On divisible design Cayley graphs*

Vladislav V. Kabanov[†]

*Krasovskii Institute of Mathematics and Mechanics,
S. Kovalevskaja st. 16, Yekaterinburg 620990, Russia, and
Sobolev Institute of Mathematics, Acad. Koptyug prospect 4, Novosibirsk 630090, Russia*

Leonid Shalaginov

*Krasovskii Institute of Mathematics and Mechanics,
S. Kovalevskaja st. 16, Yekaterinburg 620990, Russia, and
Sobolev Institute of Mathematics, Acad. Koptyug prospect 4, Novosibirsk 630090, Russia,
and Chelyabinsk State University, Brat'ev Kashirinyh st. 129, Chelyabinsk 454021, Russia*

Received 29 November 2019, accepted 16 February 2020, published online 10 March 2021

Abstract

Let v, k, b, a be integers such that $v > k \geq b \geq a \geq 0$. A *Deza graph* with parameters (v, k, b, a) is a k -regular graph on v vertices in which the number of common neighbors of any two distinct vertices takes two values a or b ($a \leq b$). A k -regular graph on v vertices is a *divisible design graph* with parameters $(v, k, \lambda_1, \lambda_2, m, n)$ when its vertex set can be partitioned into m classes of size n , such that any two distinct vertices from the same class have λ_1 common neighbors, and any two vertices from different classes have λ_2 common neighbors. It is clear, that divisible design graphs are Deza graphs.

It is shown that divisible design Cayley graphs arise only by means of divisible difference sets relative to some subgroup. Construction of a special set in an affine group over a finite field is given and shown that this set is a divisible difference set and thus its development is a divisible design Cayley graph.

Keywords: Deza graph, divisible design graph, divisible design, divisible different set, Cayley graph, affine group over a finite field.

Math. Subj. Class.: 05B05, 05C51, 51E05

*The work is supported by Mathematical Center in Akademgorodok, the agreement with Ministry of Science and High Education of the Russian Federation number 075-15-2019-1613. Both authors are also partially supported by RFBR according to the research project 17-51-560008. The authors are grateful to Vladimir Trofimov, Sergey Goryainov, Elena Konstantinova and participants of workshop “New trends in algebraic graph theory”, which was organized by Mathematical Center in Akademgorodok, for useful discussions.

[†]Corresponding author.

E-mail addresses: vvk@imm.uran.ru (Vladislav V. Kabanov), 44sh@mail.ru (Leonid Shalaginov)

1 Introduction

Let v, k, b, a be integers such that $v > k \geq b \geq a \geq 0$. A *Deza graph* with parameters (v, k, b, a) is a k -regular graph on v vertices in which the number of common neighbors of any two distinct vertices takes two values a or b ($a \leq b$). Deza graphs appeared as a generalization of strongly regular graphs in [7]. This is a wide class of graphs which includes not only strongly regular graphs but also regular $(0, \lambda)$ -graphs [10], $(0, 2)$ -graphs [5, 11] and divisible design graphs [6, 9].

A k -regular graph on v vertices is a *divisible design graph* with parameters $(v, k, \lambda_1, \lambda_2, m, n)$ when its vertex set can be partitioned into m classes of size n , such that any two distinct vertices from the same class have λ_1 common neighbors, and any two vertices from different classes have λ_2 common neighbors. For a divisible design graph the partition into classes is called a *canonical partition*.

Let G be a finite group with the identity element e . If S is a subset of G which is closed under inversion and does not contain e , then *Cayley graph* $\text{Cay}(G, S)$ is a graph with the vertex set G and two vertices x, y are adjacent if and only if $xy^{-1} \in S$.

In this paper, some divisible design graphs are constructed that arise from finite groups in the form of Cayley graphs. The following theorem is the basis for the construction.

