ISSN 2590-9770
The Art of Discrete and Applied Mathematics 4 (2021) #P3.04
https://doi.org/10.26493/2590-9770.1358.428
(Also available at http://adam-journal.eu)

Realizations of lattice quotients of Petrie-Coxeter polyhedra*

Gábor Gévay[†]
Bolyai Institute, University of Szeged, Aradi vértanúk tere 1, H-6720 Szeged, Hungary

Egon Schulte[‡]
Department of Mathematics, Northeastern University, Boston, MA 02115, USA

Received 19 February 2020, accepted 01 June 2020, published online 23 August 2021

Abstract

The Petrie-Coxeter polyhedra naturally give rise to several infinite families of finite regular maps on closed surfaces embedded into the 3-torus. For the dual pair of Petrie-Coxeter polyhedra $\{4, 6 \,|\, 4\}$ and $\{6, 4 \,|\, 4\}$, we describe highly-symmetric embeddings of these maps as geometric, combinatorially regular polyhedra (polyhedral 2-manifolds) with convex faces in Euclidean spaces of dimensions 5 and 6. In each case the geometric symmetry group is a subgroup of index 1 or 2 in the combinatorial automorphism group.

IN MEMORY OF BRANKO GRÜNBAUM.

Keywords: Regular polyhedron, regular map, Petrie-Coxeter polyhedron, polyhedral embedding, polyhedral 2-manifold, automorphism group.

Math. Subj. Class.: 51M20, 52B70.

*The authors are grateful to Marston Conder for performing computations on the automorphism groups of the regular maps studied in this paper, for genus up to 1001. The computations have provided valuable insights into the structure of these groups. We would also like to thank the referees for their careful reading of the manuscript and for their useful comments that have helped improve the paper.

[†]Corresponding author. Supported by the Hungarian National Research, Development and Innovation Office, OTKA grant No. SNN 132625.

[‡]Supported by the Simons Foundation Award No. 420718.

E-mail addresses: gevay@math.u-szeged.hu (Gábor Gévay), e.schulte@northeastern.edu (Egon Schulte)

1 Introduction

In the 1930's, Coxeter and Petrie discovered three remarkable infinite regular polyhedra (apeirohedra) with convex faces and skew vertex-figures in Euclidean 3-space $\mathbb{E}^3$ (see [7]). These polyhedra are often called the *Petrie-Coxeter polyhedra* and are known to be the only infinite regular polyhedra in $\mathbb{E}^3$ that have convex faces (see [10, 11, 16, 23]).

The Petrie-Coxeter polyhedra naturally give rise to several infinite families of finite regular maps (abstract regular polyhedra) on closed surfaces embedded into the 3-torus. In this paper we focus on the dual pair of Petrie-Coxeter polyhedra $\{4, 6 \,|\, 4\}$ and $\{6, 4 \,|\, 4\}$ (see Section 3 for notation) and investigate the maps in certain infinite families in more detail. These maps occur in dual pairs of types $\{4, 6\}$ and $\{6, 4\}$, and have genus $1 + 2t^3$ and automorphism groups $D_t^3 \rtimes D_6$, $t \geq 1$. In particular, we derive presentations for their automorphism groups and prove isomorphism with Coxeter's regular maps $\{4, 6 \,|\, 4, 2t\}$ and their duals [7, p. 57], which were rediscovered in [23, pp. 259]. For $t \geq 3$, we describe highly-symmetric polyhedral realizations as geometric polyhedra with convex faces in dimensions 5 and 6. These geometric polyhedra are free of self-intersections and are polyhedral 2-manifolds (polyhedral embeddings) in the sense of [5, 25, 26]. The 6-dimensional polyhedra are particularly remarkable, in that these are combinatorially regular polyhedra whose geometric symmetry group is a subgroup of index 1 or 2 in the full automorphism group. For each map of type $\{4, 6\}$ or $\{6, 4\}$, exactly one of the 6-dimensional realizations is a geometrically regular polyhedron (in this case the index is 1). In all other cases, the realizations are "combinatorially regular polyhedra of index 2" (that is, combinatorially regular polyhedra with the property that exactly one half of all combinatorial symmetries is realized by geometric isometries of the ambient space [9, 19, 37]).

The study of polyhedral 2-manifolds has attracted a lot of attention. Remarkable and beautiful 3-dimensional polyhedral embeddings of regular maps of relatively small genus have been discovered (see [33] for a survey of the maps up to genus 6, as well as [32] and [3]), including realizations for some well-known regular maps such as Klein's maps $\{3, 7\}_8$ [30] and $\{7, 3\}_8$ [22] of genus 3 (the latter realized with non-convex planar faces), Dyck's maps $\{3, 8\}_6$ of genus 3 [4, 1], Coxeter's maps $\{4, 6|3\}$ and $\{6, 4|3\}$ of genus 6 [31], and quite recently, Hurwitz's regular map of type $\{3, 7\}$ and genus 7 (sometimes called the Macbeath map) [2], which is the second smallest among the Hurwitz maps (regular maps of type $\{3, 7\}$), the smallest being Klein's $\{3, 7\}_8$. These realizations naturally have small symmetry groups compared with the much larger automorphism group. Appealing topological pictures of regular maps have been created in [35, 36] and [34].

There are also interesting infinite families of regular maps that have polyhedral embeddings in $\mathbb{E}^3$. In [25, 26], two remarkable infinite series of polyhedra of types $\{4, r\}$ and $\{r, 4\}$, $r \geq 3$, and genus $1 + (r - 4)2^{r-3}$ were described, and these are polyhedral embeddings of Coxeter's regular maps $\{4, r \,|\, 4^{\lfloor r/2 \rfloor - 1}\}$ and their duals [24]. The maps in each series also admit polyhedral embeddings in $\mathbb{E}^4$ as subcomplexes of the boundary complex of certain convex 4-polytopes [20, 21, 29].

2 Maps, polyhedra, and polyhedral realizations

A *map* $\mathcal{M}$ on a closed surface S is a decomposition (tessellation) of S into non-overlapping simply-connected regions, called the *faces* of $\mathcal{M}$, by arcs, called the *edges* of $\mathcal{M}$, joining pairs of points, called the *vertices* of $\mathcal{M}$, such that two conditions are satisfied: first, each edge belongs to exactly two faces; and second, if two distinct edges intersect, they meet in one vertex or in two vertices (see [15, 8]).

For most maps on surfaces, the underlying set of vertices, edges, and faces, ordered by inclusion, forms an abstract polyhedron (abstract polytope of rank 3). As we will explain in a moment, conversely, every (locally finite) abstract polyhedron gives rise to a map on a surface. Following [23] (with minor modifications), an *abstract polyhedron* $\mathcal{P}$ is a partially ordered set with a *rank* function with range $\{0, 1, 2\}$. For $j = 0, 1, 2$, an element of $\mathcal{P}$ of rank j is called a *j-face* of $\mathcal{P}$, or *vertex*, *edge*, and *face*, respectively. A *flag* of $\mathcal{P}$ is an incident triple consisting of a vertex, an edge, and a face. Two flags are *adjacent* if they differ by one element. Further, $\mathcal{P}$ is *strongly flag-connected*, meaning that any two flags Φ and Ψ of $\mathcal{P}$ can be joined by a sequence of flags $\Phi = \Phi_0, \Phi_1, \ldots, \Phi_k = \Psi$, where the flags Φ_{i-1} and Φ_i are *adjacent* for each $i \geq 1$, and $\Phi \cap \Psi \subseteq \Phi_i$ for each $i \geq 0$. Finally, every edge contains exactly two vertices and lies in exactly two faces, and every vertex of every face lies in exactly two edges of that face.

From now on, we are making the implicit assumption that all maps considered in this paper are abstract polyhedra. For a discussion about this polytopality assumption see also [13].

