


# Deterministic bootstrap percolation on trees

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## Abstract

In a graph,  $k$ -bootstrap percolation is a process by which an “infection” spreads from an initial set of infected vertices, according to the rule that on each iteration an uninfected vertex with  $k$  infected neighbors becomes infected. This process continues until either every vertex is infected or every uninfected vertex has fewer than  $k$  infected neighbors. We are particularly interested in the case where every vertex is eventually infected. The cardinality of a smallest set that results in this is the  $k$ -bootstrap percolation number of the graph. In this paper, we determine the  $k$ -bootstrap percolation number for trees of small diameter, spiders, complete  $N$ -ary trees, and caterpillars. For these graph families we also consider the smallest number of iterations needed for any smallest set to spread to the entire graph. Finally, we give an upper bound for the  $k$ -bootstrap percolation number for general trees which improves upon previous results.

*Keywords:* Bootstrap Percolation, Trees

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## 1 Introduction

Let  $G$  be a graph with vertex set  $V$  and edge set  $E$ . We define the *diameter* of a graph to be the maximum distance between any pair of vertices in  $V$  and we define the *periphery* of a graph to be the subgraph induced by all vertices in  $V$  whose distance to some other vertex in  $V$  is equal to the diameter. Since we are primarily concerned with trees, we note that any vertex on the periphery of a tree is necessarily a leaf (in other words, a vertex of degree one).

We begin with  $\mathcal{A}_k^0(G) \subseteq V(G)$ , a collection of infected vertices. On the  $t^{\text{th}}$  iteration we add newly infected vertices to  $\mathcal{A}_k^{t-1}(G)$  if they have at least  $k$  neighbors in  $\mathcal{A}_k^{t-1}(G)$ , to

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form  $\mathcal{A}_k^t(G)$ . This process is repeated until vertices not in  $\mathcal{A}_k^t(G)$  have strictly fewer than  $k$  neighbors in  $\mathcal{A}_k^t(G)$  or all vertices of  $G$  are in  $\mathcal{A}_k^t(G)$ . The above process is called *k-bootstrap percolation*. In particular, for a graph  $G$  we are interested in the size of a smallest  $\mathcal{A}_k^0(G)$  so that the entire graph is eventually infected. Throughout the paper, we call such a set a *percolating set* and we call the size of this set, denoted  $\text{bp}_k(G)$ , the *k-bootstrap percolation number* of the graph. Among all smallest percolating sets, there is one that infects all of the vertices of the graph in the minimum number of iterations. We denote this minimum value  $t_k(G)$ .

We would now like to give a brief (but far from complete) overview of the history of bootstrap percolation. In 1968, Bollobás considered an edge coloring of graphs [6] called “weak saturation,” which later came to be called “graph bootstrap percolation” [2]. Bootstrap percolation on vertices was introduced by Chalupa, Leath, and Reich [8]. Their study was motivated by a problem in magnetic systems and considered only on a special class of lattices. In the paper by Chalupa et al. and in most subsequent papers on bootstrap percolation on lattices, the initial set of infected vertices,  $\mathcal{A}_k^0(G)$ , is chosen at random. Bootstrap percolation with  $\mathcal{A}_k^0(G)$  being chosen at random has also been considered in [1, 3, 4, 5, 7, 18, 19, 20]. Alternatively, instead of choosing our initial set randomly, we choose  $\mathcal{A}_k^0(G)$  in order to insure that every vertex in a graph is eventually infected. While this deterministic approach seems to be less common historically, it has been considered in [9, 10, 12, 13, 22, 23, 24, 25] and the appendix of [4]. We should also mention that in addition to the standard bootstrap percolation considered in this paper, there are also several variants. For example, two-way bootstrap percolation, which has been considered in [21, 27, 28], and the previously mentioned bootstrap percolation on edges, which has been recently considered in [14].

In this paper, we will be primarily concerned with bootstrap percolation on trees. One primary motivation for considering trees is that they are minimally connected graphs. With this in mind, trees are natural to consider in the context of the extremal values of the *k*-bootstrap percolation number due to the fact that every graph has a spanning subtree. We should mention that our paper can most naturally be considered an extension of the work of Riedl in [25]. In [25], Riedl finds upper and lower bounds on the *k*-bootstrap percolation number for all trees<sup>1</sup>, and uses these bounds to find the precise *k*-bootstrap percolation number for certain *N*-ary trees. In this paper, we begin by obtaining exact values of the *k*-bootstrap percolation number for various commonly occurring families of trees. It should be noted that we reproduce Riedl’s formula for the *k*-bootstrap percolation number of certain *N*-ary trees, though we use a different and more concrete method. Furthermore, our main result improves upon the upper bound (See Theorem 5.1) appearing in [25].

In Section 2, we make some basic observations which will be used throughout the remainder of the paper. In Section 3.1, we consider families of trees of small diameter. In Section 3.2, we consider families of spiders. In Section 3.3, we consider complete *N*-ary trees. This result is also given in [25], since in this case the upper and lower bounds are tight, so that the ceiling of the lower bound is equal to the floor of the upper bound. In Section 4, we consider caterpillars. In Section 5, we present a sharp upper bound for  $\text{bp}_k(T)$  and compare this bound to the bounds given in [25]. In Section 6, we use this bound to give  $\text{bp}_k(T)$  for the trees on eleven vertices or less that do not fall into one of our families. Finally, we give several open problems for future avenues of research in Section 7.

<sup>1</sup>We note that the main theorem in [25] is stated incorrectly on page 3. However, it is correctly stated in their abstract.

## 2 Basic observations

In this section we state several fundamental results which will be useful for the remainder of the paper.

We begin with four observations about percolating sets that hold for all graphs.

**Observation 2.1.** Let  $G$  be a graph. (i) All vertices of degree less than  $k$  must belong to any percolating set for  $G$ . (ii) In  $k$ -bootstrap percolation, if  $u$  and  $v$  are adjacent vertices such that  $\deg(u) = \deg(v) = k$ , then at least one of  $u$  and  $v$  must be in every minimum  $k$ -bootstrap set.

*Proof.* Let  $v$  be a vertex of  $G$  with degree less than  $k$ . If  $v$  is not in a percolating set, then  $v$  can never have at least  $k$  infected neighbors. So,  $v$  will never be infected.

Likewise, suppose that  $u$  and  $v$  are adjacent vertices of degree  $k$ . If  $u$  is not in a percolating set, then  $v$  can never have at least  $k$  infected neighbors. Reversing the roles of  $u$  and  $v$  yields the result.  $\square$

Clearly, any percolating set for  $k + 1$  is also a percolating set for  $k$ . Ergo, the next proposition follows immediately from Observation 2.1.

**Observation 2.2.** If  $G$  is a connected graph with maximum degree  $\Delta$ , then

$$1 = \text{bp}_1(G) \leq \text{bp}_2(G) \leq \dots \leq \text{bp}_{\Delta+1}(G) = |V(G)|.$$

**Observation 2.3.** We have  $\text{bp}_1(G)$  is the number of connected components in  $G$ .

Based on Observation 2.3, we will only consider the case where  $k \geq 2$  for the remainder of this paper.

To obtain an upper bound for the  $k$ -bootstrap percolation number we consider the  $k$ -domination number of  $G$ . The *neighborhood* of a vertex  $x$ , denoted  $N(x)$ , is the set of all vertices adjacent to  $x$ . If  $|N(x)| = 1$ , then  $x$  is a *leaf*. A  $k$ -*domination set* is a set  $S \subseteq V(G)$  such that for all  $x \in V(G)$ , either  $x \in S$  or  $|N(x) \cap S| \geq k$ . If among all  $k$ -domination sets,  $S$  has the least number of vertices, then  $S$  is a *minimum  $k$ -domination set*. The cardinality of such a set is the  $k$ -*domination number* of  $G$ . This number is denoted  $\gamma_k(G)$ . The  $k$ -domination number was introduced by Fink and Jacobson in 1985 [11]. For more information on domination and its variations, please refer to [16, 17]. If  $\mathcal{A}_k^0(G)$  is equal to a  $k$ -domination set for  $G$ , then after a single iteration the entire graph will be infected. From this, the following bound is immediate.

