According to Lemma 3.2, a torus symmetric C_4-face-magic toroidal labeling on $C_4 \times C_4$ is equivalent to a C_4-face-magic toroidal labeling on $C_4 \times C_4$ in which the four 2×2 block sums given by $B_{i,j} = x_{i,j} + x_{i,j+2} + x_{i+2,j} + x_{i+2,j+2}$, for all $i, j = 1, 2$, also add up to the C_4-face-magic value 34.

Lemma 3.2 ([7], Lemma 8)**.** *Consider the system of linear equations $x_{i,j} + x_{i+1,j} + x_{i,j+1} + x_{i+1,j+1} = 34$ (S1), for all $i = 1, 2, 3$ and $j = 1, 2, 3$ for a C_4-face-magic labeling on $P_4 \times P_4$. Let (S2) be the system S1 together with the equations $B_{i,j} = x_{i,j} + x_{i,j+2} + x_{i+2,j} + x_{i+2,j+2} = 34$, for all $i, j = 1, 2$. If the labeling $\{x_{i,j}\}$ satisfies system (S2), then $\{x_{i,j}\}$ is torus symmetric. Also, let (S3) be the system (S1) together with the equations $R_1 = x_{1,1} + x_{1,2} + x_{1,3} + x_{1,4} = 34$, $C_1 = x_{1,1} + x_{2,1} + x_{3,1} + x_{4,1} = 34$, and $D_4 = x_{1,1} + x_{2,2} + x_{3,3} + x_{4,4} = 34$. Then, (S2) is equivalent to (S3).*

Definition 3.3. Consider the natural embedding of $C_{2n} \times C_{2n}$ in the torus. We say that two torus symmetric C_4-face-magic toroidal labelings on $C_{2n} \times C_{2n}$ are *torus equivalent* if there is a homeomorphism of the torus that maps $C_{2n} \times C_{2n}$ onto itself such that the first C_4-face-magic toroidal labeling on $C_{2n} \times C_{2n}$ is mapped to the second C_4-face-magic toroidal labeling on $C_{2n} \times C_{2n}$.

By Lemma 3.4, a torus symmetric C_4-face-magic toroidal labeling on $C_4 \times C_4$ is antipodal balanced.

Lemma 3.4 ([7], Lemma 13)**.** *Let $\{x_{i,j}\}$ be a torus symmetric C_4-face-magic toroidal labeling on $C_4 \times C_4$. Then, for all i and j, we have $x_{i,j} + x_{i+2,j+2} = 17$ where the indices are taken modulo four. In other words, the labeling $\{x_{i,j}\}$ is antipodal balanced.*

All torus symmetric C_4-face-magic labelings on $C_4 \times C_4$ (up to torus equivalence) are given in the following theorem.

Theorem 3.5 ([7], Theorem 10)**.** *There are three distinct torus nonequivalent torus symmetric C_4-face-magic toroidal labelings on $C_4 \times C_4$. These three distinct torus nonequivalent torus symmetric C_4-face-magic toroidal labelings on $C_4 \times C_4$ are given below:*

1	8	13	12
14	11	2	7
4	5	16	9
15	10	3	6

Table 1: Torus symmetric C_4-face-magic toroidal labeling A on $C_4 \times C_4$.

1	8	11	14
12	13	2	7
6	3	16	9
15	10	5	4

Table 2: Torus symmetric C_4-face-magic toroidal labeling B on $C_4 \times C_4$.

1	12	7	14
8	13	2	11
10	3	16	5
15	6	9	4

Table 3: Torus symmetric C_4-face-magic toroidal labeling C on $C_4 \times C_4$.

An interesting observation about these three labelings is that the row sums and column sums of any two labelings are the 16 C_4-face sums in the third labeling. In the remark below, we indicate how the three torus nonequivalent torus symmetric C_4-face-magic toroidal labelings in Theorem 3.5 can be regarded as coming from one particular labeling on $C_4 \times C_4$.

Remark 3.6 ([7], Remark 24)**.** Label the vertices of $C_4 \times C_4$ with the elements from the set $\{0, 1\}^4$ so that the labelings on each C_4 face adds up to $(2, 2, 2, 2)$. This labeling is given in Table 4. Then the corresponding C_4-face-magic torus labelings on $C_4 \times C_4$ are given by $x_{i,j} = x_{1,1} + a_1 d_1 + a_2 d_2 + a_3 d_3 + a_4 d_4$ where $x_{1,1} = 1$, (a_1, a_2, a_3, a_4) is the labeling on vertex (i, j) in $C_4 \times C_4$ given in Table 4, and (d_1, d_2, d_3, d_4) is one of the three choices of either $(1, 2, 4, 8)$, $(1, 4, 2, 8)$ or $(1, 8, 2, 4)$. The choices of $(1, 2, 4, 8)$, $(1, 4, 2, 8)$ or $(1, 8, 2, 4)$ for (d_1, d_2, d_3, d_4) result in the labelings A, B and C, respectively, in Theorem 3.5.

$(0,0,0,0)$	$(1,1,1,0)$	$(0,0,1,1)$	$(1,1,0,1)$
$(1,0,1,1)$	$(0,1,0,1)$	$(1,0,0,0)$	$(0,1,1,0)$
$(1,1,0,0)$	$(0,0,1,0)$	$(1,1,1,1)$	$(0,0,0,1)$
$(0,1,1,1)$	$(1,0,0,1)$	$(0,1,0,0)$	$(1,0,1,0)$

Table 4: C_4-face-magic toroidal labeling with elements from $\{0,1\}^4$ on $C_4 \times C_4$.

4 Results on $C_{2m} \times C_{2n}$

We first consider antipodal balanced C_4-face-magic toroidal labelings on $C_6 \times C_6$.

Lemma 4.1. *Let $\{x_{i,j}\}$ be an antipodal balanced toroidal C_4-face-magic labeling on $C_6 \times C_6$. Let $d_i = x_{2i+1,3i+1} - x_{2i-1,3i-2}$, for $i = 1,2,3$, and let $d_{j+3} = d'_j = x_{3j+1,2j+1} - x_{3j-2,2j-2}$, for $j = 1,2,3$. Then the value of $x_{i,j}$ is determined by the values of $x_{1,1}$ and d_i, for $i = 1,2,3,4,5,6$ as displayed in Table 5.*

Proof. By Lemma 2.11, for integers i and j, we have

$$x_{2i+1,3i+j} = x_{2i-1,3i-3+j} + (-1)^{j+1} d_i,$$

when $1 \le i \le 3$ and $1 \le j \le 6$ and

$$x_{2i+4,3i+j} = x_{2i+2,3i-3+j} + (-1)^{j+1} d_i,$$

when $1 \le i \le 3$ and $1 \le j \le 6$. Thus $x_{1,4} = x_{1,1} + d_1 + d_2 + d_3$. Furthermore, it suffices to establish the result for the expressions $x_{1,j}$ and $x_{4,j}$, for $j = 1,2,3$. Also, by Lemma 4.1, for integers i and j, we have

$$x_{3j+i,2j+1} = x_{3j-3)+i,2j-1} + (-1)^{i+1} d_{j+3},$$

when $1 \le i \le 6$ and $1 \le j \le 3$ and

$$x_{3j+i,2j+4} = x_{3j-3+i,2j+2} + (-1)^{i} d_{j+3},$$

when $1 \le i \le 6$ and $1 \le j \le 3$. Thus $x_{4,1} = x_{1,1} + d_4 + d_5 + d_6$ and $x_{4,3} = x_{1,1} + d_4$. From $x_{1,4} = x_{1,1} + d_1 + d + 2 + d_3$ and $x_{1,4} = x_{4,2} - d_6$, we have $x_{4,2} = x_{1,1} + d_1 + d_2 + d_3 + d_6$. From $x_{4,1} = x_{1,1} + d_4 + d_5 + d_6$ and $x_{1,3} = x_{4,1} - d_4$, we have $x_{1,1} = x_{1,1} + d_5 + d_6$. Lastly, from $x_{1,4} = x_{1,1} + d_1 + d + 2 + d_3$, $x_{6,3} = x_{3,1} + d_4$, $x_{3,5} = x_{6,3} + d_5$ and $x_{1,2} = x_{3,5} - d_1$, we have $x_{1,2} = x1,1 + d_1 + d_2 + d_3 + d_4 + d_5$. □

We next consider antipodal balanced C_4-face-magic toroidal labelings on $C_8 \times C_8$.