Theorem 1.1. *Let $\text{Cay}(G, S)$ be a Deza graph with parameters (v, k, b, a) . Let also A, B and $\{e\}$ be a partition of G and SS^{-1} be a multiset such that $SS^{-1} = aA + bB + k\{e\}$. If either $A \cup \{e\}$ or $B \cup \{e\}$ is a subgroup of G , then $\text{Cay}(G, S)$ is a divisible design graph and the right cosets of this subgroup give a canonical partition of the graph. Conversely, if $\text{Cay}(G, S)$ is a divisible design graph, then the class of its canonical partition which contains the identity of G is a subgroup of G and classes of the canonical partition of divisible design graph coincide with the cosets of this subgroup.*

Proof. Let $N = A \cup \{e\}$ be a subgroup of G and let $G = Na_1 \cup \dots \cup Na_r$ be a partition of G by the right cosets of N . For every $g, h \in N$ and $1 \leq i \leq r, 1 \leq j \leq r$, the set of neighbors of ha_i is Sha_i and the set of neighbors of ga_j is Sga_j . Thus, the set of common neighbors for ha_i and ga_j coincides with $Sha_i \cap Sga_j$. The number $|Sha_i \cap Sga_j|$ equals $|\{(s, s^*) | sha_i = s^*ga_j\}|$, where $s, s^* \in S$. Therefore, $h = s^{-1}s^*ga_ja_i^{-1}$. If $i = j$, then $h = s^{-1}s^*g$ and $s^{-1}s^* \in N$. So there are exactly a such pairs if $s^{-1}s^* \in A$ and k such pairs if $s^{-1}s^* = e$. If $i \neq j$, then $ga_ja_i^{-1} \notin N$ and hence $s^{-1}s^* \notin N$. So $s^{-1}s^* \in B$ and there are exactly b such pairs. The case of $N = B \cup \{e\}$ is viewed in a similar way.

Conversely, let $\text{Cay}(G, S)$ be a divisible design graph and N be a class of the canonical partition of divisible design graph which contains the identity of G . It's enough to prove that for any $h, g \in N$ we have $hg^{-1} \in N$. Since h and g belong to the same class of the canonical partition of $\text{Cay}(G, S)$, then $|Sh \cap Sg| = \lambda_1$. The number of pairs (s, s^*) such that $sh = s^*g$ is equal to λ_1 . Thus, $hg^{-1} = s^{-1}s^*$ is repeated λ_1 times in SS^{-1} . $\square$

Theorem 1.1 shows that Cayley divisible design graphs arise only by means of divisible difference sets relative to some subgroup. Let a finite group G of order mn have a subgroup N of order n . A subset S of G is called a *divisible difference set with exceptional subgroup* N if there are constants λ_1 and λ_2 such that every non-identity element of N can be expressed as a right quotient of elements in S in exactly λ_1 ways and every element in $G \setminus N$ can be expressed as a right quotient of elements in S in exactly λ_2 ways.

In other words, if $k = |S|$, then $SS^{-1} = k\{e\} + \lambda_1(N - \{e\}) + \lambda_2(G - N)$.

2 Construction of divisible design Cayley graphs

Let $\mathbb{F}$ be a finite field with q^r elements, where q is a prime power and $r > 1$.

Consider the group $\mathbb{G}$ of all 2×2 matrices $\begin{pmatrix} 1 & 0 \\ \alpha & \beta \end{pmatrix}$, where $\alpha \in \mathbb{F}$ and $\beta \in \mathbb{F} \setminus \{0\}$. It's clear that $\mathbb{G}$ is a semi-direct product of two subgroups N and K , where N consists of all matrices $\begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}$, and $\alpha \in \mathbb{F}$, K consists of all matrices $\begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix}$, and $\beta \in \mathbb{F} \setminus \{0\}$.

define a bijection ψ^+ between N and $\mathbb{F}$ as follows: $\psi^+(a) = \alpha$ for any $a \in N$, $a = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}$, $\alpha \in \mathbb{F}$. If

$$a_1 = \begin{pmatrix} 1 & 0 \\ \alpha_1 & 1 \end{pmatrix} \quad \text{and} \quad a_2 = \begin{pmatrix} 1 & 0 \\ \alpha_2 & 1 \end{pmatrix} \quad \alpha_1, \alpha_2 \in \mathbb{F},$$

then

$$a_1 a_2 = \begin{pmatrix} 1 & 0 \\ \alpha_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \alpha_2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \alpha_1 + \alpha_2 & 1 \end{pmatrix}.$$

Thus, N is isomorphic to the additive group $\mathbb{F}^+$ which we can consider as a linear space of dimension r over $\mathbb{F}_q$.

