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On strictly Deza graphs derived from the Berlekamp-Van Lint-Seidel graph*

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Abstract

In this paper, we find strictly Deza graphs that can be obtained from the Berlekamp-Van Lint-Seidel graph by dual Seidel switching.

Keywords: Dual Seidel switching, Berlekamp-Van Lint-Seidel graph, divisible design graph, Deza graph.

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1 Introduction

Goryainov et al. [8] gave a characterisation of strictly Deza graphs with parameters $(n, k, k-1, a)$ and $\beta = 1$. They found that such strictly Deza graphs necessarily come from strongly regular graphs having the property $\lambda - \mu = -1$ and can be obtained via two operations: strong product with an edge and the dual Seidel switching [9]. We are still far away from getting a classification of strongly regular graphs with $\lambda - \mu = -1$ [1].

It is known that if $\lambda = 0$ and $\mu = 1$, then such a strongly regular graph is either the pentagon, or the Petersen graph, or the Hoffman-Singleton graph, or a hypothetical strongly regular graph with parameters $(3250, 57, 0, 1)$.

Berlekamp et al. studied strongly regular graphs with $\lambda = 1$ and $\mu = 2$ [3]. It was shown that such a strongly regular graph has parameters either $(9, 4, 1, 2)$ (the only such a graph is 3×3-lattice), or $(99, 14, 1, 2)$, or $(243, 22, 1, 2)$, or $(6273, 112, 1, 2)$, or $(494019, 994, 1, 2)$. Berlekamp et al. further constructed a graph with parameters $(243, 22, 1, 2)$, which is known as the Berlekamp-Van Lint-Seidel graph, but its uniqueness

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as well as the existence of graphs for the other three parameter tuples remain undecided. In particular, for the tuple $(99, 14, 1, 2)$, this problem is known as the Conway's 99-graph problem.

The smallest feasible parameter tuples of strongly regular graphs with $\lambda = 2$, $\mu = 3$ and $\lambda = 3$, $\mu = 4$ are $(364, 33, 2, 3)$, $(676, 45, 2, 3)$ and $(901, 60, 3, 4)$, respectively [4], and it is unknown if strongly regular graphs with such parameters exist.

In [8], some examples of strictly Deza graphs with parameters $(n, k, k-1, a)$ and $\beta = 1$ were given. In particular, dual Seidel switching was applied to the Petersen graph, the Hoffman-Singleton graph, Paley graphs of square order. In this paper, we investigate if dual Seidel switching can be applied to the Berlekamp-Van Lint-Seidel graph or its complement.

2 Preliminaries

We consider undirected graphs without loops or multiple edges.

A k-regular graph Γ on v vertices is called *strongly regular* with parameters (v, k, λ, μ), $0 < k < v - 1$, if any two distinct vertices x, y in Γ have λ common neighbours when x, y are adjacent and μ common neighbours if x, y are non-adjacent. For a vertex x in a graph Γ, the *neighbourhood* $\Gamma(x)$ is the set of all neighbours of x in Γ.

Lemma 2.1 ([5], Theorem 1.3.1(i))**.** *Let Γ be a strongly regular graph with parameters (v, k, λ, μ), $\mu \neq 0$, $\mu \neq k$. Then Γ has three distinct eigenvalues k, r, s, where $k > r > 0 > s$ and the eigenvalues r, s satisfy the quadratic equation $x^2 + (\mu - \lambda)x + (\mu - k) = 0$.*

For a graph Γ, denote by $\overline{\Gamma}$ the complement of Γ.

Lemma 2.2 ([5], Theorem 1.3.1(vi))**.** *Let Γ be a strongly regular graph with parameters (v, k, λ, μ). Then the complement $\overline{\Gamma}$ of Γ is a strongly regular graph with parameters $(v, v - k - 1, v - 2k + \mu - 2, v - 2k + \lambda)$ and eigenvalues $v - k - 1, -s - 1, -r - 1$.*

A k-regular graph Δ on v vertices is called a *Deza graph* with parameters (v, k, b, a), $b \geq a$, if the number of common neighbours of any two distinct vertices in Δ takes on the two values a or b. A Deza graph Δ is called a *strictly Deza graph*, if it has diameter 2 and is not strongly regular. The following lemma gives a construction of strictly Deza graphs, which is known as *dual Seidel switching*.

Lemma 2.3 ([7], Theorem 3.1)**.** *Let Γ be a strongly regular graph with parameters (v, k, λ, μ), $k \neq \mu$, $\lambda \neq \mu$ and adjacency matrix M. Let P be a permutation matrix that represents an involution ϕ of Γ that interchanges only non-adjacent vertices. Then PM is the adjacency matrix of a strictly Deza graph Δ with parameters (v, k, b, a), where $b = max(\lambda, \mu)$ and $a = min(\lambda, \mu)$.*

Since ϕ in Lemma 2.3 represents an involution, the matrix PM is obtained from the matrix M by a permutation of rows in all pairs of rows with indexes i_1 and i_2, such that $\phi(i_1) = i_2$ and $\phi(i_2) = i_1$. Lemma 2.4 follows immediately from Lemma 2.3 and shows what is the neighbourhood of a vertex of the graph Δ.