Lemma 4.2. *Let $\{x_{i,j}\}$ be an antipodal balanced toroidal C_4-face-magic labeling on $C_8 \times C_8$. Let $d_i = x_{3i+1,4i+1} - x_{3i-2,4i-3}$, for $i = 1,2,3,4$, and let $d_{j+4} = d'_j = x_{4j+1,3j+1} - x_{4j-3,3j-2}$, for $j = 1,2,3,4$. Then the value of $x_{i,j}$ is determined by the values of $x_{1,1}$ and d_i, for $i = 1,2,3,4,5,6,7,8$ as displayed in Figure 1.*

$x_{1,1}$	$x_{1,2} = x_{1,1} + d_1 + d_2 + d_3 + d_4 + d_5$	$x_{1,3} = x_{1,1} + d_5 + d_6$	$x_{1,4} = x_{1,1} + d_1 + d_2 + d_3$	$x_{1,5} = x_{1,1} + d_4 + d_5$	$x_{1,6} = x_{1,1} + d_1 + d_2 + d_3 + d_5 + d_6$
$x_{2,1} = x_{1,1} + d_1 + d_2 + d_4 + d_5 + d_6$	$x_{2,2} = x_{1,1} + d_3 + d_6$	$x_{2,3} = x_{1,1} + d_1 + d_2 + d_4$	$x_{2,4} = x_{1,1} + d_3 + d_4 + d_5 + d_6$	$x_{2,5} = x_{1,1} + d_1 + d_2 + d_6$	$x_{2,6} = x_{1,1} + d_3 + d_4$
$x_{3,1} = x_{1,1} + d_2 + d_3$	$x_{3,2} = x_{1,1} + d_1 + d_4 + d_5$	$x_{3,3} = x_{1,1} + d_2 + d_3 + d_5 + d_6$	$x_{3,4} = x_{1,1} + d_1$	$x_{3,5} = x_{1,1} + d_2 + d_3 + d_4 + d_5$	$x_{3,6} = x_{1,1} + d_1 + d_5 + d_6$
$x_{4,1} = x_{1,1} + d_4 + d_5 + d_6$	$x_{4,2} = x_{1,1} + d_1 + d_2 + d_3 + d_6$	$x_{4,3} = x_{1,1} + d_4$	$x_{4,4} = x_{1,1} + d_1 + d_2 + d_3 + d_4 + d_5 + d_6$	$x_{4,5} = x_{1,1} + d_6$	$x_{4,6} = x_{1,1} + d_1 + d_2 + d_3 + d_4$
$x_{5,1} = x_{1,1} + d_1 + d_2$	$x_{5,2} = x_{1,1} + d_3 + d_4 + d_5$	$x_{5,3} = x_{1,1} + d_1 + d_2 + d_5 + d_6$	$x_{5,4} = x_{1,1} + d_3$	$x_{5,5} = x_{1,1} + d_1 + d_2 + d_4 + d_5$	$x_{5,6} = x_{1,1} + d_3 + d_5 + d_6$
$x_{6,1} = x_{1,1} + d_2 + d_3 + d_4 + d_5 + d_6$	$x_{6,2} = x_{1,1} + d_1 + d_6$	$x_{6,3} = x_{1,1} + d_2 + d_3 + d_4$	$x_{6,4} = x_{1,1} + d_1 + d_4 + d_5 + d_6$	$x_{6,5} = x_{1,1} + d_2 + d_3 + d_6$	$x_{6,6} = x_{1,1} + d_1 + d_4$

Table 5: C_4-face-magic toroidal labeling involving the differences d_1, d_2, d_3, d_4, d_5 and d_6 on $C_6 \times C_6$.

Proof. By Lemma 2.10, for integers i and j, we have

$$x_{3i+1,j} = x_{3i-2,j+4} + (-1)^{j+1} d_i,$$

when $1 \leq i \leq 4$ and $1 \leq j \leq 8$ and

$$x_{3i+1,j} = x_{3i-2,j+4} + (-1)^{j} d_{i-4},$$

when $5 \leq i \leq 8$ and $1 \leq j \leq 8$. Thus $x_{5,1} = x_{1,1} + d_1 + d_2 + d_3 + d_4$. Furthermore, we only need to verify the expressions for $x_{1,j}$ for $2 \leq j \leq 8$. Also, by Lemma 2.10, for all integers i and j such that we have

$$x_{i,3j+1} = x_{i+4,3j-2} + (-1)^{i+1} d'_j,$$

when $1 \leq i \leq 8$ and $1 \leq j \leq 4$ and

$$x_{i,3j+1} = x_{i+4,3j-2} + (-1)^{i} d'_{j-4},$$

when $1 \leq i \leq 8$ and $5 \leq j \leq 8$. Hence, $x_{1,7} = x_{1,1} + d_5 + d_6$, $x_{1,5} = x_{1,7} + d_7 + d_8$, $x_{1,3} = x_{1,5} - d_5 - d_6$. Thus $x_{1,5} = x_{1,1} + d_5 + d_6 + d_7 + d_8$ and $x_{1,3} = x_{1,1} + d_7 + d_8$. Also,

$x_{1,4} = x_{5,1} + d_5$, $x_{1,2} = x_{1,4} + d_6 + d_7$, $x_{1,8} = x_{1,2} + d_8 - d_5$ and $x_{1,6} = x_{1,8} - d_6 - d_7$. Since $x_{5,1} = x_{1,1} + d_1 + d_2 + d_3 + d_4$, we have $x_{1,4} = x_{1,1} + d_1 + d_2 + d_3 + d_4 + d_5$ $x_{1,2} = x_{1,1} + d_1 + d_2 + d_3 + d_4 + d_5 + d_6 + d_7$, $x_{1,8} = x_{1,1} + d_1 + d_2 + d_3 + d_4 + d_6 + d_7 + d_8$ and $x_{1,6} = x_{1,1} + d_1 + d_2 + d_3 + d_4 + d_8$. □

Lemma 4.3. *Let $\{x_{i,j}\}$ be an antipodal balanced C_4-face-magic toroidal labeling on $C_{4m} \times C_{4n}$. Then, we have*

$$R_i = \sum_{j=1}^{4n} x_{i,j} = 2n(16mn+1), \text{ for all } i = 1, 2, \ldots, 4m$$

and

$$C_j = \sum_{i=1}^{4m} x_{i,j} = 2m(16mn+1), \text{ for all } j = 1, 2, \ldots, 4n.$$

Furthermore, if $m = n$, then we have

$$D_i = \sum_{j=1}^{4n} x_{j,i+j} = 2n(16n^2+1), \text{ for all } i = 1, 2, \ldots, 4n$$

and

$$D'_i = \sum_{j=1}^{4n} x_{j,i-j} = 2n(16n^2+1), \text{ for all } i = 1, 2, \ldots, 4n.$$

Proof. Let $S = 2(16mn+1)$ be the C_4-face-magic value of $\{x_{i,j}\}$. We have, for all $i = 1, 2, \ldots, 4m$,

$$R_i + R_{i+1} = \sum_{j=1}^{2n} \big(x_{i,2j-1} + x_{i,2j} + x_{i+1,2j-1} + x_{i+1,2j}\big) = 2nS = 4n(16mn+1).$$

Thus it suffices to show that $R_1 = 2n(16mn+1)$. We first observe that for all $j = 1, 2, \ldots, n$,

$$x_{2m+1,2j-1+2n} = x_{1,2j-1+2n} + (d_1 + d_2 + \cdots + d_{2m}).$$

Thus,

$$\tfrac{1}{2}S = x_{1,2j-1} + x_{2m+1,2j-1+2n} = x_{1,2j-1} + x_{1,2j-1+2n} + (d_1 + d_2 + \cdots + d_{2m}).$$

We next observe that for all $j = 1, 2, \ldots, n$,

$$x_{2m+1,2j+2n} = x_{1,2j+2n} - (d_1 + d_2 + \cdots + d_{2m}).$$

Thus,

$$\tfrac{1}{2}S = x_{1,2j} + x_{2m+1,2j+2n} = x_{1,2j} + x_{1,2j+2n} - (d_1 + d_2 + \cdots + d_{2m}).$$

$x_{1,1}$	$x_{1,2} = x_{1,1} + d_1 + d_2 + d_3 + d_4 + d_5 + d_6 + d_7$	$x_{1,3} = x_{1,1} + d_7 + d_8$	$x_{1,4} = x_{1,1} + d_1 + d_2 + d_3 + d_4 + d_5$	$x_{1,5} = x_{1,1} + d_5 + d_6 + d_7 + d_8$	$x_{1,6} = x_{1,1} + d_1 + d_2 + d_3 + d_4 + d_8$	$x_{1,7} = x_{1,1} + d_5 + d_6$	$x_{1,8} = x_{1,1} + d_1 + d_2 + d_3 + d_4 + d_6 + d_7 + d_8$
$x_{2,1} = x_{1,1} + d_1 + d_2 + d_3 + d_5 + d_6 + d_7 + d_8$	$x_{2,2} = x_{1,1} + d_4 + d_8$	$x_{2,3} = x_{1,1} + d_1 + d_2 + d_3 + d_5 + d_6$	$x_{2,4} = x_{1,1} + d_4 + d_6 + d_7 + d_8$	$x_{2,5} = x_{1,1} + d_1 + d_2 + d_3$	$x_{2,6} = x_{1,1} + d_4 + d_5 + d_6 + d_7$	$x_{2,7} = x_{1,1} + d_1 + d_2 + d_3 + d_7 + d_8$	$x_{2,8} = x_{1,1} + d_4 + d_5$
$x_{3,1} = x_{1,1} + d_3 + d_4$	$x_{3,2} = x_{1,1} + d_1 + d_2 + d_5 + d_6 + d_7$	$x_{3,3} = x_{1,1} + d_3 + d_4 + d_7 + d_8$	$x_{3,4} = x_{1,1} + d_1 + d_2 + d_5$	$x_{3,5} = x_{1,1} + d_3 + d_4 + d_5 + d_6 + d_7 + d_8$	$x_{3,6} = x_{1,1} + d_1 + d_2 + d_8$	$x_{3,7} = x_{1,1} + d_3 + d_4 + d_5 + d_6$	$x_{3,8} = x_{1,1} + d_1 + d_2 + d_6 + d_7 + d_8$
$x_{4,1} = x_{1,1} + d_1 + d_5 + d_6 + d_7 + d_8$	$x_{4,2} = x_{1,1} + d_2 + d_3 + d_4 + d_8$	$x_{4,3} = x_{1,1} + d_1 + d_5 + d_6$	$x_{4,4} = x_{1,1} + d_2 + d_3 + d_4 + d_6 + d_7 + d_8$	$x_{4,5} = x_{1,1} + d_1$	$x_{4,6} = x_{1,1} + d_2 + d_3 + d_4 + d_5 + d_6 + d_7$	$x_{4,7} = x_{1,1} + d_1 + d_7 + d_8$	$x_{4,8} = x_{1,1} + d_2 + d_3 + d_4 + d_5$
$x_{5,1} = x_{1,1} + d_1 + d_2 + d_3 + d_4$	$x_{5,2} = x_{1,1} + d_5 + d_6 + d_7$	$x_{5,3} = x_{1,1} + d_1 + d_2 + d_3 + d_4 + d_7 + d_8$	$x_{5,4} = x_{1,1} + d_5$	$x_{5,5} = x_{1,1} + d_1 + d_2 + d_3 + d_4 + d_5 + d_6 + d_7 + d_8$	$x_{5,6} = x_{1,1} + d_8$	$x_{5,7} = x_{1,1} + d_1 + d_2 + d_3 + d_4 + d_5 + d_6$	$x_{5,8} = x_{1,1} + d_6 + d_7 + d_8$
$x_{6,1} = x_{1,1} + d_4 + d_5 + d_6 + d_7 + d_8$	$x_{6,2} = x_{1,1} + d_1 + d_2 + d_3 + d_8$	$x_{6,3} = x_{1,1} + d_4 + d_5 + d_6$	$x_{6,4} = x_{1,1} + d_1 + d_2 + d_3 + d_6 + d_7 + d_8$	$x_{6,5} = x_{1,1} + d_4$	$x_{6,6} = x_{1,1} + d_1 + d_2 + d_3 + d_5 + d_6 + d_7$	$x_{6,7} = x_{1,1} + d_4 + d_7 + d_8$	$x_{6,8} = x_{1,1} + d_1 + d_2 + d_3 + d_5$
$x_{7,1} = x_{1,1} + d_1 + d_2$	$x_{7,2} = x_{1,1} + d_3 + d_4 + d_5 + d_6 + d_7$	$x_{7,3} = x_{1,1} + d_1 + d_2 + d_7 + d_8$	$x_{7,4} = x_{1,1} + d_3 + d_4 + d_5$	$x_{7,5} = x_{1,1} + d_1 + d_2 + d_5 + d_6 + d_7 + d_8$	$x_{7,6} = x_{1,1} + d_3 + d_4 + d_8$	$x_{7,7} = x_{1,1} + d_1 + d_2 + d_5 + d_6$	$x_{7,8} = x_{1,1} + d_3 + d_4 + d_6 + d_7 + d_8$
$x_{8,1} = x_{1,1} + d_2 + d_3 + d_4 + d_5 + d_6 + d_7 + d_8$	$x_{8,2} = x_{1,1} + d_1 + d_8$	$x_{8,3} = x_{1,1} + d_2 + d_3 + d_4 + d_5 + d_6$	$x_{8,4} = x_{1,1} + d_1 + d_6 + d_7 + d_8$	$x_{8,5} = x_{1,1} + d_2 + d_3 + d_4$	$x_{8,6} = x_{1,1} + d_1 + d_5 + d_6 + d_7$	$x_{8,7} = x_{1,1} + d_2 + d_3 + d_4 + d_7 + d_8$	$x_{8,8} = x_{1,1} + d_1 + d_5$