**Observation 2.4.** For any graph  $G$ ,  $\text{bp}_k(G) \leq \gamma_k(G)$ .

## 3 Graph families

In this section, we restrict our attention to certain families of trees for which we can derive the exact  $k$ -bootstrap percolation number.

### 3.1 Trees of small diameter

We give the  $k$ -bootstrap percolation number for all trees with diameter less than or equal to five.

We begin with diameter two. A tree of diameter two is a *star*. This graph has a center vertex  $x$  adjacent to  $n$  leaves,  $y_1, \dots, y_n$ . This graph is denoted  $K_{1,n}$ .

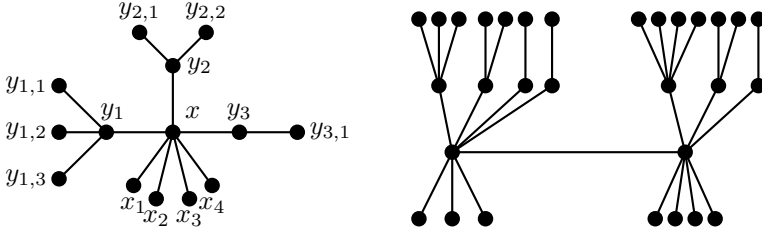


Figure 1: The graphs  $K_{1,3}(4; 3, 2, 1)$  and  $S_{4,3}(3; 3, 2, 1, 1; 4; 4, 2, 1)$

**Theorem 3.1.** Let  $k \geq 2$ . For the star  $K_{1,n}$ , we have the following:

1. If  $n \leq k - 1$ , then  $\text{bp}_k(K_{1,n}) = n + 1$  and  $t_k(K_{1,n}) = 0$ .
2. If  $n \geq k$ , then  $\text{bp}_k(K_{1,n}) = n$  and  $t_k(K_{1,n}) = 1$ .

*Proof.* By Observation 2.1, all vertices in  $\{y_1, \dots, y_n\}$  must be in every percolating set. If  $\deg(x) = n \leq k - 1$ , then  $x$  must also be in every percolating set and part 1) follows. If  $n \geq k$ , then  $x$  will get infected after one iteration and part 2) follows.  $\square$

It follows from the previous theorem that the bound in Observation 2.4 is sharp. In particular, for the star  $K_{1,k}$  we have that  $\text{bp}_k(K_{1,k}) = k = \gamma_k(K_{1,k})$ .

A tree of diameter three is a *double star*. This graph has two adjacent central vertices  $x$  and  $y$ . The vertex  $x$  is adjacent to  $r$  leaves,  $x_1, \dots, x_r$ . The vertex  $y$  is adjacent to  $s$  leaves,  $y_1, \dots, y_s$ . This graph is denoted  $S_{r,s}$ .

**Theorem 3.2.** Let  $k \geq 2$  and  $r \geq s \geq 1$ . For the double star  $T = S_{r,s}$ , we have the following:

1. If  $r \leq k - 2$ , then  $\text{bp}_k(T) = r + s + 2$  and  $t_k(T) = 0$ .
2. If  $r \geq k - 1$  and  $s \leq k - 2$ , then  $\text{bp}_k(T) = r + s + 1$  and  $t_k(T) = 1$ .
3. If  $r = s = k - 1$ , then  $\text{bp}_k(T) = r + s + 1$  and  $t_k(T) = 1$ .
4. If  $r \geq k$  and  $s = k - 1$ , then  $\text{bp}_k(T) = r + s$  and  $t_k(T) = 2$ .
5. If  $s \geq k$ , then  $\text{bp}_k(T) = r + s$  and  $t_k(T) = 1$ .

*Proof.* By Observation 1, all leaves must be initially infected, so  $\text{bp}_k(T) \geq r + s$ . If  $r \leq k - 2$ , then  $x$  and  $y$  must both be initially infected, i.e.,  $\text{bp}_k(T) = r + s + 2$  and  $t_k(T) = 0$ . If  $r \geq k - 1$  and  $s \leq k - 2$ , then  $y$  must be initially infected and  $x$  is infected after one iteration, so  $\text{bp}_k(T) = r + s + 1$  and  $t_k(T) = 1$ . If  $r = s = k - 1$  then either  $x$  or  $y$  must be initially infected, and the other is infected after one iteration, so  $\text{bp}_k(T) = r + s + 1$  and  $t_k(T) = 1$ . If  $r \geq k$  and  $s = k - 1$ , then  $x$  is infected after one iteration and  $y$  is infected after two iterations, so  $\text{bp}_k(T) = r + s$  and  $t_k(T) = 2$ . If  $s \geq k$ , then both  $x$  and  $y$  are infected after one iteration, so  $\text{bp}_k(T) = r + s$  and  $t_k(T) = 1$ .  $\square$

Any tree of diameter four can be obtained by appending leaves to the existing vertices of  $K_{1,n}$ , where  $n \geq 2$ . Suppose that we append  $c$  leaves to  $x$ , namely  $x_1, \dots, x_c$  and  $a_i \geq 1$  leaves to  $y_i$ , namely  $y_{i,1}, \dots, y_{i,a_i}$  for  $i = 1, \dots, n$ . The resulting graph will be denoted  $K_{1,n}(c; a_1, \dots, a_n)$ . Without loss of generality, assume that  $a_1 \geq \dots \geq a_n \geq 1$ . An example is shown in Figure 1.

There exist non-negative integers  $p$  and  $q$  such that the following holds:

- $a_i \geq k \geq 2$  if and only if  $i \leq p$ .
- $a_i = k - 1$  if and only if  $p + 1 \leq i \leq n - q$ .
- $a_i \leq k - 2$  if and only if  $i \geq n - q + 1$ .

Thus  $p$  is the number of  $y_i$  with at least  $k$  leaves and  $q$  is the number of  $y_i$  with at most  $k - 2$  leaves.

**Theorem 3.3.** *Let  $k \geq 2$ . For  $T = K_{1,n}(c; a_1, \dots, a_n)$ , we have the following:*

1. *If  $p + q + c \geq k$ , then  $\text{bp}_k(T) = c + a_1 + \dots + a_n + q$  and*

$$t_k(T) = \begin{cases} 1 & \text{if } q + c \geq k \text{ and } n = p + q \\ 2 & \text{if } q + c \geq k \text{ and } n \geq p + q + 1 \\ 2 & \text{if } q + c \leq k - 1 \text{ and } n = p + q \\ 3 & \text{if } q + c \leq k - 1 \text{ and } n \geq p + q + 1. \end{cases}$$

2. *If  $p + q + c \leq k - 1$ , then  $\text{bp}_k(T) = c + a_1 + \dots + a_n + q + 1$  and*

$$t_k(T) = \begin{cases} 0 & \text{if } n = q \\ 1 & \text{if } p \geq 1 \text{ or } n \geq p + q + 1. \end{cases}$$

*Proof.* By Observation 2.1, for all  $i, j$ , and  $\ell$ ,  $y_{i,j}$  and  $x_\ell$  must be in every percolating set. Further note that  $\deg(y_i) = a_i + 1$ . Thus,  $y_{n-q+1}, \dots, y_n$  must also be in the initial set. It follows that  $\text{bp}_k(T) \geq c + a_1 + \dots + a_n + q$ . Likewise,  $y_1, \dots, y_p$  will be infected after one iteration since they have at least  $k$  neighbors in the initial set. Similarly,  $y_{p+1}, \dots, y_{n-q}$  will be infected in the iteration after  $x$  is infected.

Suppose that  $p + q + c \geq k$ . If  $q + c \geq k$  and  $n = p + q$ , then  $x$  will be infected in one iteration. Since,  $n = p + q$ , then  $\{y_{p+1}, \dots, y_{n-q}\} = \emptyset$ . Thus, the entire graph is infected and  $t_k(T) = 1$ .

If  $q + c \geq k$  and  $n \geq p + q + 1$ , then  $x$  is infected in one iteration and  $y_{p+1}, \dots, y_{n-q}$  are infected in two iterations. Hence,  $t_k(T) = 2$ .