Lemma 2.4. *For the neighbourhood $\Delta(u)$ of a vertex u of the graph Δ from Lemma 2.3, the following conditions hold:*

$$\Delta(u) = \begin{cases} \Gamma(u), & \text{if } \phi(u) = u; \\ \Gamma(\phi(u)), & \text{if } \phi(u) \neq u. \end{cases}$$

In [8, Theorem 2], it was shown that the strong product with an edge and dual Seidel switching is the only method to obtain strictly Deza graphs with $k = b + 1$. Recall that the graph *strong product* of two graphs Γ_1 and Γ_2 has vertex set $V(G_1) \times V(G_2)$ and two distinct vertices (v_1, v_2) and (u_1, u_2) are connected iff they are adjacent or equal in each coordinate, i.e., for $i \in 1, 2$, either $v_i = u_i$ or $\{v_i, u_i\}$ in $E(\Gamma_i)$, where $E(\Gamma_i)$ is the edge set of Γ_i [2].

It follows from Lemma 2.2 that, if a strongly regular graph Γ has the property $\lambda - \mu = -1$, then the complementary graph $\overline{\Gamma}$ has the property $\overline{\lambda} - \overline{\mu} = -1$ as well. Thus, according to [8, Theorem 2], we concentrate on order 2 automorphisms of Γ that interchange either only non-adjacent vertices or only adjacent vertices.

Let G be a group and S be an inverse-closed identity-free subset in G. The graph on G with two vertices x, y being adjacent whenever xy^{-1} belongs to S is called the *Cayley graph* of the group G with *connection set* S and is denoted by $Cay(G, S)$.

3 The Berlekamp-Van Lint-Seidel graph and dual Seidel switching

The *Berlekamp-Van Lint-Seidel graph*, denoted by Γ, is the coset graph of the ternary Golay code [5, Section 11.3]. This graph is known to be strongly regular with parameters $(243, 22, 1, 2)$.

In this section, we deal with two more ways to define this graph and give a description of the involutions of Γ and $\overline{\Gamma}$ suitable for dual Seidel switching.

The main result of this paper is the following theorem.

Theorem 3.1. *The following statements hold.*

(1) Γ *has no order 2 automorphisms that interchange only adjacent vertices;*

(2) Γ *has the unique (up to conjugation) order 2 automorphism*

that interchanges only non-adjacent vertices.

To prove Theorem 3.1, we prove two lemmas, which imply the truth of the theorem statements.

3.1 Γ from the Mathieu group M_{11}

By ATLAS of Group Representations the Mathieu group M_{11} can be represented [10] by 5×5 matrices over $GF(3)$ as follows. Put

$$x := \begin{pmatrix} 0 & 2 & 1 & 0 & 0 \\ 2 & 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 2 & 2 \\ 1 & 0 & 2 & 2 & 1 \\ 1 & 2 & 2 & 2 & 0 \end{pmatrix}, \; y := \begin{pmatrix} 0 & 0 & 2 & 0 & 2 \\ 1 & 1 & 2 & 2 & 0 \\ 2 & 2 & 2 & 2 & 2 \\ 1 & 2 & 1 & 1 & 0 \\ 2 & 2 & 0 & 2 & 1 \end{pmatrix}.$$

Then the group $G := \langle x, y \rangle$ is isomorphic to M_{11}, where x is an involution. Let $V(5, 3)$ denote the 5-dimensional vector space of over $GF(3)$. We regard the elements of $V(5, 3)$ as rows and consider the action of G on $V(5, 3)$ by the right multiplication, which has two orbits of size 22 and 220 on the nonzero vectors. The orbit of size 22 is given by the set

$$S_1 := \{\pm(1, 0, 0, 0, 0), \pm(0, 0, 1, 0, 1), \pm(0, 1, 0, 1, 0), \pm(0, 1, 2, 0, 0),$$

$$\pm(0,0,1,2,1), \pm(0,1,0,1,2), \pm(1,1,2,0,2), \pm(1,0,0,1,2),$$

$$\pm(1,0,2,1,0), \pm(1,1,0,0,2), \pm(1,1,2,1,0)\},$$

and, moreover, Γ is isomorphic to the Cayley graph $Cay(V(5,3), S_1)$. Since G stabilises S_1 setwise, G is a subgroup in the automorphism group of Γ, which is known (see [6]) to be isomorphic to the group $3^5 : (2 \times M_{11})$. The fact that M_{11} has precisely one class of conjugate involutions implies that the automorphism group of Γ has precisely three classes of conjugate involutions. Let e be the identity matrix from G. Note that $-e$ does not belong to G, but the multiplication by $-e$ is an involution of the automorphism group of $Cay(V(5,3), S_1)$, which means that the three pairwise non-conjugate involutions of the automorphism group of $Cay(V(5,3), S_1)$ are given by the right multiplication by $-e$, x and $-x$.