Figure 1: C_4-face-magic toroidal labeling involving the differences d_1, d_2, d_3, d_4, d_5, d_6, d_7 and d_8 on $C_8 \times C_8$.

Hence, we have

$$\begin{aligned} R_1 &= \sum_{j=1}^{4n} x_{1,j} = \sum_{j=1}^{2n} \big(x_{1,2j-1} + x_{1,2j}\big) \\ &= \sum_{j=1}^{n} \big(x_{1,2j-1} + x_{1,2j-1+2n}\big) + \sum_{j=1}^{n} \big(x_{1,2j} + x_{1,2j+2n}\big) \\ &= nS = 2n(16mn+1). \end{aligned}$$

By interchanging the roles of i and j, we have

$$C_j = \sum_{i=1}^{4m} x_{i,j} = 2m(16mn+1), \text{ for all } j = 1, 2, \ldots, 4n.$$

Finally, we assume that $m = n$. Then,

$$D_i = \sum_{j=1}^{4n} x_{j,i+j} = \sum_{j=1}^{2n} \big(x_{j,i+j} + x_{j+2n,i+j+2n}\big) = (2n)\big(\tfrac{1}{2}S\big) = 2n(16n^2+1).$$

A similar argument shows that for all $i = 1, 2, \ldots, 4n$,

$$D'_i = \sum_{j=1}^{4n} x_{j,i-j} = 2n(16n^2+1).$$

□

Proposition 4.4. *Let m and n be integers where $m, n \geq 3$. Let $\{x_{i,j}\}$ be a C_4-face-magic labeling on $P_m \times P_n$ with face-magic value S. Suppose that for all integers i and j such that $1 \leq i \leq m-2$ and $1 \leq j \leq n-2$, we have*

$$x_{i,j} + x_{i,j+2} + x_{i+2,j} + x_{i+2,j+2} = S.$$

Then, $m \leq 4$ and $n \leq 4$.

Proof. For the purpose of contradiction, we assume that $m \geq 5$. We first observe that for all integers i and j such that $1 \leq i \leq m-2$ and $1 \leq j \leq n-2$, we have

$$x_{i,j} + x_{i,j+1} + x_{i+1,j} + x_{i+1,j+1} = S = x_{i,j+1} + x_{i,j+2} + x_{i+1,j+1} + x_{i+1,j+2}.$$

Thus,

$$x_{i,j} + x_{i+1,j} = x_{i,j+2} + x_{i+1,j+2}. \tag{4.1}$$

Replacing i with $i+1$ in equation (4.1) yields

$$x_{i+1,j} + x_{i+2,j} = x_{i+1,j+2} + x_{i+2,j+2}. \tag{4.2}$$

When we subtract equation (4.2) from equation (4.1) and rearrange terms, we obtain

$$x_{i,j} + x_{i+2,j+2} = x_{i+2,j} + x_{i,j+2}. \tag{4.3}$$

Since

$$x_{i,j} + x_{i,j+2} + x_{i+2,j} + x_{i+2,j+2} = S,$$

we have

$$x_{i,j} + x_{i+2,j+2} = \tfrac{1}{2}S = x_{i+2,j} + x_{i,j+2}. \tag{4.4}$$

Let $i = 1$ or 3, and $j = 1$. Then equation (4.4) yields

$$x_{1,1} + x_{3,3} = \tfrac{1}{2}S = x_{5,1} + x_{3,3}.$$

Hence, $x_{1,1} = x_{5,1}$. This is a contradiction. Therefore, $m \leq 4$. A similar argument shows that $n \leq 4$. □

Proposition 4.5. *Let m and n be even positive integers. Let y_j, for $j = 1, 2, \ldots, 2n$, be a positive integer, and let $d_1, d_2, \ldots, d_m$ be integers. We define a labeling $\{x_{i,j}\}$ on $C_{2m} \times C_{2n}$ by letting, for all integers i and j such that $0 \leq i \leq m$ and $1 \leq j \leq 2n$,*

$$x_{i(m-1)+1,in+j} = y_j + (-1)^{j+1} \sum_{k=1}^{i} d_k \textit{ and}$$
$$x_{i(m-1)+m+1,in+j} = y_j + (-1)^{j+1} \sum_{k=i+1}^{m} d_k.$$

Let $A = \{\sum_{k=1}^{i} d_k, \sum_{k=i}^{m} d_k : 1 \leq i \leq m\} \cup \{0\}$. For all integers j such that $1 \leq j \leq 2n$, let

$$A_j = \{y_j + (-1)^{j+1} a : a \in A\}.$$

Suppose that

1. *for all integers j such that $1 \leq j \leq n$, $y_j + y_{j+n} + (-1)^{j+1}\left(\sum_{k=1}^{m} d_k\right) = 4mn + 1$ and*
2. *the set $\{A_j : 1 \leq j \leq 2n\}$ forms a partition of the set $\{k \in \mathbb{Z} : 1 \leq k \leq 4mn\}$.*

Then, $\{x_{i,j}\}$ is an antipodal balanced C_4-face-magic toroidal labeling on $C_{2m} \times C_{2n}$.

Proof. We first show that the C_4-face sum is preserved for all of the relevant C_4 faces on $C_{2m} \times C_{2n}$. We note that $m - 1$ is relatively prime to $2m$. Thus, $m - 1$ is a generator of $\mathbb{Z}_{2m}$. Hence,

$$\{x_{i(m-1)+1,in+j} : i = 1, 2, \ldots, 2m \text{ and } j = 1, 2, \ldots, 2n\}$$

is the set $\{x_{i,j} : i = 1, 2, \ldots, 2m \text{ and } j = 1, 2, \ldots, 2n\}$. We observe that $(i+m)(m-1) + m + 1 = i(m-1) + 1 \pmod{2m}$. We have, for all integers i and j such that $0 \leq i \leq m$ and $1 \leq j \leq 2n$,

$$x_{i(m-1)+1,in+j} = y_j + (-1)^{j+1} \sum_{k=1}^{i} d_k \text{ and} \tag{4.5}$$

$$x_{i(m-1)+m+1,in+j} = y_j + (-1)^{j+1} \sum_{k=i+1}^{m} d_k. \tag{4.6}$$

We replace i with $i-1$ and j with $j+n$ in equation (4.6) to obtain, for all integers i and j such $1 \le i \le m+1$ and $1 \le j \le 2n$,

$$x_{(i-1)(m-1)+m+1,(i-1)n+j+n} = x_{i(m-1)+2,in+j} = y_{j+n} + (-1)^{j+1}\sum_{k=i}^{m} d_k.$$

Hence, for all integers i and j such $1 \le i \le m$ and $1 \le j \le 2n$, we have

$$\begin{aligned}
&x_{i(m-1)+1,in+j} + x_{i(m-1)+1,in+j+1} + x_{i(m-1)+2,in+j} + x_{i(m-1)+2,in+j+1}\\
&= y_j + (-1)^{j+1}\left(\sum_{k=1}^{i} d_k\right) + y_{j+1} + (-1)^{j+2}\left(\sum_{k=1}^{i} d_k\right)\\
&+ y_{j+n} + (-1)^{j+1}\left(\sum_{k=i}^{m} d_k\right) + y_{j+n+1} + (-1)^{j+2}\left(\sum_{k=i}^{m} d_k\right)\\
&= \left(y_j + y_{j+n} + (-1)^{j+1}(\sum_{k=1}^{m} d_k)\right) + \left(y_{j+1} + y_{j+n+1} + (-1)^{j+2}(\sum_{k=1}^{m} d_k)\right)\\
&= \tfrac{1}{2}S + \tfrac{1}{2}S = S.
\end{aligned}$$