If  $q + c \leq k - 1$  and  $n = p + q$ , then  $\{y_{p+1}, \dots, y_{n-q}\} = \emptyset$ . Since  $p + q + c \geq k$  but  $q + c \leq k - 1$ ,  $x$  gets infected one iteration after  $y_1, \dots, y_p$  are infected. Thus every vertex is infected after two steps and  $t_k(T) = 2$ .

Similarly, if  $q + c \leq k - 1$  and  $n \geq p + q + 1$ , then  $\{y_{p+1}, \dots, y_{n-q}\} \neq \emptyset$ . Since  $p + q + c \geq k$  but  $q + c \leq k - 1$ ,  $x$  is infected one iteration after  $y_1, \dots, y_p$  are infected. The vertices  $y_{p+1}, \dots, y_{n-q}$  are infected one iteration later. Thus every vertex is infected after three steps and  $t_k(T) = 3$ . This proves part 1).

Now, suppose that  $p + q + c \leq k - 1$ . As before,  $y_1, \dots, y_p, y_{n-q+1}, \dots, y_n$ , and  $x_1, \dots, x_c$  are either in the initial set, or (in the case of  $y_1, \dots, y_p$ ) infected after one step. Hence,  $x$  has at most  $k - 1$  neighbors that will eventually be infected. Thus  $x$  must be in the initial set.

It follows  $\text{bp}_k(T) \geq c + a_1 + \dots + a_n + q + 1$ . Thus if  $n = q$ , then  $p = 0$  and every vertex must be in the initial set. Hence  $t_k(T) = 0$ .

If  $p \geq 1$  or  $n \geq p + q + 1$ , then  $y_1, \dots, y_{n-q}$  are infected after one iteration. Thus  $t_k(T) = 1$ . This proves part 2).  $\square$

Any tree of diameter five can be obtained by appending leaves to the existing vertices of the double star. We append  $c_1$  leaves to  $x$ , namely  $w_1, \dots, w_{c_1}$ . We append  $c_2$  leaves to  $y$ , namely  $z_1, \dots, z_{c_2}$ . Similarly, we append  $a_i$  leaves to  $x_i$ , namely  $x_{i,1}, \dots, x_{i,a_i}$ , and  $b_j$  leaves to  $y_j$ , namely  $y_{j,1}, \dots, y_{j,b_j}$ . A diameter five tree with these parameters is denoted  $S_{r,s}(c_1; a_1, \dots, a_r; c_2; b_1, \dots, b_s)$  (see Figure 1). Without loss of generality, assume that  $a_1 \geq \dots \geq a_r \geq 1$  and  $b_1 \geq \dots \geq b_s \geq 1$ .

For convenience of notation, define  $X_i = \{x_{i,1}, \dots, x_{i,a_i}\}$  and  $Y_j = \{y_{j,1}, \dots, y_{j,b_j}\}$  for  $i = 1, \dots, r$  and  $j = 1, \dots, s$ . Note that the set of leaves is

$$L = \{w_1, \dots, w_{c_1}, z_1, \dots, z_{c_2}\} \cup X_1 \cup \dots \cup X_r \cup Y_1 \cup \dots \cup Y_s.$$

and that

$$|L| = c_1 + c_2 + \sum_{i=1}^r a_i + \sum_{j=1}^s b_j.$$

Given  $k \geq 2$ , there exist non-negative integers  $p_1, p_2, q_1, q_2$  such that the following holds:

- $a_i \geq k$  if and only if  $i \leq p_1$ .
- $b_j \geq k$  if and only if  $j \leq p_2$ .
- $a_i \leq k - 2$  if and only if  $i \geq r - q_1 + 1$ .
- $b_j \leq k - 2$  if and only if  $j \geq s - q_2 + 1$ .

Because our result follows in a very similar manner to the proof of Theorem 3.3, we omit the details of the proof and only provide the initial sets. In each case, it is straightforward to verify that the set in question is a minimum percolating set. While we have omitted the time parameter, this can easily be calculated from these sets.

**Theorem 3.4.** For a given  $k$ , the  $k$ -bootstrap percolation number of

$T = S_{r,s}(c_1; a_1, \dots, a_r; c_2; b_1, \dots, b_s)$  is as follows:

- (i) If  $p_1 + q_1 + c_1 \leq k - 2$  and  $p_2 + q_2 + c_2 \leq k - 2$ , then  $\text{bp}_k(T) = |L| + q_1 + q_2 + 2$ .
- (ii) If  $p_1 + q_1 + c_1 = p_2 + q_2 + c_2 = k - 1$  or at most one of  $p_1 + q_1 + c_1$  or  $p_2 + q_2 + c_2$  is less than or equal to  $k - 2$ , then  $\text{bp}_k(T) = |L| + q_1 + q_2 + 1$ .
- (iii) If  $p_1 + q_1 + c_1 \geq k - 1$  and  $p_2 + q_2 + c_2 \geq k - 1$ , with at most one of  $p_1 + q_1 + c_1$  or  $p_2 + q_2 + c_2$  equaling  $k - 1$ , then  $\text{bp}_k(T) = |L| + q_1 + q_2$ .

*Proof.* (i) Take the set  $L \cup \{x_{r-q_1+1}, \dots, x_r\} \cup \{y_{s-q_2+1}, \dots, y_s\} \cup \{x, y\}$ .

(ii) If  $p_1 + q_1 + c_1 \leq k - 2$ , then take the set  $L \cup \{x_{r-q_1+1}, \dots, x_r\} \cup \{y_{s-q_2+1}, \dots, y_s\} \cup \{x\}$ . If  $p_2 + q_2 + c_2 \leq k - 2$ , then take the set

$$L \cup \{x_{r-q_1+1}, \dots, x_r\} \cup \{y_{s-q_2+1}, \dots, y_s\} \cup \{y\}.$$

If  $p_1 + q_1 + c_1 = p_2 + q_2 + c_2 = k - 1$ , then we can take  $L \cup \{x_{r-q_1+1}, \dots, x_r\} \cup \{y_{s-q_2+1}, \dots, y_s\}$  along with either  $x$  or  $y$ . While either  $x$  or  $y$  may be chosen, we give the following procedure for choosing  $x$  and  $y$  which minimizes the number of iterations needed to completely infect the graph:

Suppose that  $r - p_1 - q_1 = 0$ . Except for  $y$ , every neighbor of  $x$  is either in the initial set or (in the case of  $x_1, \dots, x_{p_1}$ ) infected after one step. Thus, by including  $y$  in the initial set, we guarantee that every vertex is infected after one step (if  $p_1 = 0$ ) or two steps (if  $p_1 \geq 1$ ). As either  $x$  or  $y$  will not be in the initial set, this gives us the minimum number of iterations. Using an analogous argument, if  $s - p_2 - q_2 = 0$ , then we include  $x$  in the initial set.

Suppose that  $r - p_1 - q_1 \geq 1$  and  $s - p_2 - q_2 \geq 1$ . Note that this means that  $\{x_{p_1+1}, \dots, x_{r-q_1}\}$  and  $\{y_{p_2+1}, \dots, y_{s-q_2}\}$  are non-empty sets. These sets are infected one step after their corresponding center vertex. If  $p_1 = 0$ , then choosing  $y$  guarantees that  $x$  is infected on the first step and every vertex is infected in two. Note that choosing  $x$  in the case where  $r - p_1 - q_1 \geq 1$ ,  $s - p_2 - q_2 \geq 1$ ,  $p_1 = 0$ , and  $p_2 \geq 1$  will result in  $\{y_{p_2+1}, \dots, y_{s-q_2}\}$  becoming infected after three iterations. Using an analogous argument, if  $r - p_1 - q_1 \geq 1$ ,  $s - p_2 - q_2 \geq 1$ , and  $p_2 = 0$ , then we choose  $x$  for our initial set.