Define a bijection $\psi^\times$ between K and $\mathbb{F} \setminus \{0\}$ as follows: $\psi^\times(b) = \beta$ for any $b \in K$, $b = \begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix}$, $\beta \in \mathbb{F} \setminus \{0\}$. Clearly, $\psi^\times$ is an isomorphism between K and the multiplicative group of $\mathbb{F}$.

Let K be generated by matrix $f^* = \begin{pmatrix} 1 & 0 \\ 0 & \tau \end{pmatrix}$, where τ is a primitive element $\mathbb{F}$.

Also let H be a cyclic group generated by $f = (f^*)^{q-1} = \begin{pmatrix} 1 & 0 \\ 0 & \theta \end{pmatrix}$, where $\theta = \tau^{q-1}$. Thus, $G = NH$ is a subgroup of $\mathbb{G}$ of index $q - 1$ and the order of G is equal to $q^r(q^r - 1)/(q - 1)$. Furthermore, N is a normal subgroup of order q^r and index $(q^r - 1)/(q - 1)$ in G .

By the formula of Gaussian binomial coefficients, the number of $(r - 1)$ -dimensional subspaces of N equals t , where

$$t = (q^r - 1)/(q - 1).$$

Let $\mathbb{M}$ be the set of preimages of all these $(r - 1)$ -dimensional subspaces of $\mathbb{F}^+$ in N under ψ^+ . Since ψ^+ is an isomorphism, then the set $\mathbb{M}$ consists of t subgroups of order q^{r-1} from N .

Denote by M one of the subgroups from $\mathbb{M}$. Thus,

$$\mathbb{M} = \{M_i \mid \psi^+(M_i) = \psi^+(M)\tau^i, \quad i = 1, 2, \dots, t\}.$$

Let φ be a permutation on the set $\mathbb{M} = \{M_1, M_2, \dots, M_t\}$. As usual by $f^i(N \setminus M_{\varphi(i)})$ we denote the set $\{f^i a : a \in N \setminus M_{\varphi(i)}\}$.

Define a subset S of G in the following way:

$$S = \bigcup_{i=1}^t f^i(N \setminus M_{\varphi(i)}).$$

It is obvious, that S is a generating set of G . In the following lemmas, we consider the question of when this set is closed under inversion.

Lemma 2.1. *Subset S is closed under inversion in G if and only if for all integers i the following condition holds*

$$\psi^+(M_{\varphi(i)})\theta^i = \psi^+(M_{\varphi(t-i)}). \quad (*)$$

Proof. Let $s \in S$ and $s = f^i a$, for some integer i and $a \in N \setminus M_{\varphi(i)}$. Also, let $a = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}$, for some $\alpha \in \psi^+(N \setminus M_{\varphi(i)})$ and $f^i = \begin{pmatrix} 1 & 0 \\ 0 & \theta^i \end{pmatrix}$.

In such case, $a^{-1} = \begin{pmatrix} 1 & 0 \\ -\alpha & 1 \end{pmatrix}$, for $\alpha \in \psi^+(N \setminus M_{\varphi(i)})$ and $f^{-i} = \begin{pmatrix} 1 & 0 \\ 0 & \theta^{t-i} \end{pmatrix}$.

Then

$$\begin{aligned} f^i a &= \begin{pmatrix} 1 & 0 \\ 0 & \theta^i \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \alpha\theta^i & \theta^i \end{pmatrix}, \\ (f^i a)^{-1} &= \begin{pmatrix} 1 & 0 \\ -\alpha & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \theta^{t-i} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ (-\alpha\theta^i)\theta^{t-i} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \theta^{t-i} \end{pmatrix}. \end{aligned}$$

Thus, $s^{-1} \in S$ if and only if $-\alpha\theta^i \in \psi^+(M_{\varphi(t-i)})$. Hence, $S = S^{-1}$ if and only if $(*)$ holds. $\square$

The interesting question to study all permutations for which the property $(*)$ holds is done next. Let $\varphi = (\varphi_1, \dots, \varphi_t)$ be a permutation of the set $\{1, \dots, t\}$, where $\varphi(i) = \varphi_i$.