Next, we replace i with $i-1$ and j with $j+n$ in equation (4.5) to obtain, for all integers i and j such $1 \le i \le m+1$ and $1 \le j \le 2n$,

$$x_{(i-1)(m-1)+1,(i-1)n+j+n} = x_{i(m-1)+m+2,in+j} = y_{j+n} + (-1)^{j+1}\sum_{k=1}^{i-1} d_k.$$

Hence, for all integers i and j such $1 \le i \le m$ and $1 \le j \le 2n$, we have

$$\begin{aligned}
&x_{i(m-1)+m+1,in+j} + x_{i(m-1)+m+1,in+j+1} + x_{i(m-1)+m+2,in+j} + x_{i(m-1)+m+2,in+j+1}\\
&= y_j + (-1)^{j+1}\left(\sum_{k=i+1}^{m} d_k\right) + y_{j+1} + (-1)^{j+2}\left(\sum_{k=i+1}^{m} d_k\right)\\
&+ y_{j+n} + (-1)^{j+1}\left(\sum_{k=1}^{i-1} d_k\right) + y_{j+n+1} + (-1)^{j+2}\left(\sum_{k=1}^{i-1} d_k\right)\\
&= \left(y_j + y_{j+n} + (-1)^{j+1}(\sum_{k=1}^{m} d_k)\right) + \left(y_{j+1} + y_{j+n+1} + (-1)^{j+2}(\sum_{k=1}^{m} d_k)\right)\\
&= \tfrac{1}{2}S + \tfrac{1}{2}S = S.
\end{aligned}$$

Hence, $\{x_{i,j}\}$ is a C_4-face-magic toroidal labeling on $C_{2m} \times C_{2n}$.

We show that each integer k, where $1 \le k \le 4mn$, is used exactly once in the labeling $\{x_{i,j}\}$. For each integer j such that $1 \le j \le 2n$, we show that $\{x_{i(m-1)+1,in+j} : 1 \le i \le 2m\} = A_j$. We have

$$\begin{aligned}
&\{x_{i(m-1)+1,in+j} : 1 \le i \le 2m\} = \{x_{i(m-1)+1,in+j}, x_{i(m-1)+m+1,in+j} : 1 \le i \le m\}\\
&= \{y_j, y_j + (-1)^{j+1}(\sum_{k=1}^{i} d_k), y_j + (-1)^{j+1}(\sum_{k=i}^{m} d_k) : 1 \le i \le m\}\\
&= \{y_y + (-1)^{j+1}a : a \in A\} = A_j.
\end{aligned}$$

Since $\{A_j : 1 \leq j \leq 2n\}$ is a partition of the set $\{k \in \mathbb{Z} : 1 \leq k \leq 4mn\}$, each integer k, where $1 \leq k \leq 4mn$, is used exactly once in the labeling $\{x_{i,j}\}$.

Finally, we show that $\{x_{i,j}\}$ is an antipodal balanced labeling on $C_{2m} \times C_{2n}$. When we replace j with $j + n$ in equation (4.6), we have

$$x_{(i+m)(m-1)+1,(i+m)n+j+n} = x_{i(m-1)+1+m,in+j+n} = y_{j+n} + (-1)^{j+1}\Big(\sum_{k=i+1}^{m} d_k\Big). \tag{4.7}$$

When we add equations (4.5) and (4.7), we have

$$\begin{aligned} &x_{i(m-1)+1,in+j} + x_{i(m-1)+1+m,in+j+n} \\ &= y_j + (-1)^{j+1}\Big(\sum_{k=1}^{i} d_k\Big) + y_{j+n} + (-1)^{j+1}\Big(\sum_{k=i+1}^{m} d_k\Big) \\ &= y_j + y_{j+n} + (-1)^{j+1}\Big(\sum_{k=1}^{m} d_k\Big) = \tfrac{1}{2}S = 4mn + 1. \end{aligned}$$

When we replace i with $i + m$ in this equation, we have

$$\begin{aligned} &x_{(i+m)(m-1)+1+m,(i+m)n+j} + x_{(i+m)(m-1)+1,(i+m)n+j+n} \\ &= x_{i(m-1)+1,in+j} + x_{i(m-1)+1+m,in+j+n} = 4mn + 1. \end{aligned}$$

This completes the proof. □

Proposition 4.6. *We define a labeling $\{x_{i,j}\}$ on $C_{4m} \times C_{4n}$ in the following manner. For integers i and j such that $1 \leq i \leq 2m$ and $1 \leq j \leq 2n$, when j is odd, we let*

$$\begin{aligned} x_{i(2m-1)+1,j+i(2n)} &= 4m(j-1) + 2i, \\ x_{i(2m-1)+1,j+(i+1)(2n)} &= 4m(4n-j) + 2i, \\ x_{i(2m-1)+2m+1,j+i(2n)} &= 4mj - 2i + 1 \text{ and} \\ x_{i(2m-1)+2m+1,j+(i+1)(2n)} &= 4m(4n-j+1) - 2i + 1; \end{aligned}$$

and when j is even, we let

$$\begin{aligned} x_{i(2m-1)+1,j+i(2n)} &= 4mj - 2i + 1, \\ x_{i(2m-1)+1,j+(i+1)(2n)} &= 4m(4n-j+1) - 2i + 1, \\ x_{i(2m-1)+2m+1,j+i(2n)} &= 4m(j-1) + 2i \text{ and} \\ x_{i(2m-1)+2m+1,j+(i+1)(2n)} &= 4m(4n-j) + 2i. \end{aligned}$$

Then, $\{x_{i,j}\}$ is an antipodal balanced C_4-face-magic toroidal labeling on $C_{4m} \times C_{4n}$. Furthermore, by Lemma 4.3, $\{x_{i,j}\}$ is row-sum balanced and column-sum balanced, and $\{x_{i,j}\}$ is torus symmetric whenever $m = n$.

Proof. In the notation of Proposition 4.5, we have $d_1 = 1$ and $d_k = 2$, for $2 \leq k \leq 2m$. Also, for integers j such that $1 \leq j \leq 2n$, we have $y_j = 4m(j-1) + 1$ and $y_{j+2n} = 4m(4n-j) + 1$ when j is odd, and $y_j = 4mj$ and $y_{j+2n} = 4m(4n-j+1)$ when j is even. Then the labeling $\{x_{i,j}\}$ of $C_{4m} \times C_{4n}$ satisfies, for all integers i and j such that $1 \le i \le 2m$ and $1 \le j \le 2n$, when j is odd, we have

$$x_{i(2m-1)+1,i(2n)+j} = y_j + (-1)^{j+1}\Big(\sum_{k=1}^{i} d_k\Big)$$
$$= \big(4m(j-1)+1\big) + (2i-1) = 4m(j-1)+2i \text{ and}$$
$$x_{i(2m-1)+1,i(2n)+j+2n} = y_{j+2n} + (-1)^{j+1}\Big(\sum_{k=1}^{i} d_k\Big)$$
$$= \big(4m(4n-j)+1\big) + (2i-1) = 4m(4n-j)+2i,$$

and when j is even, we have

$$x_{i(2m-1)+1,i(2n)+j} = y_j + (-1)^{j+1}\Big(\sum_{k=1}^{i} d_k\Big)$$
$$= \big(4mj\big) - (2i-1) = 4mj - 2i + 1 \text{ and}$$
$$x_{i(2m-1)+1,i(2n)+j+2n} = y_{j+2n} + (-1)^{j+1}\Big(\sum_{k=1}^{i} d_k\Big)$$
$$= \big(4m(4n-j+1)\big) - (2i-1) = 4m(4n-j+1) - 2i + 1.$$

We let $i = 2m$ in each of the previous four equations. Thus, for all integers i and j such that $1 \le i \le 2m$ and $1 \le j \le 2n$, when j is odd, we have

$$x_{2m+1,j} = 4mj \text{ and } x_{2m+1,j+2n} = 4m(4n-j+1),$$

and when j is even, we have

$$x_{2m+1,j} = 4m(j-1)+1 \text{ and } x_{2m+1,j+2n} = 4m(4n-j)+1.$$

Next, we observe that the labeling $\{x_{i,j}\}$ on $C_{4m} \times C_{4n}$ satisfies, for all integers i and j such that $1 \le i \le 2m$ and $1 \le j \le 2n$, when j is odd, we have

$$x_{i(2m-1)+2m+1,i(2n)+j} = y_j + (-1)^{j+1}\Big(\sum_{k=i+1}^{2m} d_k\Big)$$
$$= \big(4m(j-1)+1\big) + 2(2m-i) = 4mj - 2i \text{ and}$$
$$x_{i(2m-1)+2m+1,i(2n)+j+2n} = y_{j+2n} + (-1)^{j+1}\Big(\sum_{k=i+1}^{2m} d_k\Big)$$
$$= \big(4m(4n-j)+1\big) + 2(2m-i) = 4m(4n-j+1) - 2i,$$

and when j is even, we have

$$\begin{aligned}
&x_{i(2m-1)+2m+1,i(2n)+j} = y_j + (-1)^{j+1}(\sum_{k=i+1}^{2m} d_k)\\
&= \big(4mj\big) - 2(2m-i) = 4m(j-1) + 2i \text{ and}\\
&x_{i(2m-1)+2m+1,i(2n)+j+2n} = y_{j+2n} + (-1)^{j+1}(\sum_{k=i+1}^{2m} d_k)\\
&= \big(4m(4n-j+1)\big) - 2(2m-i) = 4m(4n-j) + 2i.
\end{aligned}$$