Suppose that  $r - p_1 - q_1 \geq 1$ ,  $s - p_2 - q_2 \geq 1$ ,  $p_1 \geq 1$ , and  $p_2 \geq 1$ . By choosing  $x$  to be in our initial set,  $x_1, \dots, x_{r-q_1}$ ,  $y_1, \dots, y_{p_2}$  are infected after one step,  $y$  is infected on the second iteration, and  $y_{p_2+1}, \dots, y_{s-q_2}$  are infected in three steps. By reversing the roles of  $x$  and  $y$ , we see that we do no better by choosing  $y$  to be in the initial set.

(iii) Choose  $L \cup \{x_{r-q_1+1}, \dots, x_r\} \cup \{y_{s-q_2+1}, \dots, y_s\}$  as our initial set. □

### 3.2 Spiders

In this section, we consider bootstrap percolation on spiders, which are also commonly referred to as asters or starlike trees. Let  $x_1, \dots, x_e, y_1, \dots, y_o$  be positive integers with  $x_i$  even for  $1 \leq i \leq e$  and  $y_j$  odd for  $1 \leq j \leq o$ . We construct a *spider*, denoted by  $S = S(x_1, \dots, x_e, y_1, \dots, y_o)$ , as follows. First,  $S$  has a single vertex of degree larger than 2, which we denote by  $c$ . We then add an edge from  $c$  to a single leaf from each of the paths  $P_{x_i}, P_{y_j}$  for  $1 \leq i \leq e, 1 \leq j \leq o$ . Note that for  $k \geq 3$ , the  $k$ -bootstrap percolation number is straightforward to determine using Observation 2.1. However, we include it for completeness.

**Proposition 3.5.** *Suppose  $k \geq 3$  and let  $S$  be as above. Then,*

$$\text{bp}_k(S) = \begin{cases} \sum_{i=1}^e x_i + \sum_{j=1}^o y_j + 1 & \text{if } e + o \leq k - 1 \\ \sum_{i=1}^e x_i + \sum_{j=1}^o y_j & \text{if } e + o \geq k, \end{cases}$$

and

$$t_k(S) = \begin{cases} 0 & \text{if } e + o \leq k - 1 \\ 1 & \text{if } e + o \geq k. \end{cases}$$

We now proceed to determine the 2-bootstrap percolation number for  $S$ , which is somewhat more involved than the previous result.

**Theorem 3.6.** *Let  $S = S(x_1, \dots, x_e, y_1, \dots, y_o)$ . Then,*

$$\text{bp}_2(S) = \begin{cases} \sum_{i=1}^e \frac{x_i}{2} + 1 & o = 0 \\ \sum_{i=1}^e \frac{x_i}{2} + \frac{y_1+1}{2} + 1 & o = 1 \\ \sum_{i=1}^e \frac{x_i}{2} + \sum_{j=1}^o \frac{y_j+1}{2} & o \geq 2, \end{cases}$$



and

$$t_2(S) = \begin{cases} 2 & o \geq 2, e \geq 1 \\ 1 & \text{otherwise.} \end{cases}$$

*Proof.* First, we must initially infect all leaves of  $S$ . Second, for each path attached to the center vertex, we initially infect every other vertex starting from the leaf. Note that this will completely infect each odd length path after one step.

**Case 1:**  $o \geq 2$ .

In this case, the initially infected vertices given above also infect the center vertex  $c$  after one step. Note, if  $e = 0$ , then we are finished.

Now, suppose that  $e \geq 1$ . Then, after one step each even length path will have one infected end point and the other end point will be attached to the infected center vertex. Thus, the entire graph is infected after two steps. Therefore,  $\sum_{i=1}^e \frac{x_i}{2} + \sum_{j=1}^o \frac{y_j+1}{2}$  vertices are initially infected. If we initially infect fewer than this number vertices, then there will be either a leaf or a vertex of degree two which is never infected. This yields the result in this case.

**Case 2:**  $o \leq 1$ .

In this case, the initially infected vertices will not infect the center vertex  $c$ . Thus, we must initially infect one additional vertex, and we see that initially infecting  $c$  will ensure that all vertices of the even length paths are infected after one step. Then  $\sum_{i=1}^e \frac{x_i}{2} + \sum_{j=1}^o \frac{y_j+1}{2} + 1$  are initially infected and this is the smallest possible percolating set. This yields the result in this case.  $\square$

### 3.3 $N$ -ary trees

In this section we consider  $N$ -ary trees and give a formula for the  $k$ -bootstrap percolation number of a complete  $N$ -ary tree.

For  $N \geq 1$ , we say that a tree  $T$  is an  $N$ -ary tree of height  $h$  if  $T$  is a rooted tree in which each vertex has no more than  $N$  children and no child can be further than distance  $h$  from the root. Note that when  $N = 1$ , a  $N$ -ary tree of height  $h$  is simply a path on  $h + 1$  vertices. For example, the path  $P_2$  is a 1-ary tree of height  $h = 1$ . With this in mind we begin with the following result.

**Theorem 3.7.** *For a path on  $n$  vertices,  $P_n$ , we have*

$$\text{bp}_k(P_n) = \begin{cases} \lceil \frac{n+1}{2} \rceil & \text{if } k = 2 \\ n & \text{if } k \geq 3, \end{cases}$$

and

$$t_k(P_n) = \begin{cases} 1 & \text{if } k = 2 \text{ and } n \geq 3 \\ 0 & \text{if } k \geq 3 \text{ or } n = 1 \text{ or } n = 2. \end{cases}$$

*Proof.* First, suppose that  $k \geq 3$ . Then, the degree of each vertex of  $P_n$  is less than  $k$ . By Observation 2.1, a percolating set for  $P_n$  contains every vertex of  $P_n$ . So  $\text{bp}_k(P_n) = n$ .

Second, suppose that  $k = 2$ . Note, if  $\mathcal{A}_2^0(P_n) \subseteq V(P_n)$  is of size  $\lceil \frac{n+1}{2} \rceil - 1$ , then there must be a vertex in  $V(P_n) \setminus \mathcal{A}_2^0(P_n)$  which does not have two neighbors in  $\mathcal{A}_2^0(P_n)$ . Such a vertex would never be infected. Thus, a percolating set has cardinality at least  $\lceil \frac{n+1}{2} \rceil$ .

Now, label the vertices of  $P_n$  by  $\{v_1, \dots, v_n\}$ . Define the following subset of  $V(P_n)$ ,

$$\mathcal{A}_2^0(P_n) = \begin{cases} \{v_1, v_3, \dots, v_{n-3}, v_{n-1}, v_n\} & \text{if } n \equiv 0 \pmod{2} \\ \{v_1, v_3, \dots, v_{n-2}, v_n\} & \text{if } n \equiv 1 \pmod{2}. \end{cases}$$



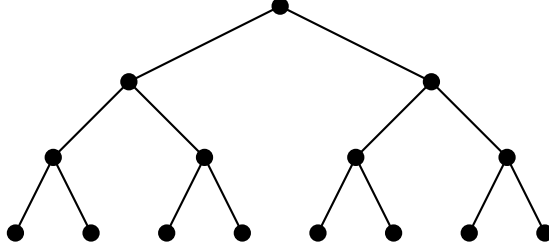


Figure 2: The complete binary tree,  $T_{2,3}$

Note,  $|\mathcal{A}_2^0(P_n)| = \lceil \frac{n+1}{2} \rceil$ . Furthermore, it is clear that after a single iteration, every vertex of  $P_n$  will be infected. Thus,  $\text{bp}_k(P_n) = \lceil \frac{n+1}{2} \rceil$  and  $t_k(P_n) = 1$  if  $n \geq 3$ .  $\square$

We now consider the case where  $N > 1$ . An  $N$ -ary tree is *complete* if every vertex has either 0 or  $N$  children and all leaves are distance  $h$  from the root. Note, for fixed  $N$  and  $h$ , there is one complete  $N$ -ary tree of height  $h$ . We denote this graph  $T_{N,h}$ . A complete 2-ary tree (also called a binary tree) of height three is given in Figure 2.