Lemma 3.2. *The following statements hold.*

(1) The involution $-e$ interchanges adjacent vertices as well as non-adjacent ones;

(2) The involution $-x$ interchanges adjacent vertices as well as non-adjacent ones.

Proof. (1) This involution fixes the zero vector and moves all non-zero vectors by swapping every two elements that are additive inverses of each other. In the graph $Cay(V(5,3), S_1)$, two additive inverses are adjacent whenever both of them belong to S_1. It means that the involution $-e$ interchanges adjacent vertices as well as non-adjacent ones.

(2) On the one hand, the involution $-x$ swaps the vertices $(0,1,0,1,0)$ and $(0,2,0,2,0)$, which are adjacent in $Cay(V(5,3), S_1)$. On the other hand, $-x$ swaps the vertices $(1,0,0,0,0)$ and $(0,2,1,0,0)$, which are not adjacent in $Cay(V(5,3), S_1)$. □

In view of Lemma 3.2, it remains to check the inner involution x. In the next subsection, we explore one more definition of the Berlekamp-Van Lint-Seidel graph and give a very natural description of the involution x.

3.2 Specific parity-check matrix

Recall that, for a positive integer n and a prime power q, $V(n,q)$ denotes the n-dimensional vector space over the finite field $\mathbb{F}_q$. The ternary Golay code can be constructed as the 6-dimensional subspace in $V(11,3)$ consisting of all row-vectors $\mathbf{c}$ such that the equality $H\mathbf{c}^T = \mathbf{0}$ holds, where

$$H := \begin{bmatrix} 1 & 1 & 1 & 2 & 2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 1 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 2 & 0 & 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 2 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

is the specific parity check matrix of this code. Let $x_1, x_2, x_3, \ldots, x_{11}$ denote the vectors from $V(5,3)$ that correspond to the columns of H. There are 22 vectors of type $\pm x_i$ and 220 vectors of type $\pm x_i \pm x_j$ where $i \neq j; i,j = 1,2,\ldots,11$. The Cayley graph $Cay(V(5,3), S_2)$, where $S_2 := \{\pm x_1, \ldots, \pm x_{11}\}$, is known to be isomorphic with the Berlekamp-Van Lint-Seidel graph (see [3]).

Lemma 3.3. *The reversal of vectors is an involution of $Cay(V(5,3), S_2)$ that interchanges only non-adjacent vertices.*

Proof. Obviously, the reversal of vectors is a permutation of the vertex set of Γ. For a vector $\gamma \in V(5,3)$, denote by γ^r the reversed vector. Note that $(S_2)^r = S_2$ holds. Since, for any two vertices γ_1, γ_2 in Γ, we have $\gamma_1^r - \gamma_2^r = (\gamma_1 - \gamma_2)^r$, the reversal is an automorphism of $Cay(V(5,3), S_2)$.

For a vector $(a,b,c,d,e) \in V(5,3)$, consider the difference

$$(a,b,c,d,e) - (a,b,c,d,e)^r = (a-e, b-d, 0, d-b, e-a).$$

Note that the first and the fifth coordinates and the second and fourth ones are additive inverses. Since S_2 has no such vectors with zero third coordinate, the reversal interchanges only non-adjacent vertices. □

4 Concluding remarks

The following three strictly Deza graphs can be derived from the Berlekamp-Van Lint-Seidel graph Γ.

First, Lemma 2.3 and Theorem 1(2) give a strictly Deza graph with parameters $(243, 22, 2, 1)$. It has spectrum $\{22^1, 5^{48}, 4^{72}, (-4)^{60}, (-5)^{62}\}$ and its automorphism group of order 2592 is a subgroup in the automorhism group of Γ.

Further, in view of [8, Construction 1], the strong product $\Gamma[K_2]$ of the Berlekamp-Van Lint-Seidel graph with an edge is a strictly Deza graph with parameters $(486, 45, 44, 4)$. It has spectrum $\{45^1, 9^{132}, (-1)^{243}, (-9)^{110}\}$.

Finally, the order 2 automorphism from Theorem 1(2) induces an order 2 automorphism of $\Gamma[K_2]$ that interchanges only non-adjacent vertices. Applying the dual Seidel switching to $\Gamma[K_2]$, we obtain one more strictly Deza graph with parameters $(486, 45, 44, 4)$, which has spectrum $\{45^1, 9^{120}, 1^{108}, (-1)^{135}, (-9)^{122}\}$.

In the connection with the results from [8], we point out that both graphs with parameters $(486, 45, 44, 4)$ are divisible design graphs.

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