We show that condition (1) of Proposition 4.5 is satisfied. Let j be an integer such that $1 \leq j \leq 2n$. When j is odd, we have

$$y_j+y_{j+2n}+(-1)^{j+1}(\sum_{k=1}^{2m} d_k) = \big(4m(j-1)+1\big)+\big(4m(4n-j)+1\big)+(4m-1) = 16mn+1.$$

When j is even, we have

$$y_j + y_{j+2n} + (-1)^{j+1}(\sum_{k=1}^{2m} d_k) = \big(4mj\big) + \big(4m(4n-j+1)\big) - (4m-1) = 16mn + 1.$$

We show that condition (2) of Proposition 4.5 is satisfied. We first observe that

$$\begin{aligned}
&A = \{\sum_{k=1}^{i} d_k, \sum_{k=i}^{2m} d_k : 1 \leq k \leq 2m\} \cup \{0\}\\
&= \{2i-1 : 1 \leq i \leq 2m\} \cup \{4m-2i+2 : 2 \leq i \leq 2m\} \cup \{0\}\\
&= \{k \in \mathbb{Z} : 0 \leq k \leq 4m-1\}.
\end{aligned}$$

Let j be an integer such that $1 \leq j \leq 2n$. When j is odd, we have

$$\begin{aligned}
&A_j = \{y_j + (-1)^{j+1}a : a \in A\} = \{4m(j-1)+1+i : i \in \mathbb{Z}, 0 \leq i \leq 4m-1\}\\
&= \{k \in \mathbb{Z} : 4m(j-1)+1 \leq j \leq 4mj\}
\end{aligned}$$

and

$$\begin{aligned}
&A_{j+2n} = \{y_{j+2n} + (-1)^{j+1}a : a \in A\}\\
&= \{4m(4n-j)+1+i : i \in \mathbb{Z}, 0 \leq i \leq 4m-1\}\\
&= \{k \in \mathbb{Z} : 4m(4n-j)+1 \leq j \leq 4m(4n-j+1)\}.
\end{aligned}$$

When j is even, we have

$$\begin{aligned}
&A_j = \{y_j + (-1)^{j+1}a : a \in A\} = \{4mj - i : i \in \mathbb{Z}, 0 \leq i \leq 4m-1\}\\
&= \{k \in \mathbb{Z} : 4m(j-1)+1 \leq j \leq 4mj\}
\end{aligned}$$

and

$$\begin{aligned} A_{j+2n} &= \{y_{j+2n} + (-1)^{j+1}a : a \in A\} \\ &= \{4m(4n-j+1) - i : i \in \mathbb{Z}, 0 \le i \le 4m-1\} \\ &= \{k \in \mathbb{Z} : 4m(4n-j)+1 \le j \le 4m(4n-j+1)\}. \end{aligned}$$

Hence, for all integers j, when $1 \le j \le 2n$, we have

$$A_j = \{k \in \mathbb{Z} : 4m(j-1)+1 \le j \le 4mj\}$$

and when $2n+1 \le j \le 4n$, we have

$$A_j = \{k \in \mathbb{Z} : 4m(4n-j)+1 \le j \le 4m(4n-j+1)\}.$$

Therefore, $\{A_j : 1 \le j \le 4n\}$ is a partition of the set $\{k \in \mathbb{Z} : 1 \le k \le 16mn\}$. This completes the proof. □

Example 4.7. We consider the example of Proposition 4.6 where $m = n = 2$. This example is given in Table 6.

1	16	17	32	57	56	41	40
62	51	46	35	6	11	22	27
5	12	21	28	61	52	45	36
58	55	42	39	2	15	18	31
8	9	24	25	64	49	48	33
59	54	43	38	3	14	19	30
4	13	20	29	60	53	44	37
63	50	47	34	7	10	23	26

Table 6: An antipodal balanced C_4-face-magic toroidal labeling on $C_8 \times C_8$.

We next prove the converse to Proposition 4.5.

Proposition 4.8. *Let m and n be even positive integers. Let $\{x_{i,j} : (i,j) \in V(C_{2m} \times C_{2n})\}$ be an antipodal balanced C_4-face-magic toroidal labeling on $C_{2m} \times C_{2n}$. For all integers j such that $1 \le j \le 2n$, let $y_j = x_{1,j}$. For all integers i such that $1 \le i \le m$, let*

$$d_i = x_{i(m-1)+1,in+1} - x_{(i-1)(m-1)+1,(i-1)n+1}.$$

Then, for all integers i and j such that $0 \le i \le m$ and $1 \le j \le 2n$,

$$x_{i(m-1)+1,in+j} = y_j + (-1)^{j+1} \sum_{k=1}^{i} d_k \text{ and} \tag{4.8}$$

$$x_{i(m-1)+m+1,in+j} = y_j + (-1)^{j+1} \sum_{k=i+1}^{m} d_k. \tag{4.9}$$

Let $A = \{\sum_{k=1}^{i} d_k, \sum_{k=i}^{m} d_k : 1 \leq i \leq m\} \cup \{0\}$. *For all integers* j *such that* $1 \leq j \leq 2n$, *let* $A_j = \{y_j + (-1)^{j+1} a : a \in A\}$. *Then*

1. *for all integers* j *such that* $1 \leq j \leq n$, $y_j + y_{j+n} + (-1)^{j+1}\left(\sum_{k=1}^{m} d_k\right) = 4mn + 1$ *and*
2. *the set* $\{A_j : 1 \leq j \leq 2n\}$ *forms a partition of the set* $\{k \in \mathbb{Z} : 1 \leq k \leq 4mn\}$.

Proof. By Lemma 2.10, for all integers i and j such that $1 \leq i \leq m$ and $1 \leq j \leq 2n$, we have

$$x_{i(m-1)+1, in+j} = x_{(i-1)(m-1)+1,(i-1)n+j} + (-1)^{j+1} d_i. \tag{4.10}$$

For all integers i and j such that $1 \leq i \leq m$ and $1 \leq j \leq 2n$, repeated use of equation (4.10) yields

$$x_{i(m-1)+1, in+j} = x_{1,j} + (-1)^{j+1} \sum_{k=1}^{i} d_k. \tag{4.11}$$

Thus equation (4.8) holds. When we let $i = m$ in equation (4.11), we have

$$x_{m+1,j} = x_{1,j} + (-1)^{j+1} \sum_{k=1}^{m} d_k. \tag{4.12}$$

By Lemma 2.10, for all integers i and j such that $m + 1 \leq i \leq 2m$ and $1 \leq j \leq 2n$, we have

$$x_{i(m-1)+1, in+j} = x_{(i-1)(m-1)+1,(i-1)n+j} + (-1)^{j} d_{i-m}. \tag{4.13}$$

Let $1 \leq i \leq m$. Replacing i with $i + m$ in equation (4.13) yields

$$x_{i(m-1)+m+1, in+j} = x_{(i-1)(m-1)+m+1,(i-1)n+j} + (-1)^{j} d_i. \tag{4.14}$$

For all integers i and j such that $1 \leq i \leq m$ and $1 \leq j \leq 2n$, repeated use of equation (4.14) yields

$$x_{i(m-1)+m+1, in+j} = x_{m+1,j} + (-1)^{j} \sum_{k=1}^{i} d_k. \tag{4.15}$$

When we combine equations (4.12) and (4.15), we have

$$x_{i(m-1)+m+1, in+j} = x_{1,j} + (-1)^{j+1} \sum_{k=i+1}^{m} d_k. \tag{4.16}$$

Thus equation (4.9) holds.

Since $\{x_{i,j}\}$ is an antipodal balanced C_4-face-magic toroidal labeling on $C_{2m} \times C_{2n}$, we have $x_{1,n+j} + x_{m+1,j} = 4mn + 1$ for all integers j such that $1 \leq j \leq n$. By equation (4.12), we have $x_{m+1,j} = y_j + (-1)^{j+1} \sum_{k=1}^{m} d_k$. Since $x_{1,n+j} = y_{j+n}$, we have $y_j + y_{j+n} + (-1)^{j+1}\left(\sum_{k=1}^{m} d_k\right) = x_{1,n+j} + x_{m+1,j} = 4mn + 1$.

By equations (4.8) and (4.9), for integers j such that $1 \leq j \leq 2n$, we have

$$A_j = \{x_{i(m-1)+1, in+j}, x_{i(m-1)+m+1, in+j} : 0 \leq i \leq m - 1\}.$$

Thus the set $\{A_j : 1 \leq j \leq 2n\}$ forms a partition of the set $\{x_{i,j} : 1 \leq i \leq 2m$ and $1 \leq j \leq 2n\} = \{k \in \mathbb{Z} : 1 \leq k \leq 4mn\}$. ☐

Proposition 4.9. *Let m and n be odd positive integers. Let y_j and z_j, for $j = 1, 2, \ldots, n$, be a positive integers, and let $d_1, d_2, \ldots, d_m$ be integers. We define a labeling $\{x_{i,j}\}$ on $C_{2m} \times C_{2n}$ by letting, for all integers i and j such $0 \le i \le m$ and $1 \le j \le n$,*

$$x_{i(m-1)+1,in+j} = y_j + (-1)^{j+1} \sum_{k=1}^{i} d_k, \tag{4.17}$$

$$x_{i(m-1)+1,(i+1)n+j} = y_j + (-1)^{j+1} \sum_{k=i+1}^{m} d_k, \tag{4.18}$$

$$x_{i(m-1)+m+1,in+j} = z_j + (-1)^{j+1} \sum_{k=1}^{i} d_k \text{ and} \tag{4.19}$$

$$x_{i(m-1)+m+1,(i+1)n+j} = z_j + (-1)^{j+1} \sum_{k=i+1}^{m} d_k. \tag{4.20}$$

Let $A = \{\sum_{k=1}^{i} d_k, \sum_{k=i}^{m} d_k : 1 \le i \le m\} \cup \{0\}$. For all integers j such that $1 \le j \le n$, let

$$A_j = \{y_j + (-1)^{j+1} a : a \in A\} \text{ and } A_{j+n} = \{z_j + (-1)^{j+1} a : a \in A\}.$$

Suppose

1. *for all integers j such that $1 \le j \le n$, $y_j + z_j + (-1)^{j+1}\left(\sum_{k=1}^{m} d_k\right) = 4mn + 1$, and*
2. *the set $\{A_j : 1 \le j \le 2n\}$ forms a partition of the set $\{k \in \mathbb{Z} : 1 \le k \le 4mn\}$.*

Then, $\{x_{i,j}\}$ is an antipodal balanced C_4-face-magic toroidal labeling on $C_{2m} \times C_{2n}$.