Lemma 2.2. *For any $t > 2$ there is at least one permutation φ satisfying*

$$\psi^+(M_{\varphi(i)})\theta^i = \psi^+(M_{\varphi(t-i)})$$

for all integers i .

Proof. Let $\varphi = (\varphi_1, \dots, \varphi_t)$. If t is an odd integer, then

$$\varphi = (1, t-1, t-3, \dots, 2, t, t-2, t-4, \dots, 3).$$

If t is an even integer, then

$$\begin{aligned} \varphi &= (1, t-1, t-3, \dots, t/2+2, t/2-1, t/2-3, \dots, 2, t/2, t-2, t-4, \dots, \\ &\quad \dots, t/2+1, t, t/2-2, t/2-4, \dots, 3). \end{aligned}$$

$\square$

For example, if $t = 7$, then there are exactly three permutations

$$(1, 6, 4, 2, 7, 5, 3), (1, 6, 3, 7, 2, 4, 5), \text{ and } (1, 6, 2, 3, 5, 7, 4)$$

which satisfy the condition $(*)$ in Lemma 2.1.

Construction 2.3. Let Γ be a Cayley graph $\text{Cay}(G, S)$ whose vertices are elements of the group G defined as above and

$$S = \bigcup_{i=1}^t f^i(N \setminus M_{\varphi(i)}).$$

In the next section we prove that if S satisfies the condition $(*)$, then Γ is a divisible design graph. Construction 2.3 gives us an infinite series of divisible design graphs which are Cayley graphs. Only the first graph among them is known and given in [9, Construction 4.20]. This divisible design Cayley graph with parameters $(12, 6, 2, 3, 3, 4)$ is the line graph of the octahedron and can be obtained as a Cayley graph from the alternating group of degree 4 (See Example 4.1 in Section 4).

3 Main theorem

The main goal of our article is to prove the following theorem.

Theorem 3.1. *Let Γ be a Cayley graph from Construction 2.3. If S satisfies the condition $(*)$, then Γ is a divisible design graph with parameters $(v, k, \lambda_1, \lambda_2, m, n)$, where*

$$\begin{aligned} v &= q^r(q^r - 1)/(q - 1), \quad k = q^{r-1}(q^r - 1), \\ \lambda_1 &= q^{r-1}(q^r - q^{r-1} - 1), \quad \lambda_2 = q^{r-2}(q - 1)(q^r - 1), \\ m &= (q^r - 1)/(q - 1), \quad n = q^r. \end{aligned}$$

Proof. It is clear, that Γ is an undirected graph on $v = q^r(q^r - 1)/(q - 1)$ vertices of valency

$$\begin{aligned} k &= |S| = \left| \bigcup_{i=0}^{t-1} f^i(N \setminus M_{\varphi(i)}) \right| = \sum_{i=0}^{t-1} (q^r - q^{r-1}) = \\ &= (q^r - q^{r-1})(q^r - 1)/(q - 1) = q^{r-1}(q^r - 1). \end{aligned}$$

Calculate a number of common adjacent vertices for any two distinct vertices from any coset. Since Γ is a Cayley graph, then it is enough to calculate this number for the identity element of G and any non-identity element from N . Let e be the identity element of G , $a \neq e$ and $a \in N$. It is important to note that a belongs to exactly $t_1 = (q^{r-1} - 1)/(q - 1)$ subgroups of N from $\mathbb{M}$.

