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Rotary one-facet maniplexes*

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Abstract

Maniplexes are combinatorial objects that generalize, simultaneously, maps on surfaces and abstract polytopes. We are interested on studying *rotary* maniplexes, that is, those having maximal 'rotational' symmetry.

This note classifies rotary 4-dimensional maniplexes with the property of having exactly one facet, and gives examples and related results.

Keywords: Graph, automorphism group, symmetry, polytope, maniplex, map, flag, transitivity, rotary, reflexible, chiral.

Math. Subj. Class.: 05E18, 05C25.

1 Introduction

Maps on surfaces are structures which are said to be of rank 3 since they have three kinds of objects: vertices, edges and faces. We have been particularly interested in those exhibiting a good deal of symmetry. Such maps have been studied from topological, algebraic, geometric and combinatorial points of view (see for example [9, 10, 15, 17]).

Abstract polytopes are partially ordered sets satisfying some of the main properties of the face-lattices of convex polytopes. Every abstract polytope of rank 3 can be regarded as a map. Conversely, many maps on surfaces can be regarded as abstract polytopes of rank 3, but many others violate some of the axioms for polytopality. In this sense, the concept of

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map on a surface is more general than that of an abstract 3-polytope. Again, we are most interested in those possessing substantial symmetry.

Maniplexes are combinatorial objects that generalize both, maps on surfaces and abstract polytopes. Recently, many results on maps and polytopes have been strenghtened being stated now for maniplexes.

In recent years there has been attention on finding the smallest polytopes with given properties (see for example [1, 2, 5]); partially motivated by the gigantic size of the chiral polytopes of ranks 6 and higher known so far ([3, 6, 13]).

As we begin to study any combinatorial structure, we always want to know what the smallest example is of such an object. In this work we study n-maniplexes having rotary symmetry and only one facet, for n up to 4. Intuitively, these will be 'small' maniplexes, and the constructions shown below may yield the smallest 4-maniplexes that satisfy certain sets of properties.

2 Maniplexes

2.1 Definitions

We consider the notion, introduced in [16], of a *maniplex*. An *(n+1)-dimensional maniplex* $\mathcal{M}$ is a pair $(\Omega, [r_0, r_1, \ldots, r_n])$, where Ω is a set of things called *flags* and each r_i is a partition of Ω into unordered pairs. We often use algebraic language, and refer to each r_i as an involutory permutation on Ω, where "$r_i x = y$" means that the pair $\{x, y\}$ appears in r_i. We require these to be such that (1) the *connection group* $C = \langle r_0, r_1, \ldots, r_n \rangle$ acts transitively on Ω, and (2) for all $0 \leq i < j - 1 \leq n - 1$, we have that $(r_i r_j)^2$ is trivial. One can easily verify that every map on a surface is a 3-maniplex with Ω being its set of (triangular) flags. Furthermore, every 3-maniplex can be realised as a map on a surface. When we desire to avoid degeneracies, such as semi-edges or maps on a surface with boundary, we also require that (3) each r_i and $r_i r_j$ are fixed-point-free, whenever $i \neq j$.

The *type* of a maniplex is the sequence $\{p_1, p_2, \ldots, p_n\}$, where each p_i is the order of $r_{i-1} r_i$ in C. The cube, then, is of type $\{4, 3\}$, the simplex is of type $\{3, 3, \ldots, 3\}$, and the 600-cell is of type $\{3, 3, 5\}$ (see [4, Chapter VII]).

Let C_i be the subgroup of C generated by all of the r_j's except r_i. Then an orbit of flags under C_i is called an *i-face*. A 0-face is a *vertex,* a 1-face is an *edge*, a 2-face is a *face*, an n-face is a *facet*. A facet of a facet is a *subfacet*; this is an orbit under $\langle r_0, r_1, \ldots, r_{n-2} \rangle$. The restriction to a subfacet of the permutation r_n acts as an isomorphism from that subfacet to some subfacet. We often view the $(n + 1)$-maniplex as being assembled from a collection of n-maniplexes, and think of r_n as 'glueing' them together along their subfacets.