Proof. We first show that the C_4-face sum is preserved for all relevant C_4 faces on $C_{2m} \times C_{2n}$. Since $\gcd(2m, m-1) = 2$, $m - 1$ generates the subgroup $\langle 2 \rangle$ of $\mathbb{Z}_{2m}$. Hence,

$$\{x_{i(m-1)+1,in+j}, x_{(i+1)(m-1)+1,in+j}, x_{i(m-1)+m+1,in+j}, x_{(i+1)(m-1)+m+1,in+j} : \\ i = 1, 2, \ldots, m \text{ and } j = 1, 2, \ldots, n\}$$

is the set

$$\{x_{i,j} : i = 1, 2, \ldots 2, m \text{ and } j = 1, 2, \ldots, 2n\}.$$

We replace i with $i - 1$ in equations (4.17), (4.18), (4.19) and (4.20) to obtain

$$x_{i(m-1)+m+2,(i+1)n+j} = y_j + (-1)^{j+1} \sum_{k=1}^{i-1} d_k, \tag{4.21}$$

$$x_{i(m-1)+m+2,in+j} = y_j + (-1)^{j+1} \sum_{k=i}^{m} d_k, \tag{4.22}$$

$$x_{i(m-1)+2,(i+1)n+j} = z_j + (-1)^{j+1} \sum_{k=1}^{i-1} d_k \text{ and} \tag{4.23}$$

$$x_{i(m-1)+2,in+j} = z_j + (-1)^{j+1} \sum_{k=i}^{m} d_k. \tag{4.24}$$

We replace j with $j+1$ in equations (4.17), (4.18), (4.19), (4.20), (4.21), (4.22), (4.23) and (4.24) to obtain

$$x_{i(m-1)+1,in+j+1} = y_{j+1} + (-1)^{j+2}\sum_{k=1}^{i} d_k, \tag{4.25}$$

$$x_{i(m-1)+1,(i+1)n+j+1} = y_{j+1} + (-1)^{j+2}\sum_{k=i+1}^{m} d_k, \tag{4.26}$$

$$x_{i(m-1)+m+1,in+j+1} = z_{j+1} + (-1)^{j+2}\sum_{k=1}^{i} d_k, \tag{4.27}$$

$$x_{i(m-1)+m+1,(i+1)n+j+1} = z_{j+1} + (-1)^{j+2}\sum_{k=i+1}^{m} d_k, \tag{4.28}$$

$$x_{i(m-1)+m+2,(i+1)n+j+1} = y_{j+1} + (-1)^{j+2}\sum_{k=1}^{i-1} d_k, \tag{4.29}$$

$$x_{i(m-1)+m+2,in+j+1} = y_{j+1} + (-1)^{j+2}\sum_{k=i}^{m} d_k, \tag{4.30}$$

$$x_{i(m-1)+2,(i+1)n+j+1} = z_{j+1} + (-1)^{j+2}\sum_{k=1}^{i-1} d_k \text{ and} \tag{4.31}$$

$$x_{i(m-1)+2,in+j+1} = z_{j+1} + (-1)^{j+2}\sum_{k=i}^{m} d_k. \tag{4.32}$$

When we add equations (4.17), (4.25), (4.24) and (4.32) together, for $1 \leq i \leq m$ and $1 \leq j \leq n-1$, we obtain

$$\begin{aligned} x_{i(m-1)+1,in+j} &+ x_{i(m-1)+1,in+j+1} + x_{i(m-1)+2,in+j} + x_{i(m-1)+2,in+j+1} \\ &= \Big(y_j + (-1)^{j+1}(\sum_{k=1}^{i} d_k)\Big) + \Big(y_{j+1} + (-1)^{j+2}(\sum_{k=1}^{i} d_k)\Big) \\ &+ \Big(z_j + (-1)^{j+1}(\sum_{k=i}^{m} d_k)\Big) + \Big(z_{j+1} + (-1)^{j+2}(\sum_{k=i}^{m} d_k)\Big) \\ = \Big(y_j + z_j &+ (-1)^{j+1}(\sum_{k=1}^{m} d_k)\Big) + \Big(y_{j+1} + z_{j+1} + (-1)^{j+2}(\sum_{k=1}^{m} d_k)\Big) = S. \end{aligned}$$

When we add equations (4.18), (4.26), (4.23) and (4.31) together, for $1 \leq i \leq m$ and $1 \leq j \leq n-1$, we obtain

$$\begin{aligned} &x_{i(m-1)+1,(i+1)n+j} + x_{i(m-1)+1,(i+1)n+j+1} \\ &+ x_{i(m-1)+2,(i+1)n+j} + x_{i(m-1)+2,(i+1)n+j+1} = S. \end{aligned}$$

Similar C_4-face-magic sums occur when we add equations (4.19), (4.49), (4.22), and (4.52) together; or when we add equations (4.20), (4.50), (4.21) and (4.51) together.

Suppose i is odd and let $j = n$ in equations (4.18), (4.20), (4.21), and (4.23), we have

$$x_{i(m-1)+1,n} = y_n + (-1)^{n+1} \sum_{k=i+1}^{m} d_k, \tag{4.33}$$

$$x_{i(m-1)+m+1,n} = z_n + (-1)^{n+1} \sum_{k=i+1}^{m} d_k, \tag{4.34}$$

$$x_{i(m-1)+m+2,n} = y_n + (-1)^{n+1} \sum_{k=1}^{i-1} d_k \text{ and} \tag{4.35}$$

$$x_{i(m-1)+2,n} = z_n + (-1)^{n+1} \sum_{k=1}^{i-1} d_k. \tag{4.36}$$

Suppose i is odd and let $j = 1$ in equations (4.17), (4.19), (4.22), and (4.24), we have

$$x_{i(m-1)+1,n+1} = y_1 + (-1)^2 \sum_{k=1}^{i} d_k, \tag{4.37}$$

$$x_{i(m-1)+m+1,n+1} = z_1 + (-1)^2 \sum_{k=1}^{i} d_k, \tag{4.38}$$

$$x_{i(m-1)+m+2,n+1} = y_1 + (-1)^2 \sum_{k=i}^{m} d_k \text{ and} \tag{4.39}$$

$$x_{i(m-1)+2,n+1} = z_1 + (-1)^2 \sum_{k=i}^{m} d_k. \tag{4.40}$$

When we add equations (4.33), (4.37), (4.36) and (4.40) together, we have

$$\begin{aligned} x_{i(m-1)+1,n} + x_{i(m-1)+1,n+1} + x_{i(m-1)+2,n} + x_{i(m-1)+2,n+1} \\ = \left(y_n + (\sum_{k=i+1}^{m} d_k)\right) + \left(y_1 + (\sum_{k=1}^{i} d_k)\right) \\ + \left(z_n + (\sum_{k=1}^{i-1} d_k)\right) + \left(z_1 + (\sum_{k=i}^{m} d_k)\right) \\ = \left(y_1 + y_n + (\sum_{k=1}^{m} d_k)\right) + \left(z_1 + z_n + (\sum_{k=1}^{m} d_k)\right) = S. \end{aligned}$$

When we add equations (4.34), (4.38), (4.35) and (4.39) together, we have

$$x_{i(m-1)+m+1,n} + x_{i(m-1)+m+1,n+1} + x_{i(m-1)+m+2,n} + x_{i(m-1)+m+2,n+1} = S.$$

Suppose i is even and let $j = n$ in equations (4.17), (4.19), (4.22) and (4.24), we have

$$x_{i(m-1)+1,n} = y_n + (-1)^{n+1} \sum_{k=1}^{i} d_k, \tag{4.41}$$

$$x_{i(m-1)+m+1,n} = z_n + (-1)^{n+1} \sum_{k=1}^{i} d_k, \tag{4.42}$$

$$x_{i(m-1)+m+2,n} = y_n + (-1)^{n+1} \sum_{k=i}^{m} d_k \text{ and} \tag{4.43}$$

$$x_{i(m-1)+2,n} = z_n + (-1)^{n+1} \sum_{k=i}^{m} d_k. \tag{4.44}$$

Suppose i is even and let $j = 1$ in equations (4.18), (4.20), (4.21) and (4.23), we have

$$x_{i(m-1)+1,n+1} = y_1 + (-1)^2 \sum_{k=i+1}^{m} d_k, \tag{4.45}$$

$$x_{i(m-1)+m+1,n+1} = z_1 + (-1)^2 \sum_{k=i+1}^{m} d_k, \tag{4.46}$$

$$x_{i(m-1)+m+2,n+1} = y_1 + (-1)^2 \sum_{k=1}^{i-1} d_k \text{ and} \tag{4.47}$$

$$x_{i(m-1)+2,n+1} = z_1 + (-1)^2 \sum_{k=1}^{i-1} d_k. \tag{4.48}$$

When we add equations (4.41), (4.45), (4.44) and (4.48) together, we have

$$\begin{aligned} x_{i(m-1)+1,n} + x_{i(m-1)+1,n+1} + x_{i(m-1)+2,n} + x_{i(m-1)+2,n+1} \\ = \Big(y_n + (\sum_{k=1}^{i} d_k)\Big) + \Big(y_1 + (\sum_{k=i+1}^{m} d_k)\Big) \\ + \Big(z_n + (\sum_{k=i}^{m} d_k)\Big) + \Big(z_1 + (\sum_{k=1}^{i-1} d_k)\Big) \\ = \Big(y_1 + y_n + (\sum_{k=1}^{m} d_k)\Big) + \Big(z_1 + z_n + (\sum_{k=1}^{m} d_k)\Big) = S. \end{aligned}$$