Every (locally finite) abstract polyhedron lives naturally as a map on a closed surface. This surface is the underlying topological space of the order complex (“combinatorial barycentric subdivision”) of $\mathcal{P}$. Recall that the *order complex* is the 2-dimensional abstract simplicial complex, whose vertices are the vertices, edges and faces of $\mathcal{P}$, and whose simplices are the chains (subsets of flags) of $\mathcal{P}$ (see [23, Ch. 2C]). The maximal simplices are in one-to-one correspondence with the flags of $\mathcal{P}$, and are 2-dimensional (triangles). Adjacency of flags in $\mathcal{P}$ corresponds to adjacency of triangles on the surface.

We require further terminology and notation that applies to both abstract polyhedra and maps. For $0 \leq j \leq 2$, every flag Φ is adjacent to just one flag, denoted Φ^j, differing by the j-face; the flags Φ and Φ^j are said to be *j-adjacent* to each other. For integers $j_1, \ldots, j_l$ (with $l \geq 2$ and $0 \leq j_1, \ldots, j_l \leq 2$) we inductively define the new flag

$$\Phi^{j_1 \cdots j_l} := (\Phi^{j_1 \cdots j_{l-1}})^{j_l}.$$

Note that $\Phi, \Phi^{j_1}, \Phi^{j_1 j_2}, \ldots, \Phi^{j_1 \cdots j_l}$ is a sequence of successively adjacent flags.

If each face of an abstract polyhedron $\mathcal{P}$ (or a map $\mathcal{M}$) has p vertices and each vertex lies in q faces, then $\mathcal{P}$ (or $\mathcal{M}$) is said to be of (*Schläfli*) *type* $\{p, q\}$.

An abstract polyhedron $\mathcal{P}$ (or a map $\mathcal{M}$) is called *regular* if its automorphism group $\Gamma(\mathcal{P})$ (or $\Gamma(\mathcal{M})$) is transitive on the flags. Suppose $\mathcal{P}$ is an abstract regular polyhedron and $\Phi := \{F_0, F_1, F_2\}$ is a (fixed) *base* flag of $\mathcal{P}$. Then $\Gamma(\mathcal{P})$ is generated by *distinguished generators* ρ_0, ρ_1, ρ_2 (*with respect to* Φ), where ρ_j is the unique automorphism which fixes all elements of Φ but the j-face. Thus $\rho_j(\Phi) = \Phi^j$ for each $j = 0, 1, 2$. These generators satisfy the standard Coxeter-type relations

$$\rho_0^2 = \rho_1^2 = \rho_2^2 = (\rho_0\rho_1)^p = (\rho_1\rho_2)^q = (\rho_0\rho_2)^2 = 1 \tag{2.1}$$

determined by the type $\{p, q\}$; in general there are also other independent relations.

A *k-hole* of an abstract polyhedron $\mathcal{P}$ (or a map $\mathcal{M}$) is an edge-path which leaves a vertex by the k-th edge from which it entered, in the same sense (that is, always keeping to the left, say, in some local orientation). If a regular polyhedron $\mathcal{P}$ has k-holes of length h_k, then the distinguished generators of $\Gamma(\mathcal{P})$ satisfy the additional relation

$$(\rho_0\rho_1(\rho_2\rho_1)^{k-1})^{h_k} = 1. \tag{2.2}$$

When $k = 2$ we refer to a k-hole simply as a *hole*. Thus a hole of a regular polyhedron of length h contributes the relation $(\rho_0\rho_1\rho_2\rho_1)^h = 1$. A *Petrie polygon* of $\mathcal{P}$ (or $\mathcal{M}$) is an edge-path with the property that any two successive edges, but not three, are edges of a face of $\mathcal{P}$. A Petrie polygon of length t contributes the relations $(\rho_0\rho_1\rho_2)^t = 1$.

For the purpose of this paper, a *geometric polyhedron* P is a closed surface embedded in a Euclidean space made up of finitely many *convex* polygons, the *faces* of P, such that any two distinct polygons intersect, if at all, in a common vertex or a common edge (see [15, 5]). Thus P has convex faces and is free of self-intersections. We require additionally that no two faces with a common edge lie in the same 2-dimensional plane. In [5], these polyhedra were called *polyhedral 2-manifolds*. We usually identify P with the map on the underlying surface, which, in our applications, will be orientable, or with the abstract polyhedron consisting of the vertices and edges (of the polygons) and the faces, partially ordered by inclusion. We also call P a *polyhedral realization* of the map or abstract polyhedron. By $G(P)$ we denote the geometric symmetry group consisting of the isometries of the ambient space that map P to itself. A geometric polyhedron P is *geometrically regular* if $G(P)$ acts transitively on the flags.

The edge-graph of a (geometric or an abstract) polyhedron is also called the *1-skeleton* of the polyhedron; its vertices and edges are formed by the vertices and edges of the polyhedron. Similarly, for a convex d-polytope K and $0 \leq k \leq d$, the *k-skeleton* of K is the subcomplex of the face-lattice of K consisting of the faces of dimension less than or equal to k.

A convex hexagon is called *semi-regular* if its symmetry group acts transitively on the vertices. In a semi-regular hexagon which is not regular, the edges are of two kinds and alternate in length.

3 The Petrie-Coxeter polyhedra revisited

Among the geometrically regular polyhedra in Euclidean 3-space $\mathbb{E}^3$, the three Petrie-Coxeter polyhedra $\{4, 6\,|\,4\}$, $\{6, 4\,|\,4\}$ and $\{6, 6\,|\,3\}$ are characterized as the infinite polyhedra (apeirohedra) with convex faces (see [7, 23]). Each polyhedron $\{p, q\,|\,h\}$ forms a periodic polyhedral surface which bounds a pair of congruent non-compact "polyhedral handlebodies" (the inside and outside) which tile $\mathbb{E}^3$. Each polyhedron has p-gonal convex faces and q-gonal skew vertex-figures, and has 2-holes of length h. The polyhedra were discovered by Petrie and Coxeter in 1930's (see [7]).

Like any geometrically regular polyhedron in $\mathbb{E}^3$, a Petrie-Coxeter polyhedron P has a flag-transitive symmetry group $G = G(P)$ isomorphic to the combinatorial automorphism group $\Gamma(P)$. The translation subgroup $T(P)$ of $G(P)$ is generated by three translations in independent directions and can be identified with the 3-dimensional *translation lattice* $\Sigma = \Sigma(P)$ in $\mathbb{E}^3$ spanned by the three corresponding translation vectors.

In this paper we focus on the dual pair of polyhedra $\{4, 6\,|\,4\}$ and $\{6, 4\,|\,4\}$. The duality is very explicit in this case, in that, up to similarity, either polyhedron could be constructed from the other by choosing the vertices at the face centers of the other; if related in this fashion, the polyhedra share the same symmetry group and translation subgroup. The translation lattice Σ for either polyhedron is the body-centered cubic lattice and thus contains a translation sublattice $\Lambda = \Lambda(P)$ of index 4 generated by three orthogonal vectors of equal lengths. When convenient, we may take Λ to be the standard integral lattice $\mathbb{Z}^3$, and Σ to be the lattice generated by the vectors $(1, 0, 0)$, $(0, 1, 0)$, $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ (denoted $\Lambda_{(1,1,1)}$ in [23, Section 6D] and [28]). Note that Λ is also a normal subgroup of $G(P)$.

The polyhedron $\{6,4\,|\,4\}$ is built from copies of an Archimedean truncated octahedron (or rather, parts of its surface). An Archimedean truncated octahedron Q has eight regular hexagons and six squares as faces [12]. Its eight regular hexagons (and their vertices and edges) form a subcomplex of the boundary complex of Q which we will call a *hexagonal octet complex*, or simply, an *HO complex* (see Figure 1). Clearly, both Q and its HO complex are naturally inscribed in a cube C bounded by the planes containing the six square faces of Q. Then the polyhedron $\{6,4\,|\,4\}$ can be constructed from the standard cubical tessellation $\mathcal{T}$ in $\mathbb{E}^3$ by translates of C by inscribing copies of the HO complex into the cubical tiles of $\mathcal{T}$ (see Figure 2). The copies of the HO complex placed into adjacent cubical tiles attach along the "missing square faces" of the corresponding truncated octahedra, and thus a polyhedral surface is formed. Note that the symmetry group of an HO complex coincides with that of its underlying truncated octahedron.