Since $\Gamma(e) \cap \Gamma(a) = S \cap Sa$, then

$$\begin{aligned} \Gamma(e) \cap \Gamma(a) &= \left(\bigcup_{i=0}^{t-1} f^i(N \setminus M_{\varphi(i)}) \right) \cap \left(\bigcup_{i=0}^{t-1} f^i(N \setminus M_{\varphi(i)})a \right) = \\ &= \left(\bigcup_{a \in M_{\varphi(i)}} f^i(N \setminus M_{\varphi(i)}) \right) \cup \left(\bigcup_{a \notin M_{\varphi(i)}} f^i(N \setminus M_{\varphi(i)}) \cap f^i(N \setminus M_{\varphi(i)})a \right). \end{aligned}$$

Hence

$$|\Gamma(e) \cap \Gamma(a)| = \sum_{i=0}^{t_1-1} (q^r - q^{r-1}) + \sum_{i=t_1}^{t-1} (q^r - (q - 2)q^{r-1}) = q^{r-1}(q^r - q^{r-1} - 1).$$

Calculate a number of common adjacent vertices for any two vertices from any two distinct cosets. Since Γ is a Cayley graph, then it is enough to calculate this number for the identity element e from G and any element g from Nf^i , where $i \not\equiv 0 \pmod{t}$. Let $g = f^x a_g$, $x \not\equiv 0 \pmod{t}$. We have $f^x a_g = [f^{-x}, a_g^{-1}] a_g f^x$, where $[f^{-x}, a_g^{-1}]$ is the commutator of elements f^{-x} and a_g^{-1} . It is easy to verify, that if $b_g = [f^{-x}, a_g^{-1}] a_g$, then $b_g \in N$.

Since $\Gamma(e) \cap \Gamma(g) = S \cap Sg$, then

$$\Gamma(e) \cap \Gamma(g) = \left(\bigcup_{i=0}^{t-1} f^i(N \setminus M_{\varphi(i)}) \right) \cap \left(\bigcup_{i=0}^{t-1} f^i(N \setminus M_{\varphi(i)})g \right).$$

Taking into account that N is a normal subgroup of G and $b_g \in N$ we have

$$f^i(N \setminus M_{\varphi(i)}) \cap f^j(N \setminus M_{\varphi(j)})b_g f^x \neq \emptyset \quad \text{if and only if} \quad i - x = j.$$

Thus, we have

$$\begin{aligned} & \bigcup_{i=0}^{t-1} (f^i(N \setminus M_{\varphi(i)}) \cap f^{i-x}(N \setminus M_{\varphi(i-x)})f^x) = \\ & = \bigcup_{i=0}^{t-1} f^{i-x} (f^x(N \setminus M_{\varphi(i)}) \cap (N \setminus M_{\varphi(i-x)})f^x). \end{aligned}$$

If $h \in f^x(N \setminus M_{\varphi(i)}) \cap (N \setminus M_{\varphi(i-x)})f^x$, then there are some

$$\alpha_1 \in \psi^+(N \setminus M_{\varphi(i)}), \quad \alpha_2 \in \psi^+(N \setminus M_{\varphi(i-x)})$$

such that

$$h = \begin{pmatrix} 1 & 0 \\ 0 & \theta^x \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \alpha_1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \alpha_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \theta^x \end{pmatrix}.$$

Therefore, $\theta^x \alpha_1 = \alpha_2$.

Since bijection ψ^+ is an isomorphism between N and $\mathbb{F}^+$, then

$$\begin{aligned} & |f^x(N \setminus M_{\varphi(i)}) \cap (N \setminus M_{\varphi(i-x)})f^x| = \\ & = |\psi^+(N \setminus M_{\varphi(i)}) \cap \theta^{-x} \psi^+((N \setminus M_{\varphi(i-x)}))| = \\ & = |\psi^+(N) \setminus \psi^+(M_{\varphi(i)}) \cup \theta^{-x} \psi^+(M_{\varphi(i-x)})| = \\ & = (q^r - 2q^{r-1} + q^{r-2}). \end{aligned}$$

Thus,

$$\begin{aligned} |\Gamma(e) \cap \Gamma(g)| &= \sum_{i=0}^{t-1} (q^r - 2q^{r-1} + q^{r-2}) = \\ &= (q^r - 2q^{r-1} + q^{r-2})(q^r - 1)/(q - 1) = \\ &= q^{r-2}(q - 1)(q^r - 1). \end{aligned}$$

Hence, Γ is a divisible design graph. □

4 Examples

All examples in this section except Example 4.1 were found using computer search.