Furthermore, we have that the number of vertices of  $T_{N,h}$  is

$$|V(T_{N,h})| = \sum_{i=0}^h N^i = \frac{N^{h+1} - 1}{N - 1},$$

and the number of leaves is  $N^h$ . For convenience, we will denote the root vertex of  $T_{N,h}$  by  $v_0$ , the set of children of  $v_0$  by  $S_1$ , and so on until we have that set of all leaves of  $T_{N,h}$  is denoted  $S_h$ . We now present the main result of this section.

**Theorem 3.8.** *Let  $k, N \geq 2$ . Then,*

$$\text{bp}_k(T_{N,h}) = \begin{cases} N^h & \text{if } k \leq N \\ \frac{N^{h+2}-1}{N^2-1} & \text{if } k = N+1, h \equiv 0 \pmod{2} \\ \frac{N^{h+2}-1}{N^2-1} + \frac{N}{N+1} & \text{if } k = N+1, h \equiv 1 \pmod{2} \\ \frac{N^{h+1}-1}{N-1} & \text{if } k \geq N+2, \end{cases}$$

and

$$t_k(T_{N,h}) = \begin{cases} h & \text{if } k \leq N \\ 1 & \text{if } k = N+1 \\ 0 & \text{if } k \geq N+2. \end{cases}$$

*Proof.* First, suppose that  $k \leq N$ . By Observation 2.1, each leaf must be in  $\mathcal{A}_k^0(T_{N,h})$ , i.e.,  $S_h \subseteq \mathcal{A}_k^0(T_{N,h})$ , so  $\text{bp}_k(T_{N,h}) \geq N^h$ . Furthermore, since each non-leaf has  $N$  children, we see that after one iteration all of the vertices in  $S_{h-1}$  will be infected, after a second iteration all of the vertices in  $S_{h-2}$  will be infected, and the process repeats  $h$  times until the entire tree is infected. Hence, we have only the leaves in  $\mathcal{A}_k^0(T_{N,h})$ , so  $\text{bp}_k(T_{N,h}) = N^h$  and  $t_k(T_{N,h}) = h$ .

Second, suppose that  $k \geq N + 2$ . Then, every vertex of  $T_{N,h}$  has degree strictly less than  $k$ , hence every vertex of  $T_{N,h}$  must be in  $\mathcal{A}_k^0(T_{N,h})$ . Since  $|V(T_{N,h})| = \frac{N^{h+1}-1}{N-1}$ , the result follows.

Finally, suppose that  $k = N + 1$ . Since every vertex of degree strictly less than  $k$  must be in  $\mathcal{A}_k^0(T_{N,h})$ , we have  $S_h \cup \{v_0\} \subseteq \mathcal{A}_k^0(T_{N,h})$ . We begin by proving that

$$\text{bp}_k(T_{N,h}) \leq \begin{cases} \frac{N^{h+2}-1}{N^2-1} & \text{if } h \equiv 0 \pmod{2} \\ \frac{N^{h+2}-1}{N^2-1} + \frac{N}{N+1} & \text{if } h \equiv 1 \pmod{2}. \end{cases}$$

Suppose that  $h$  is even. Let  $\mathcal{A}_k^0(T_{N,h}) = S_h \cup S_{h-2} \cup \dots \cup S_2 \cup \{v_0\}$ . Then, after a single iteration we have that every vertex in  $T_{N,h}$  is infected. Thus,

$$\text{bp}_k(T_{N,h}) \leq |\mathcal{A}_k^0(T_{N,h})| = \sum_{i=0}^{\frac{h}{2}} |S_{2i}| = \sum_{i=0}^{\frac{h}{2}} N^{2i} = \frac{N^{h+2}-1}{N^2-1}.$$

Suppose that  $h$  is odd. Let  $\mathcal{A}_k^0(T_{N,h}) = S_h \cup S_{h-2} \cup \dots \cup S_3 \cup S_1 \cup \{v_0\}$ . Then, after a single iteration we have that every vertex in  $T_{N,h}$  is infected. Thus,

$$\text{bp}_k(T_{N,h}) \leq 1 + \sum_{i=0}^{\frac{h-1}{2}} |S_{2i+1}| = 1 + \sum_{i=0}^{\frac{h-1}{2}} N^{2i+1} = \frac{N^{h+2}-1}{N^2-1} + \frac{N}{N+1}.$$

We want to show that the set  $\mathcal{A}_k^0(T_{N,h})$  above has the smallest possible size. Every edge can be used at most once to infect a neighboring vertex, and at least  $N + 1$  edges must be used to infect one vertex. The number of edges in  $T(N, h)$  is  $\frac{N^{h+1}-1}{N-1} - 1 = \frac{N^{h+1}-N}{N-1}$ , therefore at most  $\lfloor \frac{N^{h+1}-N}{N-1} \frac{1}{N+1} \rfloor = \lfloor \frac{N^{h+1}-N}{N^2-1} \rfloor$  new vertices can be infected. Therefore the cardinality of the percolating set must be at least  $\frac{N^{h+1}-1}{N-1} - \lfloor \frac{N^{h+1}-N}{N^2-1} \rfloor$ . If  $h$  is even, then  $\frac{N^{h+1}-N}{N^2-1} = N \sum_{i=0}^{\frac{h}{2}-1} N^{2i}$  is an integer, and this lower bound on the size of the  $k$ -bootstrap set equals the upper bound above. If  $h$  is odd, then  $\frac{N^{h+1}-N}{N^2-1} = N \sum_{i=0}^{\frac{h-3}{2}} N^{2i+1} + \frac{N}{N+1}$  is not an integer. In this case, taking the floor reduces the total by  $\frac{N}{N+1}$ , again giving a lower bound which equals the upper bound above.

Therefore, the given example of a percolating set is minimum. □

Recall that in Observation 2.4, we showed that  $\text{bp}_k(G) \leq \gamma_k(G)$ . To see that  $\gamma_k(G) - \text{bp}_k(G)$  can be made arbitrarily large, consider  $T_{k,h}$ , where  $h$  is sufficiently large. As shown in Theorem 3.8,  $\text{bp}_k(T_{k,h}) = k^h$ . However,

$$\gamma_k(T_{k,h}) = \sum_{i=0}^{\lfloor h/2 \rfloor} k^{h-2i} = \begin{cases} \frac{k^{h+2}-1}{k^2-1} & h \equiv 0 \pmod{2} \\ \frac{k^{h+2}-1}{k^2-1} + \frac{k}{k+1} - 1 & h \equiv 1 \pmod{2}. \end{cases}$$

To see this, note that a  $k$ -domination set of minimum size must contain the leaves of the tree and every vertex that is of even distance from its closest leaf. Hence as  $h$  increases, this difference becomes arbitrarily large. It is interesting to note that  $\gamma_k(T_{k,h}) = \text{bp}_{k+1}(T_{k,h})$  when  $h$  is even and  $\gamma_k(T_{k,h}) + 1 = \text{bp}_{k+1}(T_{k,h})$  when  $h$  is odd. This shows that the difference  $\text{bp}_{k+1}(G) - \text{bp}_k(G)$  in Observation 2.2 can be made arbitrarily large.

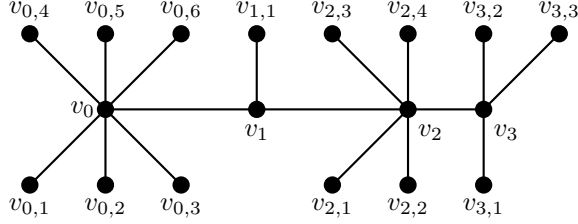


Figure 3: The caterpillar  $P_4(6, 1, 4, 3)$

## 4 Caterpillars

In this section, we give a closed formula for the  $k$ -bootstrap percolation number of a caterpillar. A caterpillar is obtained from the path on  $r$  vertices by appending leaves to the existing vertices of the path. The vertices of the original path, which are called the *spine* of the caterpillar, are labeled  $v_1, \dots, v_r$  in the natural way, and we call  $r$  the *spine length*. We append  $x_i$  leaves to  $v_i$  for  $1 \leq i \leq r$ . The caterpillar with parameters  $r, x_1, \dots, x_r$  will be denoted  $P_r(x_1, \dots, x_r)$  (see Figure 3). Without loss of generality, we will assume that for  $i \in \{1, r\}$ ,  $x_i \geq 1$ .