When we add equations (4.42), (4.46), (4.43) and (4.47) together, we have

$$x_{i(m-1)+m+1,n} + x_{i(m-1)+m+1,n+1} + x_{i(m-1)+m+2,n} + x_{i(m-1)+m+2,n+1} = S.$$

For $1 \le j \le n$, by equations (4.17) and (4.18), we have

$$\begin{aligned}
&\{x_{i(m-1)+1,in+j} : i = 0, 1, \ldots, 2m-1\} \\
&= \{x_{i(m-1)+1,in+j}, x_{(i+m)(m-1)+1,(i+m)n+j} : i = 0, 1, \ldots, m-1\} \\
&= \{x_{i(m-1)+1,in+j}, x_{i(m-1)+1,(i+1)n+j} : i = 1, 2, \ldots, m\} \\
&= \{y_j + (-1)^{j+1} \sum_{k=1}^{i} d_k, y_j + (-1)^{j+1} \sum_{k=i+1}^{m} d_k : i = 0, 1, \ldots, m-1\} \\
&= \{y_j + (-1)^{j+1} a : a \in A\} = A_j.
\end{aligned}$$

For $1 \le j \le n$, by equations (4.19) and (4.20), we have

$$\begin{aligned}
&\{x_{i(m-1)+m+1,in+j} : i = 0, 1, \ldots, 2m-1\} \\
&= \{x_{i(m-1)+m+1,in+j}, x_{(i+m)(m-1)+m+1,(i+m)n+j} : i = 0, 1, \ldots, m-1\} \\
&= \{x_{i(m-1)+m+1,in+j}, x_{i(m-1)+m+1,(i+1)n+j} : i = 1, 2, \ldots, m\} \\
&= \{z_j + (-1)^{j+1} \sum_{k=1}^{i} d_k, z_j + (-1)^{j+1} \sum_{k=i+1}^{m} d_k : i = 0, 1, \ldots, m-1\} \\
&= \{z_j + (-1)^{j+1} a : a \in A\} = A_{j+n}.
\end{aligned}$$

Since $\{A_j, A_{j+n} : j = 1, 2, \ldots, n\}$ is a partition of $\{k \in \mathbb{Z} : 1 \le k \le 4mn\}$, $\{x_{i,j} : i = 1, 2, \ldots, 2m \text{ and } j = 1, 2, \ldots, 2n\} = \{k \in \mathbb{Z} : 1 \le k \le 4mn\}$. By Lemma 2.2, $\{x_{i,j}\}$ is a C_4-face-magic labeling on $C_{2m} \times C_{2n}$.

We need to show that $\{x_{i,j}\}$ is antipodal balanced. Let i and j be integers such that $0 \le i \le m$ and $1 \le j \le n$. We add equations (4.17) and (4.20) together to obtain

$$\begin{aligned}
&x_{i(m-1)+1,in+j} + x_{i(m-1)+m+1,(i+1)n+j} \\
&= y_j + (-1)^{j+1} \sum_{k=1}^{i} d_k + z_j + (-1)^{j+1} \sum_{k=i+1}^{m} d_k \\
&= y_j + +z_j + (-1)^{j+1} \sum_{k=1}^{m} d_k = \tfrac{1}{2} S = 4mn + 1.
\end{aligned}$$

We add equations (4.18) and (4.19) together to obtain

$$\begin{aligned}
&x_{i(m-1)+1,(i+1)n+j} + x_{i(m-1)+m+1,in+j} \\
&= y_j + (-1)^{j+1} \sum_{k=i+1}^{i} d_k + z_j + (-1)^{j+1} \sum_{k=1}^{i} d_k \\
&= y_j + +z_j + (-1)^{j+1} \sum_{k=1}^{m} d_k = \tfrac{1}{2} S = 4mn + 1.
\end{aligned}$$

Hence, $\{x_{i,j}\}$ is an antipodal balanced C_4-face-magic toroidal labeling on $C_{2m} \times C_{2n}$. □

Proposition 4.10. *Suppose m and n are odd positive integers. Suppose i and j are integers such that $1 \le i \le m$, $1 \le j \le n$, and j is odd. Let*

$$\begin{aligned} x_{i(m-1)+1,in+j} &= 2m(j-1)+2i, \\ x_{i(m-1)+1,(i+1)n+j} &= 2mj-2i+1, \\ x_{i(m-1)+m+1,in+j} &= 2m(2n-j)+2i \text{ and} \\ x_{i(m-1)+m+1,(i+1)n+j} &= 2m(2n-j+1)-2i+1. \end{aligned}$$

Suppose i and j are integers such that $1 \le i \le m$, $1 \le j \le n$, and j is even. Let

$$\begin{aligned} x_{i(m-1)+1,in+j} &= 2mj-2i+1, \\ x_{i(m-1)+1,(i+1)n+j} &= 2m(j-1)+2i, \\ x_{i(m-1)+m+1,in+j} &= 2m(2n-j+1)-2i+1 \text{ and} \\ x_{i(m-1)+m+1,(i+1)n+j} &= 2m(2n-j)+2i. \end{aligned}$$

Then, $\{x_{i,j}\}$ is an antipodal balanced C_4-face-magic toroidal labeling on $C_{2m} \times C_{2n}$.

Proof. This labeling $\{x_{i,j}\}$ corresponds to the labeling in Proposition 4.9 where for integers j such that $1 \le j \le n$ and j is odd, $y_j = 2m(j-1)+1$ and $z_j = 2m(2n-j)+1$; for integers j such that $1 \le j \le n$ and j is even, $y_j = 2mj$ and $z_j = 2m(2n-j+1)$; and $d_1 = 1$ and $d_k = 2$ for integers k such that $2 \le k \le m$.

Let j be an integer such that $1 \le j \le n$ and j is odd. Then,

$$\begin{aligned} y_j + z_j + (-1)^{j+1}\Big(\sum_{k=1}^{m} d_k\Big) &= \big(2m(j-1)+1\big) \\ &+\big(2m(2n-j)+1\big) + (2m-1) = 4mn+1. \end{aligned}$$

Let j be an integer such that $1 \le j \le n$ and j is even. Then,

$$y_j + z_j + (-1)^{j+1}\Big(\sum_{k=1}^{m} d_k\Big) = 2mj + 2m(2n-j+1) - (2m-1) = 4mn+1.$$

Thus condition (1) of Proposition 4.9 is satisfied.

We have

$$\begin{aligned} A &= \{\sum_{k=1}^{i} d_k, \sum_{k=i+1}^{m} d_k : i = 1,2,\ldots,m\} \cup \{0\} \\ &= \{2i-1, 2(m-i) : i = 1,2,\ldots,m\} \cup \{0\} = \{k \in \mathbb{Z} : 0 \le k \le 2m-1\}. \end{aligned}$$

Let j be an integer such that $1 \le j \le n$ and j is odd. Then,

$$A_j = \{y_j + (-1)^{j+1}a : a \in A\} = \{2m(j-1)+1+a : a \in A\}$$

and

$$A_{j+n} = \{z_j + (-1)^{j+1}a : a \in A\} = \{2m(2n-j)+1+a : a \in A\}.$$

Let j be an integer such that $1 \le j \le n$ and j is even. Then,

$$A_j = \{y_j + (-1)^{j+1}a : a \in A\} = \{2mj - a : a \in A\}$$
$$= \{2m(j-1) + 1 + a' : a' \in A\}$$

and

$$A_{j+n} = \{z_j + (-1)^{j+1}a : a \in A\} = \{2m(2n - j + 1) - a : a \in A\}$$
$$= \{2m(2n-j) + 1 + a' : a' \in A\}.$$

Hence, $\{A_j : 1 \le j \le 2n\}$ forms a partition of $\{k \in \mathbb{Z} : 1 \le k \le 4mn\}$. Thus condition (2) of Proposition 4.9 is satisfied. Therefore by Proposition 4.9, $\{x_{i,j}\}$ is an antipodal balanced C_4-face-magic toroidal labeling on $C_{2m} \times C_{2n}$. □

Example 4.11. We consider the example of Proposition 4.10 where $m = n = 3$. This example is given in Table 7.

1	12	13	6	7	18
34	27	22	33	28	21
5	8	17	2	11	14
31	30	19	36	25	24
4	9	16	3	10	15
35	26	23	32	29	20

Table 7: An antipodal balanced C_4-face-magic toroidal labeling on $C_6 \times C_6$.

We need the following converse to Proposition 4.9 in order to prove our last result in this paper.