Example 4.1. Divisible design graph with parameters $(12, 6, 2, 3, 3, 4)$.

There is the unique example of divisible design Cayley graph with parameters $(12, 6, 2, 3, 3, 4)$ based on the alternating group Alt_4 . We can choose the set

$$S = \{(13)(24), (12)(34), (123), (132), (234), (243)\}$$

as its generating set. A fragment of the Cayley table of Alt_4 below shows us the necessary properties.

	(13)(24)	(12)(34)	(123)	(132)	(234)	(243)
(13)(24)	e	(14)(23)	(243)	(124)	(143)	(123)
(12)(34)	(14)(23)	e	(134)	(234)	(132)	(142)
(123)	(142)	(243)	(132)	e	(13)(24)	(143)
(132)	(234)	(143)	e	(123)	(142)	(12)(34)
(234)	(132)	(124)	(12)(34)	(134)	(243)	e
(243)	(134)	(123)	(124)	(13)(24)	e	(234)

Example 4.2. Divisible design graphs with parameters $(36, 24, 15, 16, 4, 9)$.

It was found in [8] that there are three non-isomorphic divisible design graphs with parameters $(36, 24, 15, 16, 4, 9)$. From our Construction 2.3 with permutations $(1, 3, 4, 2)$ and $(1, 3, 2, 4)$ we have two of that non-isomorphic divisible design graphs, which are based on subgroup of index 2 of $AG(9)$. It is important to note that, if t is even, then $t/2$ and t can be in any place in φ according to the condition $(*)$.

Example 4.3. Divisible design graphs with parameters $(56, 28, 12, 14, 7, 8)$.

It was found in [8] that there are five non-isomorphic examples of divisible design graphs with parameters $(56, 28, 12, 14, 7, 8)$ which are based on group $AG(8)$. This group can be described as follows

$$G = \langle f_1, f_2, f_3, f_4 \rangle$$

with defining group relationships

$$f_1^7 = f_2^2 = f_3^2 = f_4^2 = e,$$

$$f_2 * f_1 = f_1 * f_3, f_3 * f_1 = f_1 * f_4, f_4 * f_1 = f_1 * f_2 * f_4.$$

Below, we give the list of generating sets for these Cayley graphs.

$$S_1 = \{f_3, f_2 * f_3, f_3 * f_4, f_2 * f_3 * f_4, f_1 * f_4, f_1 * f_2 * f_4, f_1 * f_2 * f_3 * f_4, f_1 * f_3 * f_4, f_1^2 * f_2, f_1^2 * f_3, f_1^2 * f_4, f_1^2 * f_2 * f_3 * f_4, f_1^3 * f_2, f_1^3 * f_3, f_1^3 * f_2 * f_4, f_1^3 * f_3 * f_4, f_1^4 * f_2, f_1^4 * f_3, f_1^4 * f_2 * f_4, f_1^4 * f_3 * f_4, f_1^5 * f_2, f_1^5 * f_4, f_1^5 * f_2 * f_3, f_1^5 * f_3 * f_4, f_1^6 * f_3, f_1^6 * f_4, f_1^6 * f_2 * f_3, f_1^6 * f_2 * f_4\};$$

$$S_2 = \{f_2, f_4, f_2 * f_3, f_3 * f_4, f_1, f_1 * f_3, f_1 * f_4, f_1 * f_3 * f_4, f_1^2 * f_3, f_1^2 * f_4, f_1^2 * f_2 * f_3, f_1^2 * f_2 * f_4, f_1^3 * f_2 * f_4, f_1^3 * f_3 * f_4, f_1^3 * f_2 * f_3, f_1^4 * f_1^4 * f_2, f_1^4 * f_1^4 * f_4, f_1^4 * f_1^4 * f_2 * f_3\}$$

$$f_4, f_1^5 * f_2, f_1^5 * f_3, f_1^5 * f_2 * f_4, f_1^5 * f_3 * f_4, f_1^6, f_1^6 * f_2, f_1^6 * f_3, f_1^6 * f_2 * f_3\};$$