For the caterpillar  $C = P_r(x_1, \dots, x_r)$ , our initial percolating set must contain every vertex of degree less than  $k$  by Observation 2.1. Thus for  $k \geq 2$ , this set must contain every leaf. Further, it must contain all  $v_i$  such that  $x_i \leq k - 3$  for  $1 \leq i \leq r$ . Likewise, if  $x_1 \leq k - 2$ , then  $v_1$  is in the set. Similarly, if  $x_r \leq k - 2$ , then  $v_r$  is in the set. Note that if  $x_i \geq k$ , then  $v_i$  will not be included in our percolating set as these vertices will be infected after one step.

The above discussion tells us nothing about the following vertices:

- $v_1$  if  $x_1 = k - 1$ .
- $v_r$  if  $x_r = k - 1$ .
- $v_i$  if  $x_i \in \{k - 2, k - 1\}$  and  $2 \leq i \leq r - 1$ .

We call these vertices *sensitive*. We partition the sensitive vertices into two sets,  $S_1$  and  $S_2$ , as follows. We let  $S_1$  consist of all  $v_i$  satisfying  $x_i = k - 1$  and  $1 \leq i \leq r$ . We let  $S_2$  consist of all  $v_i$  satisfying  $x_i = k - 2$  and  $2 \leq i \leq r - 1$ .

Consider the subgraph induced by  $S_1 \cup S_2$ . Label the connected components of this subgraph  $L_1, \dots, L_m$ . We call these connected components *sensitive strings*. By definition, two sensitive strings are separated by at least one vertex whose inclusion in the initial set is decided according to the above discussion. For this reason, we may consider each sensitive string individually. Our goal for each sensitive string is to determine the minimum number of vertices to include in our initial set so that the entire string is eventually infected. We denote this number  $w(L_i)$  for  $1 \leq i \leq m$ .

**Lemma 4.1.** *Let  $k \geq 2$ , let  $C = P_r(x_1, \dots, x_r)$  be a caterpillar,  $L_1, \dots, L_m$  be the sensitive strings in  $C$ , and  $w(L_i)$  for  $1 \leq i \leq m$  be as above. Then, for  $1 \leq i \leq m$  we have*

1. If  $v_1 \notin V(L_i)$  and  $v_r \notin V(L_i)$ , then

$$w(L_i) = \left\lfloor \frac{|V(L_i) \cap S_2|}{2} \right\rfloor.$$

2. If  $v_1 \in V(L_i)$  or  $v_r \in V(L_i)$  but  $\{v_1, v_r\} \not\subseteq V(L_i)$ , then

$$w(L_i) = \left\lfloor \frac{1 + |V(L_i) \cap S_2|}{2} \right\rfloor.$$

3. If  $v_1 \in V(L_i)$  and  $v_r \in V(L_i)$ , then

$$w(L_i) = \left\lfloor \frac{2 + |V(L_i) \cap S_2|}{2} \right\rfloor.$$

*Proof.* Let  $1 \leq i \leq m$  be fixed and let  $S = V(L_i) \cap S_2 = \{s_1, \dots, s_t\}$  be a sequence. To prove part 1), we choose for  $A_k^0(C)$  every other  $s_j$  beginning with  $s_2$ . It is necessary to initially infect every other vertex in  $S$  because if two vertices in  $S$  are not initially infected, they must have an infected vertex between them by Observation 2.1. We choose to begin with  $s_2$  because  $S$  is flanked by vertices that are either initially infected or will eventually become infected. Now, consider a connected component of the subgraph induced by  $V(L_i) \cap S_1$ . In Case 1, where  $L_i$  contains no endpoint of the spine, if this component of  $S_1$  lies to the left of  $s_1$  or to the right of  $s_t$  it will be eventually infected by the vertex to its left (right). If, on the other hand, it lies between two connected components of  $S$  then either the  $s_j$  to its left or the  $s_j$  to its right will have an even subscript and eventually infect all of its vertices<sup>2</sup>. This establishes part 1).

To prove part 2), we assume without loss of generality that  $v_1 \in V(L_i)$  and  $v_r \notin V(L_i)$ . Note that  $L_i$  is adjacent on the right to a vertex  $w$  which is either in our initial set or will be infected eventually. Hence, our result will follow in a similar manner to the proof of part 1). However, the appropriate set of vertices to include from  $S$  is now

$$S' = \begin{cases} \{s_1, s_3, \dots, s_{t-1}\} & \text{if } t \equiv 0 \pmod{2} \\ \{s_1, s_2, s_4, \dots, s_{t-1}\} & \text{if } t \equiv 1 \pmod{2}. \end{cases}$$

Note that this gives us the desired result of

$$w(L_i) = \left\lfloor \frac{1 + |V(L_i) \cap S_2|}{2} \right\rfloor.$$

As for part 3), note that if  $v_1 \in V(L_i)$  and  $v_r \in V(L_i)$ , then the entire spine is a sensitive string. Hence, our result will follow in a similar manner to the proof of part 1). However, the appropriate set of vertices to include from  $V(L_i) \cap S_2$  is now

$$S' = \begin{cases} \{s_1, s_3, \dots, s_{t-1}, s_t\} & \text{if } t \equiv 0 \pmod{2} \\ \{s_1, s_3, \dots, s_t\} & \text{if } t \equiv 1 \pmod{2}. \end{cases}$$

<sup>2</sup>If the size of  $S$  is even, then we could just as well have initially infected the  $s_j$  with odd subscripts. If the size of  $S$  is odd, then initially infecting the  $s_i$  with even subscripts is necessary for the number of initially infected vertices to be minimum.

Note that this gives us the desired result of

$$w(L_i) = \left\lfloor \frac{2 + |V(L_i) \cap S_2|}{2} \right\rfloor.$$

The final remaining case is when the entire spine of the caterpillar is in  $S_1$ , so that  $m = 1$  and  $S$  is empty. In this case, we choose the middle vertex (or one of the two middle vertices) in  $S_1$  to include in the initially infected set. The formula in part 3) gives the correct weight of the spine as 1. □

Combining Observation 2.1 and Lemma 4.1, we obtain the main result of this section. Note that a caterpillar with spine length one is a star. The  $k$ -bootstrap percolation number of such a caterpillar was given in Theorem 3.1. For this reason, we assume that  $r \geq 2$ . For convenience of exposition, we let  $d_{\leq \ell}(C)$  denote the number of vertices in  $C$  of degree less than or equal to  $\ell$ .

**Theorem 4.2.** *Let  $k \geq 2$  and let  $C = P_r(x_1, \dots, x_r)$  with  $r \geq 2$ . Let  $L_1, \dots, L_m$  be the sensitive strings in  $C$ . For each  $L_i$  we let  $w(L_i)$  be as above. The  $k$ -bootstrap percolation number of the caterpillar is*

$$\text{bp}_k(C) = \sum_{i=1}^m w(L_i) + d_{\leq k-1}(C).$$

*Proof.* This is a straightforward combination of Observation 2.1 and Lemma 4.1. □

Note that a double star is a caterpillar with a spine of length two, so this gives an alternate proof of Theorem 3.2. Moreover, we can use the above result to show that the bound from Observation 2.4 is sharp. Consider the caterpillar  $P_n(t, \dots, t)$ , where  $k \geq 4$  and  $t \leq k - 3$ . Every vertex has degree less than  $k$ . Hence, every vertex must be in a  $k$ -domination set and in a percolating set. We also mention that we omit the time parameter in this setting due to the length and tedium of the required calculation as well as the complexity of the resulting formula.

## 5 An upper bound

In this section we present a sharp upper bound for the  $k$ -bootstrap percolation number of a tree. We then compare this bound to other known bounds.

Before stating the theorem, we define the following notation. Recall that  $d_{\leq k}(T)$  is the number of vertices in  $T$  of degree less than or equal to  $k$ . Similarly, we let  $d_k(T)$  be the number of degree  $k$  vertices in  $T$ , and we let  $d_{\geq k}(T)$  be the number of vertices in  $T$  of degree greater than or equal to  $k$ . Furthermore, for a vertex  $s \in V(T)$ , we set  $\ell(s)$  to be the number of leaves adjacent to  $s$ .