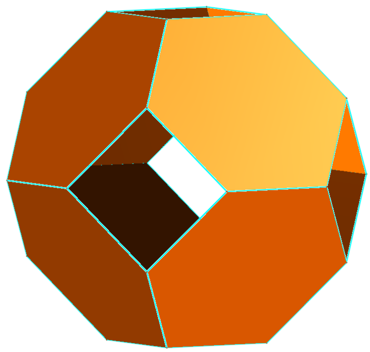

Figure 1: Hexagonal octet complex (HO complex).

Note that the polyhedron $\{6,4\,|\,4\}$ is also related to the Voronoi tiling of the body-centered cubic lattice [7], whose tiles are truncated octahedra tessellating the "outside" and the "inside" of the polyhedron. This tiling is among the 28 uniform tilings of $\mathbb{E}^3$ enumerated by Grünbaum [17]. The polyhedron then consists of all the hexagonal 2-faces of this tiling.

The dual polyhedron $\{4,6\,|\,4\}$ can also be constructed from a standard cubical tessellation $\mathcal{T}$ of $\mathbb{E}^3$. In this case, a copy of the 1-skeleton of a smaller cube of half the size is placed concentrically into each cubical tile of $\mathcal{T}$, and then the copies in adjacent cubical tiles of $\mathcal{T}$ are connected by cylindrical square-faced tunnels so that, at either end, a tunnel meets the corresponding copy of the 1-skeleton in a "missing square face" (see Figure 3). Alternatively, and less formally, each cubical tile of $\mathcal{T}$ is shrunk concentrically to half its size and then adjacent shrunk cubes are connected by cylindrical square-faced tunnels. We refer to the part of the polyhedron lying inside a single cube of $\mathcal{T}$ as a *6-elbow*. Thus a 6-elbow consists of six *half-tunnels* attached to a shrunk cube in a cross-like fashion, where each half-tunnel is formed by four rectangles with sides of lengths 1 and 1/2 emanating from a square face of the shrunk cube. The construction shows that the polyhedron has a natural *6-elbow decomposition*.

The polyhedron $\{4,6\,|\,4\}$ also admits a 3-elbow decomposition relative to the standard

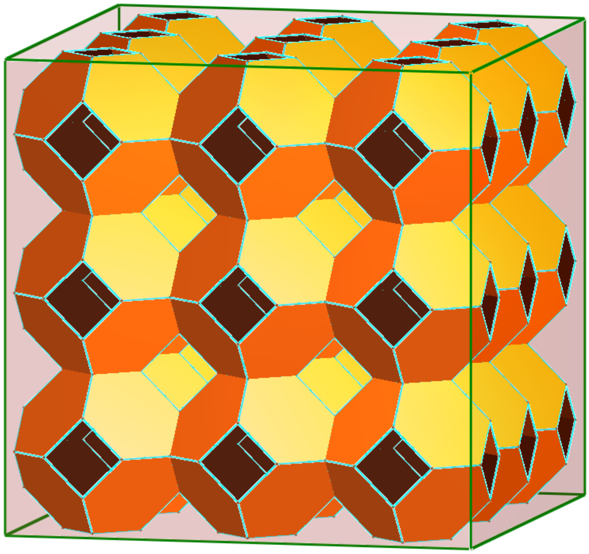

Figure 2: A $3 \times 3 \times 3$ block of the Petrie-Coxeter polyhedron $\{6, 4 \,|\, 4\}$.

cubical tessellation $\mathcal{T}$ of $\mathbb{E}^3$. Let e_1, e_2, e_3 denote the standard basis vectors of $\mathbb{E}^3$, let $a_i := \frac{1}{2} e_i$ for $i = 1, 2, 3$, and let

$$C := \{(x_1, x_2, x_3) \mid 0 \leq x_1, x_2, x_3 \leq 1\}.$$

Then $\{4, 6 \,|\, 4\}$ can be constructed as follows. The vertex set is again $\frac{1}{2}\,\mathbb{Z}^3$. The face set of $\{4, 6 \,|\, 4\}$ consists of the translates, under $\mathbb{Z}^3$, of a set of twelve particular faces contained in C. More precisely, the intersection of the entire polyhedron with C, denoted E, consists of the twelve square faces of the smaller cubes $a_i + \frac{1}{2}C$ $(i = 1, 2, 3)$ which do not lie in the planes $x_i = \frac{1}{2}$ or $x_i = 1$. Six of these square faces of E meet at the vertex $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ of the polyhedron. The entire polyhedron then consists of all translates $z + E$ with $z \in \mathbb{Z}^3$. Each translate $z + E$ coincides with the intersection of the polyhedron with the cubical tile $z + C$ of $\mathcal{T}$. We refer to these intersections as *3-elbows*. Then it is clear that the polyhedron admits a natural *3-elbow decomposition* such that every cubical tile of $\mathcal{T}$ contributes exactly one 3-elbow.

4 Petrie-Coxeter-type polyhedra in the 3-torus

The three Petrie-Coxeter polyhedra P naturally give rise to regular maps embedded in the 3-torus which usually are abstract regular polyhedra. This construction was also independently described by Montero [28]. Suppose as above that the translation subgroup $T(P)$ of the symmetry group $G(P)$ of P is identified with the translation lattice $\Sigma = \Sigma(P)$. Then the maps arise as quotients of P by subgroups of $T(P)$ which are normal in $G(P)$, or equivalently, as quotients by sublattices Ω of Σ, denoted P/Ω, which are invariant under $G(P)$.

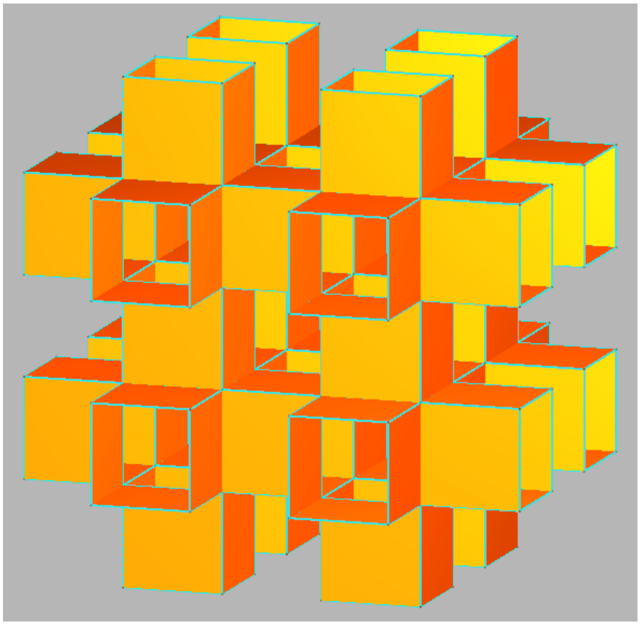

Figure 3: The Petrie-Coxeter polyhedron $\{4, 6\,|\,4\}$.

In this paper, we only consider the maps derived from the dual pair of polyhedra $\{6, 4\,|\,4\}$ and $\{4, 6\,|\,4\}$, and focus on those that later are realized polyhedrally. We choose, as sublattices Ω, certain scaled copies of the sublattice Λ of Σ (of index 4) described earlier, where here we identify Λ with $\mathbb{Z}^3$. The resulting maps are embedded in the 3-torus $\mathbb{E}^3/\Omega$ determined by Ω. We have not investigated quotients of P by other types of sublattices Ω of Σ.