$$S_3 = \{f_1, f_3, f_4, f_1^2, f_1 * f_3, f_2 * f_4, f_3 * f_4, f_1^2 * f_2, f_1^2 * f_3, f_1^2 * f_4, f_1 * f_2 * f_4, f_1 * f_3 * f_4, f_1^3 * f_3, f_1^5, f_1^4 * f_2, f_1^4 * f_4, f_1^3 * f_2 * f_3, f_1^3 * f_2 * f_4, f_1^3 * f_3 * f_4, f_1^6, f_1^5 * f_2, f_1^4 * f_2 * f_3, f_1^4 * f_2 * f_4, f_1^6 * f_2, f_1^6 * f_4, f_1^5 * f_2 * f_3, f_1^5 * f_3 * f_4, f_1^6 * f_2 * f_3\};$$

$$S_4 = \{f_1, f_2, f_3, f_1 * f_3, f_1 * f_4, f_2 * f_4, f_3 * f_4, f_1^2 * f_2, f_1^2 * f_3, f_1^2 * f_4, f_1 * f_3 * f_4, f_1^3 * f_3, f_1^4 * f_3, f_1^4 * f_4, f_1^3 * f_2 * f_3, f_1^3 * f_3 * f_4, f_1^2 * f_2 * f_3 * f_4, f_1^6, f_1^5 * f_2, f_1^5 * f_4, f_1^4 * f_2 * f_3, f_1^4 * f_2 * f_4, f_1^3 * f_2 * f_3 * f_4, f_1^6 * f_2, f_1^6 * f_3, f_1^5 * f_2 * f_3, f_1^5 * f_3 * f_4, f_1^6 * f_2 * f_3\};$$

$$S_5 = \{f_1, f_2, f_3, f_4, f_1^2, f_1 * f_3, f_1 * f_4, f_1^2 * f_4, f_1 * f_3 * f_4, f_2 * f_3 * f_4, f_1^3 * f_3, f_1^3 * f_4, f_1^2 * f_2 * f_3, f_1^5, f_1^4 * f_2, f_1^4 * f_4, f_1^3 * f_2 * f_3, f_1^3 * f_2 * f_4, f_1^2 * f_2 * f_3 * f_4, f_1^6, f_1^5 * f_2, f_1^5 * f_4, f_1^4 * f_2 * f_3, f_1^4 * f_2 * f_4, f_1^3 * f_2 * f_3 * f_4, f_1^6 * f_2, f_1^6 * f_3, f_1^5 * f_2 * f_4, f_1^6 * f_2 * f_3\}.$$

Three of them are isomorphic to the graphs we have from our Construction 2.3 with permutations which pointed out after Lemma 2.2.

Example 4.4. Divisible design graph with parameters $(80, 60, 44, 45, 5, 16)$.

There is at least one divisible design Cayley graph which is based on subgroup of index 3 of $AG(4^2)$ that we have from our Construction 2.3 with permutations $(1, 4, 2, 5, 3)$. This is the first example where q is not prime.

5 Conclusion remarks

Any divisible design graph can be considered as a symmetric group-divisible design [2, 3, 4], the vertices of which are points, and the neighborhoods of the vertices are blocks. Such a design is called the *neighborhood design*. However, non-isomorphic graphs can correspond to isomorphic designs. Examples 4.2 and 4.3 of this article give us non-isomorphic divisible design graphs which produce isomorphic group divisible designs. If group-divisible design admit a symmetric incidence matrix with zero diagonal, then it corresponds to divisible design graph (see [9, Section 4.3]).

There is a great possibility to construct divisible designs from groups. Let G be a group of order mn containing a subgroup N of order n . A k -subset D of G is called a *divisible* $(m, n, k, \lambda_1, \lambda_2)$ difference set (*divisible by cosets of subgroup N*) if the list of elements xy^{-1} with $x, y \in D$ contains all non-identity elements in N exactly λ_1 times and all elements in $G \setminus N$ exactly λ_2 times. In case that $N = \{0\}$, the definition of a divisible difference set coincides with the definition of a *difference set* in the usual sense. In case that $\lambda_1 = 0$, the definition of a divisible difference set coincides with the definition of a *relative difference set* [2, 12, 13].