We wish to assign colors, red and white, to flags so that for any given two i-adjacent flags, either one is colored red (and not white) and one is colored white (and not red), or both flags are colored both red and white. Choose a *root* flag (sometimes called also *base flag*) and call it I. Let $\mathcal{R}_0 = \{I\}$. Recursively let $\mathcal{W}_i$ be the set of all flags adjacent to flags in $\mathcal{R}_i$, and let $\mathcal{R}_{i+1}$ be the set of all flags adjacent to flags in $\mathcal{W}_i$. Finally, let $\mathcal{R}$ be the union of all $\mathcal{R}_i$'s and similarly let $\mathcal{W}$ be the union of all $\mathcal{W}_i$'s. We often say this differently: let C^+ be the subgroup of C generated by all products of the form $r_i r_j$. Then $\mathcal{R}$ is the orbit of I under C^+ and $\mathcal{W}$ is the orbit of $r_0 I$ under C^+. Consider these as assignments of the colors red and white, respectively to the flags. There are two possibilities for the result:

1. it could happen that $\mathcal{R}$ and $\mathcal{W}$ are disjoint; in this case we say that $\mathcal{M}$ is *orientable*;

2. otherwise it must happen that $\mathcal{R} = \mathcal{W} = \Omega$, and in this case we say that $\mathcal{M}$ is *non-orientable*.

See [11] for more information about bi-colorings of flags.

2.2 Symmetry

We define a *symmetry* of a maniplex $\mathcal{M}$ as a permutation of the flags which preserves the connections. We write symmetries on the right, so that the image of the flag f under the symmetry α is $f\alpha$. We denote the group of symmetries of $\mathcal{M}$ by $\mathrm{Aut}(\mathcal{M})$, and the notation gives the nice statement that for all $i \in \{0, 1, 2, \dots, n\}$ and all $\alpha \in \mathrm{Aut}(\mathcal{M})$, we have that

$$(r_i f)\alpha = r_i(f\alpha).$$

There are two levels of symmetry that are particularly interesting in maps and maniplexes. First, we say that $\mathcal{M}$ is *rotary* provided that $\mathrm{Aut}(\mathcal{M})$ acts transitively on $\mathcal{R}$, the set of red flags. Also, $\mathcal{M}$ is *reflexible* provided that $\mathrm{Aut}(\mathcal{M})$ acts transitively on Ω. It follows trivially, then, that if $\mathcal{M}$ is rotary and non-orientable, then it is reflexible. If $\mathcal{M}$ is rotary but not reflexible, we say it is *chiral*. If $\mathcal{M}$ is orientable, it is often useful to consider $\mathrm{Aut}^+(\mathcal{M})$; this is the group of all symmetries which send $\mathcal{R}$ (the set of red flags) to itself (and so send $\mathcal{W}$ to itself). For orientable $\mathcal{M}$, we refer to symmetries in $\mathrm{Aut}^+(\mathcal{M})$ as 'preserving orientation' and those not in $\mathrm{Aut}^+(\mathcal{M})$ as 'reversing orientation'.

The group $\mathrm{Aut}^+(\mathcal{M})$ acts semi-regularly on Ω. That is, for any two flags, there is at most one symmetry sending one to the other. Thus, for any flag f and any symmetries β, γ, if $f\beta = f\gamma$, then $\beta = \gamma$.

2.3 One Facet

The purpose of this note is to discuss when a rotary n-maniplex might have just one facet for $n \leq 4$.

In the case $n = 1$, there is only one isomorphism class of maniplex and it has two facets.

For $n = 2$, a maniplex is a polygon, a facet is an edge and a polygon with one edge is the degenerate maniplex that consists of a single vertex joined to itself by a loop.

For $n = 3$, a maniplex is a map. In [18] it was noted that a rotary one-face map with k edges must be one of the two reflexible maps called M_k, δ_k in that paper.

The rest of this paper is devoted to classifying rotary 4-maniplexes with one facet. Whereas some reflexible one-facet n-maniplexes can be constructed with techniques similar to the ones presented here, a formal treatment of all rotary n-maniplexes for $n > 4$ appears to present difficulties not encountered in lower dimensions. It is not known yet whether there are any chiral n-maniplexes for $n > 4$ which have just one facet.