4.1 The polyhedron $\mathcal{P}_t := \{6, 4\,|\,4\}/t\mathbb{Z}^3$

We begin with the polyhedron $P := \{6, 4\,|\,4\}$. Suppose P has been constructed from a standard cubical tessellation $\mathcal{T}$ by inscribing HO complexes into the cubical tiles of $\mathcal{T}$ so that the HO complexes in adjacent cubes are attached along "missing squares". Further, suppose that the origin o lies at the center of one of these HO complexes, at $\mathrm{HO_o}$ (say), and that Λ is generated by the translations by e_1, e_2, e_3, the standard basis vectors of $\mathbb{E}^3$. Thus $\Lambda = \mathbb{Z}^3$.

Now let $t \geq 1$ be an integer, and let $\Lambda_t := t\Lambda = t\mathbb{Z}^3$. Then Λ_t is invariant under $G(P)$ and hence

$$\mathcal{P}_t := P/\Lambda_t = \{6, 4\,|\,4\}/\Lambda_t$$

is a regular map of type $\{6, 4\}$ embedded in the 3-torus $\mathbb{E}^3/\Lambda_t$. The numbers of vertices, edges and faces of $\mathcal{P}_t$ are given by $12t^3$, $24t^3$ and $8t^3$, respectively. Thus the Euler characteristic is

$$\chi(\mathcal{P}_t) := 12t^3 - 24t^3 + 8t^3 = -4t^3,$$

and since $\mathcal{P}_t$ is orientable (as we will see), it must have genus $1 + 2t^3$. Clearly, since $\mathcal{P}_t$ is a regular map with 4-valent vertices, the order of the automorphism group $\Gamma(\mathcal{P}_t)$ of $\mathcal{P}_t$ is given by $96t^3 = 8 \cdot 12t^3$. The map is an abstract polyhedron for each $t \geq 1$. When $t = 1$ the map has the property that any two hexagon faces that meet in an edge also meet in the opposite edge, so in particular the map cannot admit a polyhedral realization with convex faces in any Euclidean space.

The structure of $\Gamma(\mathcal{P}_t)$ can be determined more explicitly. First note that

$$\Gamma(\mathcal{P}_t) = \Gamma(P/\Lambda_t) \cong \Gamma(P)/\Lambda_t \cong G(P)/\Lambda_t; \tag{4.1}$$

the second equality follows, for example, from [23, 2E18]. Clearly, by construction, $\Gamma(\mathcal{P}_t)$ contains a normal abelian subgroup $\mathbb{Z}_t^3$, where $\mathbb{Z}_t := \mathbb{Z}/t\mathbb{Z}$. This subgroup is generated by the three "translations" corresponding to three generators of Λ/Λ_t. Its normality is inherited from the normality of Λ_t in $G(P)$. It follows directly from (4.1) that a presentation for $\Gamma(\mathcal{P}_t)$ can be obtained from a presentation of $G(P) = \Gamma(P)$ by adding a single extra relation determining the quotient of P by Λ_t. More explicitly, since Λ_t is generated by the conjugates of the translation by the single vector te_1 (or by te_2, or by te_3), denoted T (say), it suffices to express this translation in terms of the standard generators of $G(P)$ and add the corresponding relation to the standard relations for $G(P)$. The details are as follows.

We begin by choosing a base flag $\Phi := \{F_0, F_1, F_2\}$ of $P = \{6, 4 \,|\, 4\}$. Then $G(P)$ is generated by three involutory isometries R_0, R_1, R_2 of $\mathbb{E}^3$ defined by the conditions $R_j(F_i) = F_i$, $R_i(F_i) \neq F_i$, for $i, j = 0, 1, 2$, $j \neq i$. The generators R_0 and R_2 are plane reflections and R_1 is a half-turn. In terms of these generators, $G(P)$ has the presentation

$$R_0^2 = R_1^2 = R_2^2 = (R_0R_1)^6 = (R_1R_2)^4 = (R_0R_2)^2 = (R_0R_1R_2R_1)^4 = 1. \tag{4.2}$$

Note that the relator $R_0R_1R_2R_1$ of the last relation shifts a 2-hole of P one step along itself and hence has period 4.

To explain the additional relation for the automorphism group $\Gamma(\mathcal{P}_t)$ of the quotient $\mathcal{P}_t$ we use the notation for flags introduced in Section 2. Consider the flags Φ and $\Phi^{101012101012}$ of the original polyhedron P, as well as the corresponding implied sequence of successively adjacent flags of P joining them,

$$\Phi, \Phi^1, \Phi^{10}, \Phi^{101}, \Phi^{1010}, \ldots, \Phi^{10101210101}, \Phi^{101012101012}.$$

On the underlying surface of P, flags correspond to triangles of the barycentric subdivision, and adjacency of flags corresponds to adjacency of triangles. Thus the sequence of flags translates into a sequence of triangles that starts with the triangle for the base flag, and inspection shows that the last triangle in the sequence is just the translate of the first triangle under the translation T. The expression of T in terms of the generators R_0, R_1, R_2 then can be derived from the flag sequence. In fact, taking the sequence of indices in reverse order we see that

$$T = R_2R_1R_0R_1R_0R_1R_2R_1R_0R_1R_0R_1 = (R_2R_1(R_0R_1)^2)^2$$

and hence $T^t = (R_2R_1(R_0R_1)^2)^{2t}$. Thus, a presentation for $\Gamma(\mathcal{P}_t)$ consists of the relations in (4.2) and the single extra relation

$$(R_2R_1(R_0R_1)^2)^{2t} = 1. \tag{4.3}$$

For a justification of the method employed see [23, Sect. 2B] or [27]. (The proper setting for the above argument is the monodromy group (connection group) of P, but since P

regular, this group is isomorphic to the automorphism group of P.) Note that each defining relation of $\Gamma(\mathcal{P}_t)$ involves an even number of generators. In particular this shows that $\mathcal{P}_t$ is orientable.

In Section 4.2 we will show that $\mathcal{P}_t$ is isomorphic to the orientable regular map $\{4,6\,|\,4,2t\}^*$, the dual of Coxeter's regular map $\{4,6\,|\,4,2t\}$, with automorphism group $D_t^3 \rtimes D_6$.

The length of the Petrie polygons of $\mathcal{P}_t$ is given by the order of $R_0R_1R_2$ in $\Gamma(\mathcal{P}_t)$, which is $6t$. In fact, the element $(R_0R_1R_2)^k$ of $G(P)$ lies in Λ if and only if $6\,|\,k$, and hence $(R_0R_1R_2)^k$ lies in Λ_t if and only if $6t\,|\,k$. This can either be seen directly geometrically, or by using a coordinate representation for $G(P)$ and its generators, for example the presentation of [23, p. 231]. Thus the element $R_0R_1R_2$ of $\Gamma(\mathcal{P}_t)$ has order $6t$.

Recall from [23, Sect. 6D] that the 3-torus $\mathbb{E}^3/\Lambda_t$ admits a regular tessellation by t^3 cubes obtained as the quotient of the standard cubical tessellation $\{4,3,4\}$ of $\mathbb{E}^3$ by Λ_t; this quotient, $\{4,3,4\}/\Lambda_t$, is called a *cubic toroid* and is denoted $\{4,3,4\}_{(t,0,0)}$. More informally, $\{4,3,4\}_{(t,0,0)}$ is obtained from a $t\times t\times t$ block of cubes by identifying opposite sides of the block. Note that we can think of $\mathcal{P}_t$ as being constructed from $\{4,3,4\}_{(t,0,0)}$ by inscribing copies of the HO complex into the cubical tiles, in the same way in which copies of the HO complex were inscribed into the cubical tiles of the cubical tessellation of $\mathbb{E}^3$ to construct P.