Proposition 4.12. *Let m and n be odd positive integers. Let $\{x_{i,j} : (i,j) \in V(C_{2m} \times C_{2n})\}$ be an antipodal balanced C_4-face-magic toroidal labeling on $C_{2m} \times C_{2n}$. For all integers j such that $1 \le j \le n$, let $y_j = x_{1,j}$ and $z_j = x_{m+1,j}$. For all integers i such that $1 \le i \le m$, let*

$$d_i = x_{i(m-1)+1,in+1} - x_{(i-1)(m-1)+1,(i-1)n+1}.$$

Then, for all integers i and j such that $0 \le i \le m$ and $1 \le j \le n$, we have

$$x_{i(m-1)+1,in+j} = y_j + (-1)^{j+1} \sum_{k=1}^{i} d_k, \tag{4.49}$$

$$x_{i(m-1)+1,(i+1)n+j} = y_j + (-1)^{j+1} \sum_{k=i+1}^{m} d_k, \tag{4.50}$$

$$x_{i(m-1)+m+1,in+j} = z_j + (-1)^{j+1} \sum_{k=1}^{i} d_k \text{ and} \tag{4.51}$$

$$x_{i(m-1)+m+1,(i+1)n+j} = z_j + (-1)^{j+1} \sum_{k=i+1}^{m} d_k. \tag{4.52}$$

Let $A = \{\sum_{k=1}^{i} d_k, \sum_{k=i}^{m} d_k : 1 \leq i \leq m\} \cup \{0\}$*. For all integers* j *such that* $1 \leq j \leq n$*, let*

$$A_j = \{y_j + (-1)^{j+1}a : a \in A\} \text{ and } A_{j+n} = \{z_j + (-1)^{j+1}a : a \in A\}.$$

Then

1. *for all integers* j *such that* $1 \leq j \leq n$*,* $y_j + z_j + (-1)^{j+1}\left(\sum_{k=1}^{m} d_k\right) = 4mn + 1$*, and*
2. *the set* $\{A_j : 1 \leq j \leq 2n\}$ *forms a partition of the set* $\{k \in \mathbb{Z} : 1 \leq k \leq 4mn\}$*.*

Proof. By Lemma 2.11, for all integers i and j such that $1 \leq i \leq m$ and $1 \leq j \leq 2n$, we have

$$x_{i(m-1)+1,in+j} = x_{(i-1)(m-1)+1,(i-1)n+j} + (-1)^{j+1}d_i \text{ and} \tag{4.53}$$

$$x_{i(m-1)+m+1,in+j} = x_{(i-1)(m-1)+m+1,(i-1)n+j} + (-1)^{j+1}d_i. \tag{4.54}$$

For integers i and j such that $1 \leq i \leq m$ and $1 \leq j \leq n$, repeated use of equation (4.53) yields

$$x_{i(m-1)+1,in+j} = x_{1,j} + (-1)^{j+1}\sum_{k=1}^{i} d_k. \tag{4.55}$$

Thus equation (4.49) holds. When we let $i = m$ in equation (4.55), we have

$$x_{1,n+j} = x_{1,j} + (-1)^{j+1}\sum_{k=1}^{m} d_k. \tag{4.56}$$

When we replace j with $j + n$ in equation (4.53), for integers i and j such that $1 \leq i \leq m$ and $1 \leq j \leq n$, we have

$$x_{i(m-1)+1,(i+1)n+j} = x_{(i-1)(m-1)+1,in+j} + (-1)^{j}d_i. \tag{4.57}$$

For integers i and j such that $1 \leq i \leq m$ and $1 \leq j \leq n$, repeated use of equation (4.57) yields

$$x_{i(m-1)+1,(i+1)n+j} = x_{1,n+j} + (-1)^{j}\sum_{k=1}^{i} d_k. \tag{4.58}$$

Combining equation (4.58) with equation (4.56) yields

$$x_{i(m-1)+1,(i+1)n+j} = x_{1,j} + (-1)^{j+1}\sum_{k=i+1}^{m} d_k.$$

Thus equation (4.50) holds. A similar proof using equation (4.54) shows that equations (4.51) and (4.52) hold.

A proof similar to that in Proposition 4.8 shows that, for all integers j such that $1 \leq j \leq n$, we have $y_j + z_j + (-1)^{j+1}\left(\sum_{k=1}^{m} d_k\right) = 4mn + 1$, and the set $\{A_j : 1 \leq j \leq 2n\}$ forms a partition of the set $\{k \in \mathbb{Z} : 1 \leq k \leq 4mn\}$. □

Proposition 4.13. *Let* m *and* n *be positive odd integers. Then,* $C_{2m} \times C_{2n}$ *has no antipodal balanced* C_4*-face-magic toroidal labeling that is both row-sum balanced and column-sum balanced.*

Proof. Let $\{x_{i,j}\}$ be an antipodal balanced C_4-face-magic toroidal labeling on $C_{2m} \times C_{2n}$ that is both row-sum balanced and column-sum balanced. For all integers j such that $1 \le j \le n$, let $y_j = x_{1,j}$ and $z_j = x_{m+1,j}$. For all integers i such that $1 \le i \le m$, let

$$d_i = x_{i(m-1)+1,in+1} - x_{(i-1)(m-1)+1,(i-1)n+1}.$$

By Proposition 4.12, for all integers i such that $1 \le i \le m$, equations (4.49), (4.50), (4.51), and (4.52) hold.

Let

$$T = \sum_{j=1}^{n} \Big(y_j + z_j + (-1)^{j+1} \big(\sum_{k=1}^{m} d_k\big) \Big).$$

By Proposition 4.12, we have $y_j + z_j + (-1)^{j+1}\big(\sum_{k=1}^{m} d_k\big) = 4mn + 1$. Then

$$\begin{aligned} T &= \sum_{j=1}^{n} \Big(y_j + z_j + (-1)^{j+1} \big(\sum_{k=1}^{m} d_k\big) \Big) \\ &= \sum_{j=1}^{n} (4mn+1) = n(4mn+1) \equiv 1 \pmod 2. \end{aligned}$$

For $i = 1, 2, \dots, 2m$, let $R_i = \sum_{j=1}^{2n} x_{i,j}$ be the row sum of the labels on the vertices in row i. Also, for $j = 1, 2, \dots, 2n$, let $C_j = \sum_{i=1}^{2m} x_{i,j}$ be the column sum of the labels on the vertices in column j. Let m_1 be the integer such that $m = 2m_1 + 1$. Then for any integer j such that $1 \le j \le n$, we have

$$\begin{aligned} C_j &= \sum_{i=1}^{m} \big(x_{i(m-1)+1,j} + x_{i(m-1)+m+1,j} \big) \\ &= \sum_{i=1}^{m_1+1} \big(x_{(2i-1)(m-1)+1,j} + x_{(2i-1)(m-1)+m+1,j} \big) \\ &\quad + \sum_{i=1}^{m_1} \big(x_{(2i)(m-1)+1,j} + x_{(2i)(m-1)+m+1,j} \big) \\ &= \sum_{i=1}^{m_1+1} \Big(y_j + (-1)^{j+1} \big(\sum_{k=2i}^{m} d_k\big) + z_j + (-1)^{j+1} \big(\sum_{k=2i}^{m} d_k\big) \Big) \\ &\quad + \sum_{i=1}^{m_1} \Big(y_j + (-1)^{j+1} \big(\sum_{k=1}^{2i} d_k\big) + z_j + (-1)^{j+1} \big(\sum_{k=1}^{2i} d_k\big) \Big) \\ &= m(y_j + z_j) + (-1)^{j+1} \big(\sum_{k=1}^{m} (m + (-1)^k) d_k\big). \end{aligned}$$

We also have

$$\begin{aligned}C_{n+j} &= \sum_{i=1}^{m}\big(x_{i(m-1)+1,n+j} + x_{i(m-1)+m+1,n+j}\big)\\ &= \sum_{i=1}^{m_1+1}\big(x_{(2i-1)(m-1)+1,n+j} + x_{(2i-1)(m-1)+m+1,n+j}\big)\\ &\quad + \sum_{i=1}^{m_1}\big(x_{(2i)(m-1)+1,n+j} + x_{(2i)(m-1)+m+1,n+j}\big)\\ &= \sum_{i=1}^{m_1+1}\Big(y_j + (-1)^{j+1}\big(\sum_{k=1}^{2i-1} d_k\big) + z_j + (-1)^{j+1}\big(\sum_{k=1}^{2i-1} d_k\big)\Big)\\ &\quad + \sum_{i=1}^{m_1}\Big(y_j + (-1)^{j+1}\big(\sum_{k=2i+1}^{m} d_k\big) + z_j + (-1)^{j+1}\big(\sum_{k=2i+1}^{m} d_k\big)\Big)\\ &= m(y_j+z_j) + (-1)^{j+1}\big(\sum_{k=1}^{m}(m+(-1)^{k+1})d_k\big).\end{aligned}$$

Since $C_j = C_{n+j}$, we have $\sum_{k=1}^{m}(-1)^k d_k = 0$.

We have

$$R_1 = \sum_{j=1}^{n}\big(x_{1,j} + x_{1,n+j}\big) = \sum_{j=1}^{n}\Big(y_j + y_j + (-1)^{j+1}\sum_{k=1}^{m} d_k\Big) = 2\sum_{j=1}^{n} y_j + \sum_{k=1}^{m} d_k.$$

We also have

$$\begin{aligned}R_{m+1} &= \sum_{j=1}^{n}\big(x_{m+1,j} + x_{m+1,n+j}\big) = \sum_{j=1}^{n}\Big(z_j + z_j + (-1)^{j+1}\sum_{k=1}^{m} d_k\Big)\\ &= 2\sum_{j=1}^{n} z_j + \sum_{k=1}^{m} d_k.\end{aligned}$$

Since $R_1 = R_{m+1}$, we have $\sum_{j=1}^{n} y_j = \sum_{j=1}^{n} z_j$. Thus

$$\begin{aligned}T &= \sum_{j=1}^{n}\Big(y_j + z_j + (-1)^{j+1}\big(\sum_{k=1}^{m} d_k\big)\Big)\\ &= \sum_{j=1}^{n} y_j + \sum_{j=1}^{n} z_j + \sum_{k=1}^{m} d_k + \sum_{k=1}^{m}(-1)^k d_k\\ &\equiv 2\big(\sum_{j=1}^{n} y_j\big) + \sum_{i=1}^{m_1} 2d_{2i} \equiv 0 \pmod 2.\end{aligned}$$

This contradicts $T \equiv 1 \pmod 2$. □

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