3 Constructions

In all of this section, suppose that $\mathcal{F} = (\Omega, [r_0, r_1, r_2])$ is a rotary 3-maniplex having root flag $I \in \mathcal{R}$. Let ρ be the symmetry of $\mathcal{F}$ sending I to $r_0 r_1 I$. Let G stand for $\mathrm{Aut}(\mathcal{F})$, G^+ stand for $\mathrm{Aut}^+(\mathcal{F})$, and let e be the identity in those groups. If $\mathcal{F}$ is reflexible, let α be the symmetry of $\mathcal{F}$ such that $I\alpha = r_0 I$. A symmetry $\tau \in G$ is *blue* provided that both τ and $\tau\rho$ are of order two.

For a blue τ, we define two kinds of elligibility for the pair $(\mathcal{F}, \tau)$ based on the flag $f = r_0 I \tau$:

1. We will say that the pair $(\mathcal{F}, \tau)$ is *white-elligible* provided that f is not red. This can happen only if (a) $\mathcal{F}$ is chiral or (b) $\mathcal{F}$ is orientable and reflexible, and τ preserves orientation.
2. We will say that the pair $(\mathcal{F}, \tau)$ is *red-elligible* provided first that f *is* red. This can happen only if (c) $\mathcal{F}$ is non-orientable (and hence reflexible) or (d) $\mathcal{F}$ is orientable and reflexible, and τ reverses orientation. In both case (c) and case (d), we further require that $\tau\alpha$ is of order 2.

Note that the blue condition shows that

$$r_0 r_1 f = r_0 r_1 r_0 I \tau = r_0 I \rho^{-1} \tau = r_0 I \tau \rho = f \rho.$$

Theorem 3.1 (One-Facet Maniplexes). *If the pair $(\mathcal{F}, \tau)$ is white-elligible, let $r_3 = \{I, f\}G^+$. Then $\mathcal{M} = (\Omega, [r_0, r_1, r_2, r_3])$ is an orientable, rotary maniplex, invariant under G^+. It is reflexible if $\mathcal{F}$ is reflexible and $\tau\alpha$ is of order 2.*

If the pair $(\mathcal{F}, \tau)$ is red-elligible, let $r_3 = \{I, f\}G$. Then $\mathcal{M} = (\Omega, [r_0, r_1, r_2, r_3])$ is a non-orientable, reflexible maniplex, invariant under G.

Proof. First suppose that $(\mathcal{F}, \tau)$ is white-elligible. Note that, for any pair $\{g, h\}$ in r_3, we may assume with no loss of generality that for some symmetry β in G^+, $g = I\beta$ is the red flag and $h = f\beta$ is the white.

As G^+ is transitive on each color, the union of pairs in r_3 is all of Ω. And as for each red flag, there is only one element of G^+ sending I there, r_3 is a partition of Ω. Now, r_3 contains the pair $\{I, f\}\tau = \{I\tau, f\tau\} = \{r_0 f, r_0 I\}$. Thus, for every pair $\{g, h\}$ in r_3, the pair $\{r_0 g, r_0 h\}$ is also in r_3, showing that $r_0 r_3 = r_3 r_0$. Further,

$$r_1 g = r_1 I \beta = r_0 r_0 r_1 I \beta = r_0 I \rho \beta = f \tau \rho \beta$$

and

$$r_1 h = r_1 f \beta = r_0 r_0 r_1 f \beta = r_0 f \rho \beta = r_0 r_0 I \tau \rho \beta = I \tau \rho \beta,$$

and so the pair $\{r_1 g, r_1 h\} = \{I, f\}\tau\rho\beta$ is in r_3. Thus r_1 and r_3 commute, and so $\mathcal{M}$ is a maniplex.

Every symmetry in G^+ is a symmetry of $\mathcal{M}$, so $\mathrm{Aut}(\mathcal{M})$ is transitive on the red flags of $\mathcal{F}$. Since $\mathcal{F}$ is orientable, and every pair in r_3 is of two colors, $\mathcal{M}$ is orientable. The red flags of $\mathcal{M}$ are exactly the red flags of $\mathcal{F}$, and so $\mathcal{M}$ is rotary.

If $\mathcal{F}$ is reflexible, assume that $\tau\alpha$ is of order 2, so that τ and α commute. Then

$$f\alpha = r_0 I \tau \alpha = r_0 I \alpha \tau = I \tau = r_0 f.$$

Consider the typical element $\{g, h\}$ of r_3, with $g = I\beta$ and $h = f\beta$. Since G^+ is normal in G, $\alpha\beta\alpha = \beta'$, for some β' in G^+. Then

$$g\alpha = I\beta\alpha = I\alpha\beta' = r_0 I \beta',$$

and, similarly,

$$h\alpha = f\beta\alpha = f\alpha\beta' = r_0 f \beta'.$$

Now, $\{I\beta', f\beta'\}$ is in r_3 and so $\{g\alpha, h\alpha\}$ is, as well. Thus α acts on $\mathcal{M}$ as a symmetry and so $\mathcal{M}$ is reflexible.