4.2 The polyhedron $\mathcal{Q}_t := \{4,6\,|\,4\}/t\mathbb{Z}^3$

For the dual polyhedron $Q := \{4,6\,|\,4\}$ of $P := \{6,4\,|\,4\}$, we may assume that its vertices lie at the face centers of P. Then $G(Q) = G(P)$ and $\Lambda = \Lambda(Q) = \Lambda(P)$. The distinguished generators of $G(Q)$ are just the distinguished generators of $G(P)$ taken in reverse order. If we label these new generators of $G(Q)$ by R_0, R_1, R_2 so that $R_j(F_i) = F_i$, $R_i(F_i) \neq F_i$, for $i,j = 0,1,2$, $j \neq i$, and some base flag $\{F_0, F_1, F_2\}$ of Q, then $G(Q)$ admits the presentation

$$R_0^2 = R_1^2 = R_2^2 = (R_0R_1)^4 = (R_1R_2)^6 = (R_0R_2)^2 = (R_0R_1R_2R_1)^4 = 1, \qquad (4.4)$$

where the order of the relator $R_0R_1R_2R_1$ gives the length of the 2-hole of Q, which again is 4.

If we set again $\Lambda_t := t\Lambda = t\mathbb{Z}^3$ for $t \geq 1$, then

$$Q_t := Q/\Lambda_t = \{4,6\,|\,4\}/\Lambda_t$$

is a regular map of type $\{4,6\}$ and genus $1+2t^3$ embedded in the 3-torus $\mathbb{E}^3/\Lambda_t$, and is the dual of $\mathcal{P}_t := P/\Lambda_t$. The numbers of vertices, edges, and faces of $\mathcal{Q}_t$ are given by $8t^3$, $24t^3$, and $12t^3$, respectively. The automorphism groups of $\mathcal{Q}_t$ and $\mathcal{P}_t$ are isomorphic. In particular, if we reverse the order of the generators in (4.2) and (4.3), we find that a presentation of $\Gamma(\mathcal{Q}_t)$ is given by the relations in (4.4) and the extra relation

$$(R_0R_1(R_2R_1)^2)^{2t} = 1. \qquad (4.5)$$

Now recall from Coxeter [7, p. 57] that the set of relations in (4.4) and (4.5) abstractly define the automorphism group Γ (say) of the regular map $\{4,6\,|\,4,2t\}$ determined by the lengths 4 and $2t$ of the 2-holes and 3-holes, respectively. Thus $\{4,6\,|\,4,2t\}$ is the "universal" regular map with 2-holes and 3-holes of lengths 4 and $2t$, respectively, and by

[7, p. 57] and [23, p. 259] is known to be finite and have an automorphism group isomorphic to $D_t^3 \rtimes D_6$, of order $96t^3$. As $\Gamma(\mathcal{Q}_t)$ satisfies all the defining relations of Γ, it must be a quotient of Γ, and since $\Gamma(\mathcal{Q}_t)$ has the same order as Γ, it must coincide with Γ. Thus $\Gamma(\mathcal{Q}_t) = \Gamma$ and

$$\mathcal{Q}_t = \{4, 6 \,|\, 4, 2t\},$$

and by duality,

$$\mathcal{P}_t = \{4, 6 \,|\, 4, 2t\}^*.$$

For $t = 2$ we obtain $Q_2 = \{4, 6 \,|\, 4, 4\}$, which also occurs (with $r = 6$) as the first non-toroidal map in Coxeter's infinite series of regular maps $\{4, r \,|\, 4^{\lfloor r/2 \rfloor - 1}\}$ mentioned in the Introduction. The Petrie polygons of the regular map $\mathcal{Q}_t$ are of course all of length $6t$, just like those of $\mathcal{P}_t$.

Note that we can construct $\mathcal{Q}_t$ from $\{4, 3, 4\}_{(t,0,0)}$ by inscribing copies of a 6-elbow into the cubical tiles, in the same way in which Q can be obtained by inscribing copies of a 6-elbow into the cubical tiles of the cubical tessellation of $\mathbb{E}^3$.

5 Embeddings in $\mathbb{E}^6$

The polyhedral embeddings of the Petrie-Coxeter type polyhedra $\mathcal{P}_t$ and $\mathcal{Q}_t$, $t \geq 3$, in Euclidean 6-space $\mathbb{E}^6$ will be constructed from an embedding of the 3-torus as a 3-dimensional cubical subcomplex in the boundary complex of a convex 6-polytope K_t. Then, via Schlegel diagrams [18], polyhedral embeddings can also be constructed in Euclidean 5-space $\mathbb{E}^5$. We begin by describing the polytope K_t in $\mathbb{E}^6$. Our construction requires that $t \geq 3$.

5.1 A polyhedral embedding of the cubic toroid

It is well-known that the 3-torus admits a natural embedding into the 5-sphere $\sqrt{3}\ \mathbb{S}^5$ (of radius $\sqrt{3}$) as the submanifold M consisting of all points $x := (x_1, \dots, x_6)$ of $\mathbb{E}^6$ satisfying the equations

$$x_1^2 + x_2^2 = x_3^2 + x_4^2 = x_5^2 + x_6^2 = 1.$$

This expresses M as the product $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$ lying in $\mathbb{E}^6$.

Let $t \geq 3$. We now construct a convex 6-polytope K_t in $\mathbb{E}^6$ with vertices in M and exploit its structure to find polyhedral realizations of the regular maps $\mathcal{P}_t$ and $\mathcal{Q}_t$ constructed in the previous section. For $j = 0, \dots, t-1$, set $a_j := 2\pi j/t$ and define the points

$$\begin{aligned} u_j &:= (\cos a_j, \sin a_j, 0, 0, 0, 0), \\ v_j &:= (0, 0, \cos a_j, \sin a_j, 0, 0), \\ w_j &:= (0, 0, 0, 0, \cos a_j, \sin a_j), \end{aligned}$$

all of which are contained in M. Then the convex 6-polytope

$$K_t := \operatorname{conv}\{u_j + v_k + w_l \,|\, j, k, l = 0, \dots, t-1\} \tag{5.1}$$

is the cartesian product of the three regular convex t-gons K_t^1, K_t^2 and K_t^3 with vertex-sets $\{u_0, \dots, u_{t-1}\}$, $\{v_0, \dots, v_{t-1}\}$ or $\{w_0, \dots, w_{t-1}\}$, respectively, lying in the 2-dimensional coordinate subspaces containing the first two, the third and the fourth, or the last two standard coordinate axes of $\mathbb{E}^6$. Thus

$$K_t = K_t^1 \times K_t^2 \times K_t^3.$$

Note that K_t is a 6-cube when $t = 4$. Now recall that the nonempty faces of a cartesian product of convex polytopes are just the cartesian products of the nonempty faces of its component polytopes [18]. In particular, the cartesian products of three edges from different component polygons K_t^j give t^3 3-dimensional faces of K_t that are 3-cubes. When $t \neq 4$ there are no other 3-faces of K_t which are 3-cubes; however, when $t = 4$ the polytope K_t is the 6-cube and thus each of its 3-faces is a 3-cube. In any case, the union of these t^3 3-faces obtained as products of three edges from different components forms a subcomplex $\mathcal{C}_t$ of the boundary complex of K_t which is homeomorphic to the 3-torus. This follows from the fact that $\mathcal{C}_t$ is topologically the product of the three unit circles determined by the boundary complexes of K_t^1, K_t^2 and K_t^3. Thus $\mathcal{C}_t$ is a 3-dimensional cubical complex embedded into the boundary complex of K_t and is isomorphic to the regular cubic toroid $\{4,3,4\}_{(t,0,0)}$.

When $t \neq 4$ the symmetry group of K_t is isomorphic to the wreath product $D_t \wr S_3$ ($\cong D_t^3 \rtimes S_3$), of order $48t^3$. However, when $t = 4$ the symmetry group of K_t is isomorphic to $C_2 \wr S_6$, of order $2^6\,6!$; this group still contains a subgroup isomorphic to $D_4 \wr S_3$, which is of index 15. In any case, for each $t \geq 3$, the subgroup $D_t \wr S_3$ of the symmetry group of K_t leaves the cubical complex $\mathcal{C}_t$ invariant. In fact, $D_t \wr S_3$ is the symmetry group of $\mathcal{C}_t$, and being of the right order, is also the full automorphism group of $\mathcal{C}_t$. It acts flag-transitively on $\mathcal{C}_t$.