**Theorem 5.1.** *Let  $T$  be a tree and  $k \geq 2$ . Then,  $\text{bp}_k(T) \leq d_{\leq k-1}(T) + \left\lfloor \frac{d_k(T)}{2} \right\rfloor$ .*

*Proof.* We proceed by induction on  $n$ , the number of vertices of the tree  $T$ .

Up to isomorphism, there is only one tree on two vertices and one tree on three vertices. The result is easily verified in both cases.

Suppose for induction that the result is true for all trees with at most  $n$  vertices.

Let  $T$  be a tree with  $n + 1$  vertices and choose a leaf  $v \in V(T)$  on the periphery of  $T$  with unique neighbor  $s$ . As shown in Theorem 3.1, this result holds for stars. For this reason, we will assume that  $T$  is not a star. Note, since  $v$  is on the periphery and  $T$  is not a star, we have that  $\deg(s) = \ell(s) + 1$ . We now consider several cases.

**Case 1:**  $\ell(s) \leq k - 2$  or  $\ell(s) \geq k + 1$ .

In this case, we remove the leaf  $v$  from  $T$  and denote the resulting tree  $T'$ . Note,  $d_{\leq k-1}(T') = d_{\leq k-1}(T) - 1$  and  $d_k(T') = d_k(T)$ . By the induction hypothesis,  $T'$  has a percolating set, denoted  $S$ , of cardinality at most

$$d_{\leq k-1}(T') + \left\lfloor \frac{d_k(T')}{2} \right\rfloor = d_{\leq k-1}(T) - 1 + \left\lfloor \frac{d_k(T)}{2} \right\rfloor.$$

Thus,  $S \cup \{v\}$  eventually infects all of  $T$  and has cardinality at most  $d_{\leq k-1}(T) + \left\lfloor \frac{d_k(T)}{2} \right\rfloor$ .

**Case 2:**  $\ell(s) = k - 1$  and  $d_k(T)$  is even.

In this case, we remove the leaf  $v$  from  $T$  and denote the resulting tree  $T'$ . Note,  $d_{\leq k-1}(T') = d_{\leq k-1}(T)$  and  $d_k(T') = d_k(T) - 1$  since the degree of  $s$  in  $T$  is  $k$  and has been decreased by one in  $T'$ . By the induction hypothesis,  $T'$  has a percolating set, denoted  $S$ , of cardinality at most

$$\begin{aligned} d_{\leq k-1}(T') + \left\lfloor \frac{d_k(T')}{2} \right\rfloor &= d_{\leq k-1}(T) + \left\lfloor \frac{d_k(T) - 1}{2} \right\rfloor \\ &= d_{\leq k-1}(T) + \left\lfloor \frac{d_k(T)}{2} \right\rfloor - 1, \end{aligned}$$

where we have used that  $d_k(T)$  is even in the second equality. Thus,  $S \cup \{v\}$  eventually infects all of  $T$  and has cardinality at most  $d_{\leq k-1}(T) + \left\lfloor \frac{d_k(T)}{2} \right\rfloor$ .

**Case 3:**  $\ell(s) = k - 1$  and  $d_k(T)$  is odd.

First, we label the leaves of  $s$  by  $L = \{v = v_1, v_2, \dots, v_{k-1}\}$ . Furthermore, let  $t \in V(T)$  be a non-leaf with  $st \in E(T)$ , which is possible since  $T$  is not a star. We now remove  $s$  and its  $k - 1$  adjacent leaves from  $T$  to obtain a tree  $T'$ .

If  $\deg(t) \leq k - 1$  or  $\deg(t) \geq k + 2$  in  $T$ , then  $d_{\leq k-1}(T') = d_{\leq k-1}(T) - (k - 1)$  and  $d_k(T') = d_k(T) - 1$ . This follows because we have removed  $k - 1$  leaves, a vertex of degree  $k$ , and decreased the degree of  $t$  by one. Then,  $T'$  has a percolating set, denoted  $S$ , of cardinality at most

$$\begin{aligned} d_{\leq k-1}(T') + \left\lfloor \frac{d_k(T')}{2} \right\rfloor &= d_{\leq k-1}(T) - (k - 1) + \left\lfloor \frac{d_k(T) - 1}{2} \right\rfloor \\ &= d_{\leq k-1}(T) - (k - 1) + \left\lfloor \frac{d_k(T)}{2} \right\rfloor, \end{aligned}$$

where we have used that  $d_k(T)$  is odd in the second equality. Note,  $t \in S$ , and hence  $S \cup L$  eventually infects all of  $T$  and has cardinality at most  $d_{\leq k-1}(T) + \left\lfloor \frac{d_k(T)}{2} \right\rfloor$ .

If  $\deg(t) = k$ , then  $d_{\leq k-1}(T') = d_{\leq k-1}(T) - (k - 2)$  and  $d_k(T') = d_k(T) - 2$ . This follows because we have removed  $k - 1$  leaves, a vertex of degree  $k$ , and decreased the

degree of  $t$  by one. Then,  $T'$  has a percolating set, denoted  $S$ , of cardinality at most

$$\begin{aligned} d_{\leq k-1}(T') + \left\lfloor \frac{d_k(T')}{2} \right\rfloor &= d_{\leq k-1}(T) - (k-2) + \left\lfloor \frac{d_k(T) - 2}{2} \right\rfloor \\ &= d_{\leq k-1}(T) - (k-1) + \left\lfloor \frac{d_k(T)}{2} \right\rfloor. \end{aligned}$$

Note,  $t \in S$ , and hence  $S \cup L$  eventually infects all of  $T$  and has cardinality at most  $d_{\leq k-1}(T) + \left\lfloor \frac{d_k(T)}{2} \right\rfloor$ .

If  $\deg(t) = k+1$ , then  $d_{\leq k-1}(T') = d_{\leq k-1}(T) - (k-1)$  and  $d_k(T') = d_k(T)$ . This follows because we have removed  $k-1$  leaves, a vertex of degree  $k$ , and decreased the degree of  $t$  by one. Then,  $T'$  has a percolating set, denoted  $S$ , of cardinality at most

$$d_{\leq k-1}(T') + \left\lfloor \frac{d_k(T')}{2} \right\rfloor = d_{\leq k-1}(T) - (k-1) + \left\lfloor \frac{d_k(T)}{2} \right\rfloor.$$

Note, as  $S$  eventually infects all of  $T'$ , we have that  $t$  will eventually be infected, and hence  $S \cup L$  eventually infects all of  $T$  and has cardinality at most  $d_{\leq k-1}(T) + \left\lfloor \frac{d_k(T)}{2} \right\rfloor$ .

**Case 4:**  $\ell(s) = k$  and  $d_k(T)$  is even.

In this case, we remove the leaf  $v$  from  $T$  and denote the resulting tree  $T'$ . Note,  $d_{\leq k-1}(T') = d_{\leq k-1}(T) - 1$  and  $d_k(T') = d_k(T) + 1$  since the degree of  $s$  in  $T$  is  $k+1$  and has been decreased by one in  $T'$ . By the induction hypothesis,  $T'$  has a percolating set, denoted  $S$ , of cardinality at most

$$\begin{aligned} d_{\leq k-1}(T') + \left\lfloor \frac{d_k(T')}{2} \right\rfloor &= d_{\leq k-1}(T) - 1 + \left\lfloor \frac{d_k(T) + 1}{2} \right\rfloor \\ &= d_{\leq k-1}(T) - 1 + \left\lfloor \frac{d_k(T)}{2} \right\rfloor, \end{aligned}$$

where we have used that  $d_k(T)$  is even in the second equality. Thus,  $S \cup \{v\}$  eventually infects all of  $T$  and has cardinality at most  $d_{\leq k-1}(T) + \left\lfloor \frac{d_k(T)}{2} \right\rfloor$ .