Divisible difference set D gives rise to a symmetric group-divisible design $\mathbb{D}$ with the set of blocks $\{Dg | g \in G\}$ and has the same parameters as D . This symmetric group-divisible design is called the *development of D* and admits G as a regular automorphism group (by right translation). Thus, symmetric group-divisible designs with a regular group G are equivalent to divisible difference sets in G . For having a symmetric incidence matrix with zero diagonal, the divisible difference set should be reversible (or equivalently, it must have a strong multiplier -1). There is more information on such difference sets in [1].

ORCID iDs

Vladislav V. Kabanov <https://orcid.org/0000-0001-7520-3302>

Leonid Shalaginov <https://orcid.org/0000-0001-6912-2493>

References

- [1] K. T. Arasu, D. Jungnickel and A. Pott, Divisible difference sets with multiplier -1 , *J. Algebra* **133** (1990), 35–62, doi:10.1016/0021-8693(90)90067-X.
- [2] T. Beth, D. Jungnickel and H. Lenz, *Design theory. Vol. I*, volume 69 of *Encyclopedia of Mathematics and its Applications*, Cambridge University Press, Cambridge, 2nd edition, 1999, doi:10.1017/CBO9780511549533.
- [3] R. C. Bose, Symmetric group divisible designs with the dual property, *J. Statist. Plann. Inference* **1** (1977), 87–101, doi:10.1016/0378-3758(77)90008-8.
- [4] R. C. Bose and W. S. Connor, Combinatorial properties of group divisible incomplete block designs, *Ann. Math. Statistics* **23** (1952), 367–383, doi:10.1214/aoms/1177729382.
- [5] A. E. Brouwer, Classification of small $(0, 2)$ -graphs, *J. Combin. Theory Ser. A* **113** (2006), 1636–1645, doi:10.1016/j.jcta.2006.03.023.
- [6] D. Crnković and W. H. Haemers, Walk-regular divisible design graphs, *Des. Codes Cryptogr.* **72** (2014), 165–175, doi:10.1007/s10623-013-9861-0.
- [7] M. Erickson, S. Fernando, W. H. Haemers, D. Hardy and J. Hemmeyer, Deza graphs: a generalization of strongly regular graphs, *J. Combin. Des.* **7** (1999), 395–405, doi:10.1002/(SICI)1520-6610(1999)7:6<395::AID-JCD1>3.3.CO;2-L.
- [8] S. V. Goryainov and L. V. Shalaginov, Cayley–deza graphs with less than 60 vertices (in russian), *Sib. Electron. Mat. Izv.* **11** (2014), 268–310, <http://semr.math.nsc.ru>.
- [9] W. H. Haemers, H. Kharaghani and M. A. Meulenberg, Divisible design graphs, *J. Combin. Theory Ser. A* **118** (2011), 978–992, doi:10.1016/j.jcta.2010.10.003.
- [10] M. Mulder, $(0, \lambda)$ -graphs and n -cubes, *Discrete Math.* **28** (1979), 179–188, doi:10.1016/0012-365X(79)90095-5.
- [11] S. Ode, P. Khansen and F. Messin, Classification of small regular polygons by area and perimeter, *Diskretn. Anal. Issled. Oper.* **15** (2008), 65–73, 97, doi:10.1134/S1990478909010037.
- [12] A. Pott, *Finite geometry and character theory*, volume 1601 of *Lecture Notes in Mathematics*, Springer-Verlag, Berlin, 1995, doi:10.1007/BFb0094449.
- [13] A. Pott, A survey on relative difference sets, in: *Groups, difference sets, and the Monster (Columbus, OH, 1993)*, de Gruyter, Berlin, volume 4 of *Ohio State Univ. Math. Res. Inst. Publ.*, pp. 195–232, 1996.
- [14] R.-H. Schulz and A. G. Spera, Divisible designs and groups, *Geom. Dedicata* **44** (1992), 147–157, doi:10.1007/BF00182946.
- [15] A. G. Spera, Semi-regular divisible designs and Frobenius groups, *Geom. Dedicata* **42** (1992), 285–294, doi:10.1007/BF02414067.