5.2 A polyhedral embedding of $\mathcal{P}_t = \{6,4\,|\,4\}/t\mathbb{Z}^3$

We begin by rescaling the 6-polytope so that the 3-cubes in the cubical complex become unit 3-cubes. Each 3-cube in $\mathcal{C}_t$ has edge length $2\sin(\pi/t)$, which is the edge length of a regular convex t-gon inscribed in a unit circle. Thus the new convex polytope $K_t' := rK_t$, with $r := 1/(2\sin(\pi/t))$, as well as the corresponding cubical complex $\mathcal{C}_t' := r\,\mathcal{C}_t$, have the property that all faces which are 3-cubes are unit 3-cubes. In particular, $\mathcal{C}_t'$ is a faithful realization of the regular cubic toroid $\{4,3,4\}_{(t,0,0)}$. In other words, $\mathcal{C}_t'$ is isomorphic to $\{4,3,4\}_{(t,0,0)}$, and the automorphism group of $\mathcal{C}_t'$ (of order $48t^3$) is realized by a group of Euclidean isometries, namely $D_t \wr S_3$.

Now the construction of the embedding for $\mathcal{P}_t$ is straightforward: in each 3-cube of $\mathcal{C}_t'$ (or rather its support) we inscribe a basic building block for $\mathcal{P}_t$, that is, a copy of an HO complex. Note that the insertion in each 3-cube is unique. Thus we arrive at a polyhedral 2-manifold in $\mathbb{E}^6$ with regular convex hexagons as faces, which is isomorphic to $\mathcal{P}_t$ and invariant under $D_t \wr S_3$. We will show in the next section that the latter group provides only one half of all the geometric symmetries of the polyhedral 2-manifold. In fact, the polyhedral embedding for $\mathcal{P}_t$ is a geometrically regular polyhedron in $\mathbb{E}^6$ (with regular convex hexagons as faces), that is, its symmetry group is isomorphic to the full automorphism group $D_t^3 \rtimes D_6$.

There is also a host of 6-dimensional polyhedral realizations of $\mathcal{P}_t$ in $\mathbb{E}^6$ with convex hexagonal faces which are not regular. To construct these we can proceed in the same way as above, but now from a modified HO complex. The modified HO complex can be obtained from the HO complex of Figure 1 by shrinking or expanding the "missing squares" in a uniform fashion, so that the resulting octet consists of eight congruent semi-regular convex hexagons (with two kinds of edges). The resulting polyhedral 2-manifold has $D_t \wr S_3$ as its symmetry group, that is, it is a combinatorially regular polyhedron of index 2 in $\mathbb{E}^6$.

Via a Schlegel diagram of the 6-polytope K_t' we can also produce polyhedral realizations of $\mathcal{P}_t$ in Euclidean 5-space $\mathbb{E}^5$. The (possibly modified) HO complexes sitting inside the 3-cubes of $\mathcal{C}'_t$ project to distorted HO complexes sitting inside the 3-dimensional parallelepipeds which are the projections of the 3-cubes and make up the image of $\mathcal{C}'_t$ under the projection. There are no self-intersections, since $\mathcal{C}'_t$ is a subcomplex of the boundary complex of K'_t, and Schlegel diagrams faithfully represent the boundary complex of a convex polytope.

5.3 A polyhedral embedding of $\mathcal{Q}_t = \{4, 6 \,|\, 4\}/t\mathbb{Z}^3$

We describe two possible ways of constructing polyhedral realizations of the regular maps $\mathcal{Q}_t = \{4, 6 \,|\, 4\}/t\mathbb{Z}^3$ derived from $Q := \{4, 6 \,|\, 4\}$. The first is based on the 3-elbow decomposition of Q, and the second employs the 6-elbow decomposition of Q and has a larger symmetry group.

The first construction of the polyhedral embedding for $\mathcal{Q}_t$ proceeds as follows: in each 3-cube of $\mathcal{C}'_t$ we inscribe a basic building block for $\mathcal{Q}_t$, that is, a copy of a 3-elbow. Note that the insertion in each 3-cube is uniquely determined once a 3-elbow has been placed in one 3-cube. Thus we arrive at a polyhedral 2-manifold with square faces in $\mathbb{E}^6$ which is isomorphic to $\mathcal{Q}_t$ and lies in the support of the 3-skeleton of K'_t. The symmetry group of this realization is $C_t \wr S_3$, of order $6t^3$, which is a subgroup of index 16 in the full automorphism group of $\mathcal{Q}_t$. Note that the special position of the 3-elbow at a corner inside the unit cube prevents additional symmetries from occuring.

The second construction of a polyhedral realization of $\mathcal{Q}_t$ must proceed in a different way, since adjacent 6-elbows in the 6-elbow decomposition attach along the boundary of half-tunnels, not along 4-cycles of edges as in the previous case. In this case we will construct a realization that lies in the support of the 4-skeleton of the polytope K'_t but not in the support of the 3-skeleton. However, the 3-dimensional cubical complex $\mathcal{C}'_t$ still provides the blueprint for the construction.

The basic idea is simple but the details are tedious. Let K'_t and $\mathcal{C}'_t$ be as above. We shrink all 3-cubes of $\mathcal{C}'_t$ concentrically (with respect to their centers, by a fixed scaling factor $\lambda < 1$), and then connect the shrunk copies of any two adjacent 3-cubes of $\mathcal{C}'_t$ by small cylindrical tunnels whose "bases" and "tops" are given by the shrunk copies of the square face (with sides of length λ) shared by the adjacent 3-cubes, and whose side faces are rectangles. The set of all side faces of cylindrical tunnels obtained in this way, as well as their vertices and edges, forms a polyhedral realization of the regular map $\mathcal{Q}_t$ with rectangular faces (with one side of length λ). Moreover, as we explain below, if the 3-cubes are shrunk to the right size, then the rectangular faces become squares.

The realization of $\mathcal{Q}_t$ lies inside the support of the 4-skeleton of K'_t, and we show that it is free of self-intersections. Recall that each 3-cube in $\mathcal{C}'_t$ is the cartesian product of three edges from different component polygons of K'_t. Two 3-cubes C_1 and C_2 of $\mathcal{C}'_t$ are adjacent if their sets of three edges have two edges in common, I_1 and I_2 (say), and the remaining two edges, I_3 and I_4 respectively (say), are adjacent edges of a component polygon, F (say), of K'_t meeting at a vertex z (say) of F. In the shrinking process, the common square face $I_1 \times I_2 \times \{z\}$ of C_1 and C_2 yields a pair of parallel squares of smaller size contained in C_1 and C_2 (the base and top of the cylindrical tunnel). The cylindrical tunnel connecting these smaller squares lies entirely in the cartesian product $F' := I_1 \times I_2 \times F$, which is a 4-dimensional face of K'_t that has C_1 and C_2 as a pair of adjacent facets. Note that F' is a double prism over the regular convex t-gon F. More explicitly, since $C_1 = I_1 \times I_2 \times I_3$ and $C_2 = I_1 \times I_2 \times I_4$, their shrunk copies are given by $C'_1 = I'_1 \times I'_2 \times I'_3$ and $C'_2 = I'_1 \times I'_2 \times I'_4$, respectively, where $I'_j =: [a_j, b_j] \subset I_j$ for $j = 1, 2, 3, 4$ and the labeling is such that the points a_3 and a_4 are closer to z than b_3 and b_4; here $[a_j, b_j]$ denotes the line segment with endpoints a_j and b_j. It now follows that the four rectangular side faces of the cylindrical tunnel connecting C'_1 and C'_2 are given by

$$\{a_1\} \times I'_2 \times [a_3, a_4], \quad I'_1 \times \{a_2\} \times [a_3, a_4],$$
$$\{b_1\} \times I'_2 \times [a_3, a_4], \quad I'_1 \times \{b_2\} \times [a_3, a_4].$$

Then it is clear that two cylindrical tunnels inserted during the process could only intersect nontrivially (that is, other than in vertices, or in edges lying in their bases or tops) if they lied in the same 4-face F' of K'_t. Thus the question whether or not there are unwanted intersections reduces to a 4-dimensional problem.