**Case 5:**  $\ell(s) = k$  and  $d_k(T)$  is odd.

In this case, we remove two leaves from  $T$ , say  $v$  and  $w$ , which are both supported by  $s$ , and denote the resulting tree  $T'$ . Note,  $d_{\leq k-1}(T') = d_{\leq k-1}(T) - 1$  and  $d_k(T') = d_k(T)$  since the degree of  $s$  in  $T$  is  $k+1$  and has been decreased by two in  $T'$ . By the induction hypothesis,  $T'$  has a percolating set, denoted  $S$ , of cardinality at most

$$d_{\leq k-1}(T') + \left\lfloor \frac{d_k(T')}{2} \right\rfloor = d_{\leq k-1}(T) - 1 + \left\lfloor \frac{d_k(T)}{2} \right\rfloor.$$

Note,  $s \in S$  since  $\deg(s) = k-1$  in  $T'$ . Thus,  $(S \cup \{v, w\}) \setminus \{s\}$  eventually infects all of  $T$  and has cardinality at most  $d_{\leq k-1}(T) + \left\lfloor \frac{d_k(T)}{2} \right\rfloor$ .

In all cases we have shown that  $T$  has a percolating set of cardinality at most  $d_{\leq k-1}(T) + \left\lfloor \frac{d_k(T)}{2} \right\rfloor$ , and hence  $\text{bp}_k(T) \leq d_{\leq k-1}(T) + \left\lfloor \frac{d_k(T)}{2} \right\rfloor$ . The proof follows by induction.  $\square$

We now make a few observations concerning the above bound.



1. The above bound is sharp for paths when  $k = 2$  and for the family of caterpillars of the form  $P_n(k - 2, k - 2, \dots, k - 2, k - 2)$ , where  $k \geq 3$ .
2. We can make the difference  $d_{\leq k-1}(T) + \left\lfloor \frac{d_k(T)}{2} \right\rfloor - \text{bp}_k(T)$  arbitrarily large using the family of caterpillars  $P_n(k, k - 2, k, k - 2, \dots, k - 2, k)$ .
3. For a connected graph  $G$ , we can remove edges from  $G$  to obtain a spanning tree  $T$  of  $G$ . Then the inequality  $\text{bp}_k(G) \leq \text{bp}_k(T)$  combined with the above upper bound gives an upper bound for  $\text{bp}_k(G)$ .

We conclude this section by comparing our above result with the bounds obtained by Riedl in [25]. The upper and lower bounds for  $\text{bp}_k(T)$  given by Riedl can be found in Proposition 3 (lower bound) and Theorem 4 (upper bound) of [25] and are given by

$$\frac{(k-1)n+1}{k} \leq \text{bp}_k(T) \leq \frac{kn+d_{\leq k-1}(T)}{k+1}, \quad (5.1)$$

where  $n$  is the order of the tree and  $d_{\leq k-1}(T)$  is defined before the statement of Theorem 5.1. It should be noted that our quantity  $\text{bp}_k(T)$  is denoted in [25] as  $m(T, k)$ . Moreover, the upper bound given in [25] is actually an upper bound for a different, but larger, quantity than  $\text{bp}_k(T)$ .

We first mention that following the statement of Proposition 3 in [25], Riedl mentions that for  $k = 2$  his bound is sharp for odd length paths, and for  $k > 2$  his lower bound is sharp for complete  $k$ -ary trees and complete  $k - 1$ -ary trees. Note, this is precisely the cases of Theorem 3.8 with  $k = N, N + 1$ .

With regards to the bound in Theorem 5.1, we have that this is equal to the lower bound in Equation 5.1 for paths of odd length when  $k = 2$ . Moreover, by writing  $n = d_{\leq k-1}(T) + d_{\geq k}(T)$ , we can rewrite the upper bound in Equation 5.1 as

$$\text{bp}_k(T) \leq d_{\leq k-1}(T) + \frac{kd_{\geq k}(T)}{k+1}.$$

As  $d_{\geq k}(T) \geq d_k(T)$  and  $k > 1$  we have

$$d_{\leq k-1}(T) + \frac{kd_{\geq k}(T)}{k+1} \geq d_{\leq k-1}(T) + \frac{kd_k(T)}{k+1} \geq d_{\leq k-1}(T) + \left\lfloor \frac{d_k(T)}{2} \right\rfloor,$$

which is precisely the upper bound in Theorem 5.1. Hence, Theorem 5.1 gives an improvement upon the upper bound in [25].

## 6 Trees of small order

In this section, we use the above results to complete the characterization of trees on eleven vertices or less. Throughout, we denote such a tree by  $T$ .

There are 201 non-isomorphic trees on ten vertices or less (see Harary [15] or Steinbach's "Field Guide to Simple Graphs" [26]). All but seven of these can be classified as spiders, caterpillars, or trees of diameter at most five. These seven trees all have degree sequence  $[3, 3, 2, 2, 2, 2, 1, 1, 1, 1]$ . Riedl's lower bound (Equation 5.1) shows that the 2-bootstrap percolation number satisfies  $\text{bp}_2(T) \geq 6$ . The bound given in Theorem 5.1

shows that  $\text{bp}_2(T) \leq 6$ . Therefore,  $\text{bp}_2(T) = 6$  and  $t_2(T) \leq 2$  in these cases. Except for the four cases in which the two vertices of degree three are adjacent, we have that  $\text{bp}_3(T) = 8$  and  $t_3(T) = 1$ . In the four cases in which the two vertices of degree three are adjacent, we have that  $\text{bp}_3(T) = 9$  and  $t_3(T) = 1$  by Observation 2.1.

As for the 235 non-isomorphic trees on eleven vertices, all but 42 of these can be classified as spiders, caterpillars, or trees of diameter at most five. Note that Riedl's lower bound guarantees that  $\text{bp}_2(T) \geq 6$  and  $\text{bp}_3(T) \geq 8$ .

Fifteen of these have degree sequence  $[3, 3, 2, 2, 2, 2, 2, 1, 1, 1, 1]$ . In these cases, the bound given in Theorem 5.1 shows that  $\text{bp}_2(T) \leq 6$ . Hence  $\text{bp}_2(T) = 6$ . The trivial lower bound given by Observation 2.1 shows that  $\text{bp}_3(T) \geq 9$  while the bound given in Theorem 5.1 shows that  $\text{bp}_3(T) \leq 10$ . It is straightforward to check that  $\text{bp}_3(T) = 9$  for these trees if and only if their two vertices of degree three are not adjacent. Otherwise, we have that  $\text{bp}_3(T) = 10$ .

Thirteen of these have degree sequence  $[3, 3, 3, 2, 2, 2, 1, 1, 1, 1, 1]$ . Again, the bound given in Theorem 5.1 shows that  $\text{bp}_2(T) \leq 6$ . Combining this with Riedl's lower bound yields  $\text{bp}_2(T) = 6$ . Further, due to Riedl and Theorem 5.1, we have that  $8 \leq \text{bp}_3(T) \leq 9$ . Of these, only one has no two vertices of degree three adjacent. Therefore,  $\text{bp}_3(T) = 8$  in this case. For the remaining twelve,  $\text{bp}_3(T) = 9$  due to Observation 2.1.

The fourteen remaining trees have degree sequence  $[4, 3, 2, 2, 2, 2, 1, 1, 1, 1, 1]$ . Combining Riedl's bound with Theorem 5.1 yields  $6 \leq \text{bp}_2(T) \leq 7$ . It is straightforward to check that eight of these have  $\text{bp}_2(T) = 6$  and the remaining six have  $\text{bp}_2(T) = 7$ . Note that the trivial lower bound and Theorem 5.1 guarantee that  $\text{bp}_3(T) = 9$ . Observation 2.1 shows that  $\text{bp}_4(T) = 10$  in all of these cases.

## 7 Open problems

In this section, we give open problems related to this study as possible avenues for future research.

Suppose that we want every vertex to be infected within  $t$  iterations. Among all  $k$ -bootstrap sets that will infect the graph within  $t$  iterations, choose one with minimum cardinality. What is the cardinality of such a set?

Suppose that we limit the size of the initial set. What is the maximum number of vertices that can be infected? How is this maximum changed if we also limit the number of iterations?

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