Each 4-face $F' = I_1 \times I_2 \times F$ of K'_t contains exactly t cyclindrical tunnels, one for each pair of adjacent edges of F. As the shrinking process is uniform, the subgroup $D_t \wr S_3$ of the symmetry group of K'_t preserves the realization of $\mathcal{Q}_t$. In particular, the symmetry group of F, which is isomorphic to D_t, appears as a subgroup of $D_t \wr S_3$ and permutes the t cylindrical tunnels in F' according to a standard dihedral action. Thus the collection of t cylindrical tunnels inside F' is invariant under the dihedral group D_t determined by F.

To see that nontrivial self-intersections cannot occur, choose vertices u and v of I_1 and I_2, respectively, and project F' orthogonally along the direction of $I_1 \times I_2$ onto its 2-face $\{u\} \times \{v\} \times F$. A square face shared by a pair of adjacent 3-cubes from $\mathcal{C}'_t$ that lie in F' is necessarily of the form $I_1 \times I_2 \times \{w\}$, where w is a vertex of F, and is parallel to its two shrunk copies (the base and top of the cylindrical tunnel) in the 3-cubes. Hence each cylindrical tunnel in F' projects orthogonally to a line segment strictly inside $\{u\}\times\{v\}\times F$. For the cylindrical tunnel associated with C_1 and C_2 as described above, the line segment is $[a_3, a_4]$. There are no nontrivial intersections among the t resulting line segments. Since these line segments were obtained by orthogonal projections of cylindrical tunnels, there also cannot be any nontrivial intersections among the cylindrical tunnels. Hence there are no intersections of cylindrical tunnels inside a 4-face F'. Thus the realizations of $\mathcal{Q}_t$ are free of self-intersections, for all scaling factors λ.

As will become clearer in a moment, the above analysis also shows that each realizations for $\mathcal{Q}_t$ is a subcomplex of the 3-skeleton of a convex 6-polytope (depending on λ) defined as the cartesian product of three semi-regular convex $2t$-gon (with two kinds of edges); moreover, the realization of $\mathcal{Q}_t$ has the same vertex-set as the 6-polytope. The tunnels then are formed by four rectangular faces of 3-dimensional faces (rectangular boxes) given by the cartesian product of three edges of these $2t$-gons, with one edge from each $2t$-gon and exactly two edges of the same length. This observation provides an alternative proof of the fact that the realization for $\mathcal{Q}_t$ is free of self-intersections.

As in the previous section, in all but one case only one half of the combinatorial symmetries appear as symmetries of the realization. The symmetry group of the realization for $\mathcal{Q}_t$ is $D_t \wr S_3$ (even if $t = 4$), except when the rectangular faces become squares (in which case it is $D_t^3 \rtimes S_3$), as can be seen as follows.

If the unit 3-cubes are shrunk to smaller 3-cubes of the right size, then all rectangular faces of the realization become squares. In fact, the correct scaling factor λ is the inverse ratio between the edge lengths of a convex regular t-gon and of a convex regular $2t$-gon obtained from the t-gon by vertex-truncation. It is straightforward to check that

$$\lambda = \frac{\cos\frac{\pi}{t}}{1 + \cos\frac{\pi}{t}}.$$

Thus $\mathcal{Q}_t$ admits a polyhedral embedding with square faces in $\mathbb{E}^6$ and then, via a Schlegel diagram, a polyhedral embedding with quadrangular faces in $\mathbb{E}^5$. The vertex-set of the 6-dimensional square-faced realization coincides with the vertex-set of the cartesian product of three regular $2t$-gons, namely the regular $2t$-gons obtained by vertex-truncation from the three regular t-gons appearing as factors in the cartesian product decomposition of K_t'. This also shows that this square-faced realization is identical with the realization of $\mathcal{Q}_t$ outlined in [7, p. 57] and [23, p. 259] and further described below. The square-faced realization of $\mathcal{Q}_t$ has the desirable property that all combinatorial symmetries are realized by geometric symmetries, or more exactly, that the geometric symmetry group is isomorphic to $\Gamma(\mathcal{Q}_t) = D_t^3 \rtimes D_6$.

The regular maps $\mathcal{Q}_t = \{4, 6 \,|\, 4, 2t\}$ are isomorphic to certain abstract polyhedra $2^{\{6\},\mathcal{G}(s)}$ (see [23, p. 259]). These polyhedra admit 6-dimensional realizations (in the sense of [23, Ch. 5]) as subcomplexes of the 2-skeleton of the cartesian product of three regular $2t$-gons (see [23, p. 264] and [7, p. 57]), and thus are free of self-intersections and form a polyhedral 2-manifold. Our approach to realizations of the maps $\mathcal{Q}_t$ as polyhedral 2-manifolds builds on a 6-dimensional realization of the cubic 4-toroid $\{4, 3, 4\}_{(t,0,0)}$ and exploits the quotient relationships of $\mathcal{Q}_t$ and $\mathcal{P}_t$ with the Coxeter-Petrie polyhedra $\{4, 6 \,|\, 4\}$ and $\{6, 4 \,|\, 4\}$ in a very explicit manner. In particular, we based our construction of rectangle-faced polyhedral embeddings for $\mathcal{Q}_t$ on the cartesian product of three regular t-gons, rather than of three regular $2t$-gons as in [23]. However, as hinted at before, the rectangular-faced embeddings can also be viewed as subcomplexes of the 3-skeleton of a cartesian product of three semi-regular convex $2t$-gons. On the other hand, the square-faced realization of [7, p. 57] and [23, p. 264] has the advantage that all combinatorial symmetries are realized by geometric symmetries.

Note that the 6-dimensional polyhedral embeddings for $\mathcal{Q}_t$ could also be obtained directly from those of the previous section for their combinatorial duals $\mathcal{P}_t$, by choosing as vertices of an embedding for $\mathcal{Q}_t$ the centers of the hexagonal faces of the embeddings for $\mathcal{P}_t$ and then proceeding according to duality. (This process results in rectangular faces for the realization of $\mathcal{Q}_t$. In fact, the four vertices of every face of the realization of $\mathcal{Q}_t$ must lie on four of the edges of a tetrahedron which has two of its vertices located at the centers of a pair of adjacent 3-cubes, and the other two vertices located at adjacent vertices of the common square face of these 3-cubes. These four edges of the tetrahedron form a skew equilateral 4-gon, and the vertices of the realization for $\mathcal{Q}_t$ all have the same distance from the center of the 3-cube in which they lie. Hence the vertices are those of a rectangle.) Under this dual correspondence, the polyhedral embedding with regular hexagons as faces for $\mathcal{P}_t$ corresponds to the square-faced polyhedral embedding for $\mathcal{Q}_t$, and in particular, both polyhedral embeddings are geometrically regular polyhedra in $\mathbb{E}^6$ with symmetry group $D_t^3 \rtimes D_6$. In all other cases, polyhedral embeddings with semi-regular hexagon faces (with two kinds of edges) for $\mathcal{P}_t$ correspond to rectangle-faced polyhedral embeddings for $\mathcal{Q}_t$, and both kinds give combinatorially regular polyhedra of index 2, with symmetry group $D_t \wr S_3$.

ORCID iDs

Gábor Gévay https://orcid.org/0000-0002-5469-5165
Egon Schulte https://orcid.org/0000-0001-9725-3589

References

[1] J. Bokowski, A geometric realization without self-intersections does exist for Dyck's regular map, *Discrete Comput. Geom.* **4** (1989), 583–589, doi:10.1007/BF02187748.

[2] J. Bokowski and M. Cuntz, Hurwitz's regular map $(3, 7)$ of genus 7: a polyhedral realization, *Art Discrete Appl. Math.* **1** (2018), #P1.02, 17, doi:10.26493/2590-9770.1186.258.

[3] J. Bokowski and J. M. Wills, Regular polyhedra with hidden symmetries, *Math. Intelligencer* **10** (1988), 27–32, doi:10.1007/BF03023848.

[4] U. Brehm, Maximally symmetric polyhedral realizations of Dyck's regular map, *Mathematika* **34** (1987), 229–236, doi:10.1112/S0025579300013474.

[5] U. Brehm and J. M. Wills, Polyhedral manifolds, in: *Handbook of convex geometry, Vol. A, B*, North-Holland, Amsterdam, pp. 535–554, 1993.

[6] M. D. E. Conder, Regular maps and hypermaps of Euler characteristic -1 to -200, *J. Combin. Theory Ser. B* **99** (2009), 455–459, doi:10.1016/j.jctb.2008.09.003.

[7] H. S. M. Coxeter, Regular Skew Polyhedra in Three and Four Dimension, and their Topological Analogues, *Proc. London Math. Soc. (2)* **43** (1937), 33–62, doi:10.1112/plms/s2-43.1.33.

[8] H. S. M. Coxeter and W. O. J. Moser, *Generators and relations for discrete groups*, volume 14 of *Ergebnisse der Mathematik und ihrer Grenzgebiete [Results in Mathematics and Related Areas]*, Springer-Verlag, Berlin-New York, 4th edition, 1980.

[9] A. M. Cutler and E. Schulte, Regular polyhedra of index two, I, *Beitr. Algebra Geom.* **52** (2011), 133–161, doi:10.1007/s13366-011-0015-0.

[10] A. W. M. Dress, A combinatorial theory of Grünbaum's new regular polyhedra. I. Grünbaum's new regular polyhedra and their automorphism group, *Aequationes Math.* **23** (1981), 252–265, doi:10.1007/BF02188039.

[11] A. W. M. Dress, A combinatorial theory of Grünbaum's new regular polyhedra. II. Complete enumeration, *Aequationes Math.* **29** (1985), 222–243, doi:10.1007/BF02189831.

[12] L. Fejes Tóth, *Regular figures*, A Pergamon Press Book, The Macmillan Co., New York, 1964.

[13] J. Garza-Vargas and I. Hubard, Polytopality of maniplexes, *Discrete Math.* **341** (2018), 2068–2079, doi:10.1016/j.disc.2018.02.017.

[14] G. Gévay, E. Schulte and J. M. Wills, The regular Grünbaum polyhedron of genus 5, *Adv. Geom.* **14** (2014), 465–482, doi:10.1515/advgeom-2013-0033.

[15] J. E. Goodman and J. O'Rourke (eds.), *Handbook of discrete and computational geometry*, CRC Press Series on Discrete Mathematics and its Applications, CRC Press, Boca Raton, FL, 1997.

[16] B. Grünbaum, Regular polyhedra—old and new, *Aequationes Math.* **16** (1977), 1–20, doi:10.1007/BF01836414.

[17] B. Grünbaum, Uniform tilings of 3-space, *Geombinatorics* **4** (1994), 49–56.

[18] B. Grünbaum, *Convex polytopes*, volume 221 of *Graduate Texts in Mathematics*, Springer-Verlag, New York, 2nd edition, 2003, doi:10.1007/978-1-4613-0019-9, prepared and with a preface by Volker Kaibel, Victor Klee and Günter M. Ziegler.

[19] I. Hubard, Two-orbit polyhedra from groups, *European J. Combin.* **31** (2010), 943–960, doi:10.1016/j.ejc.2009.05.007.

[20] M. Joswig and T. Rörig, Neighborly cubical polytopes and spheres, *Israel J. Math.* **159** (2007), 221–242, doi:10.1007/s11856-007-0044-4.

[21] M. Joswig and G. M. Ziegler, Neighborly cubical polytopes, *Discrete Comput. Geom.* **24** (2000), 325–344, doi:10.1007/s004540010039, the Branko Grünbaum birthday issue.

[22] D. P. McCooey, A non-self-intersecting polyhedral realization of the all-heptagon Klein map, *Symmetry Cult. Sci.* **20** (2009), 247–268.

[23] P. McMullen and E. Schulte, *Abstract regular polytopes*, volume 92 of *Encyclopedia of Mathematics and its Applications*, Cambridge University Press, Cambridge, 2002, doi:10.1017/CBO9780511546686.

[24] P. McMullen, E. Schulte and J. M. Wills, Infinite series of combinatorially regular polyhedra in three-space, *Geom. Dedicata* **26** (1988), 299–307, doi:10.1007/BF00183021.

[25] P. McMullen, C. Schulz and J. M. Wills, Equivelar polyhedral manifolds in E^3, *Israel J. Math.* **41** (1982), 331–346, doi:10.1007/BF02760539.

[26] P. McMullen, C. Schulz and J. M. Wills, Polyhedral 2-manifolds in E^3 with unusually large genus, *Israel J. Math.* **46** (1983), 127–144, doi:10.1007/BF02760627.

[27] B. Monson, D. Pellicer and G. Williams, Mixing and monodromy of abstract polytopes, *Trans. Amer. Math. Soc.* **366** (2014), 2651–2681, doi:10.1090/S0002-9947-2013-05954-5.

[28] A. Montero, Regular polyhedra in the 3-torus, *Adv. Geom.* **18** (2018), 431–450, doi:10.1515/advgeom-2018-0017.

[29] T. Rörig and G. M. Ziegler, Polyhedral surfaces in wedge products, *Geom. Dedicata* **151** (2011), 155–173, doi:10.1007/s10711-010-9524-5.

[30] E. Schulte and J. M. Wills, A polyhedral realization of Felix Klein's map $\{3, 7\}_8$ on a Riemann surface of genus 3, *J. London Math. Soc. (2)* **32** (1985), 539–547, doi:10.1112/jlms/s2-32.3.539.

[31] E. Schulte and J. M. Wills, On Coxeter's regular skew polyhedra, *Discrete Math.* **60** (1986), 253–262, doi:10.1016/0012-365X(86)90017-8.

[32] E. Schulte and J. M. Wills, Combinatorially regular polyhedra in three-space, in: *Symmetry of discrete mathematical structures and their symmetry groups*, Heldermann, Berlin, volume 15 of *Res. Exp. Math.*, pp. 49–88, 1991.

[33] E. Schulte and J. M. Wills, Convex-faced combinatorially regular polyhedra of small genus, *Symmetry* **4** (2012), 1–14, doi:10.3390/sym4010001.

[34] C. H. Séquin, My search for symmetrical embeddings of regular maps, in: *Proceedings of Bridges 2010: Mathematics, Music, Art, Architecture, Culture (Pécs, Hungary, 2010)*, North-Holland, Amsterdam, pp. 85–94, 1993.

[35] J. J. van Wijk, Symmetric tiling of closed surfaces: visualization of regular maps, *ACM Transactions on Graphics* **28** (2009), Article 49.

[36] J. J. van Wijk, Visualization of regular maps: The chase continues, *IEEE Transactions on Visualization and Computer Graphics* **20** (2014), 2614–2623.

[37] J. M. Wills, The combinatorially regular polyhedra of index 2, *Aequationes Math.* **34** (1987), 206–220, doi:10.1007/BF01830672.

[38] J. M. Wills, Equivelar polyhedra, in: *The Coxeter legacy*, Amer. Math. Soc., Providence, RI, pp. 121–128, 2006.