On the other hand, suppose that $(\mathcal{F}, \tau)$ is red-elligible. Then $r_3 = \{I, f\}G$, and the symmetries τ and α commute. Suppose that $\{g, h\}$ and $\{g, h'\}$ are distinct pairs in r_3. Then we can assume that for some symmetries β and γ, we have

$$g = I\beta = f\gamma, h = f\beta, h' = I\gamma.$$

Then

$$I\beta = f\gamma = r_0 I\tau\gamma = I\alpha\tau\gamma,$$

and so $\beta = \alpha\tau\gamma$. Thus, $\gamma = \tau\alpha\beta$. Then

$$h' = I\gamma = I\tau\alpha\beta = r_0 f\alpha\beta = f\beta = h.$$

Thus, r_3 is a partition of Ω.

As in the first case, r_0 and r_1 commute with r_3, and so $\mathcal{M}$ is a maniplex. Because r_3 connects flags which have, in $\mathcal{F}$, the same color, all flags of $\mathcal{M}$ have both colors, and so $\mathcal{M}$ is non-orientable. As each r_i is a partition invariant under G, which acts transitively on Ω, all of G acts on $\mathcal{M}$, and so $\mathcal{M}$ is reflexible. □

Theorem 3.2. *If $\mathcal{M}$ is any rotary 4-maniplex, let τ in $\mathrm{Aut}^+(\mathcal{M})$ be the symmetry such that $I\tau = r_3 r_0 I$. If $\mathcal{M}$ has exactly one facet $\mathcal{F}$, then the pair $(\mathcal{F}, \tau)$ is white- or red-elligible, and $\mathcal{M}$ is isomorphic to one of the maniplexes constructed from $\mathcal{F}$ and τ in Theorem 3.1.*

Proof. Suppose that $\mathcal{M} = (\Omega, [r_0, r_1, r_2, r_3])$ is a rotary 4-maniplex containing exactly one facet, $\mathcal{F} = (\Omega, [r_0, r_1, r_2])$, and that $I \in \mathcal{R}$ is the root flag of each. Let f be $r_3 I = r_0 I\tau$.

Then $I\tau = r_3 r_0 I$ and so I is fixed by τ^2, making τ of order 2.Further, $I\rho\tau = r_0 r_1 I\tau = r_0 r_1 r_0 r_3 I$. As r_3 commutes with both r_0 and r_1, the flag I is fixed by $(\rho\tau)^2$. Thus, $\tau\rho$ is also an involution and so τ is blue.

In the case in which $\mathcal{M}$ is orientable, $\mathrm{Aut}^+(\mathcal{M})$ acts regularly on the red flags, and on the white flags, and each of the r_i's is an orbit of the red-white pair $\{I, r_i I\}$. Thus, such an $\mathcal{M}$ is constructed in the first half of Theorem 3.1.

In the case in which $\mathcal{M}$ is non-orientable, and thus reflexible, there is $\alpha \in \mathrm{Aut}(\mathcal{M})$ such that $I\alpha = r_0 I$. Then

$$I\tau\alpha = r_3 r_0 I\alpha = r_3 I = f = r_0 I\tau = I\alpha\tau.$$

Thus, $\tau\alpha = \alpha\tau$ and so $(\mathcal{F}, \tau)$ is red-elligible. Further, $\mathrm{Aut}(\mathcal{M})$ acts regularly on the flags, and each r_i is an orbit of $\{I, r_i I\}$ under $\mathrm{Aut}(\mathcal{M})$. Thus, such an $\mathcal{M}$ is constructed in the second half of Theorem 3.1. □

4 Examples and results

We present here some examples of and results about one-facet maniplexes to display some of the variety possible:

4.1 Opposites

If $\mathcal{M} = (\Omega, [r_0, r_1, r_2, \ldots, r_n])$ is an $n{+}1$-maniplex, let $s_2 = r_0r_2$. Then $(\Omega, [r_0, r_1, s_2, \ldots, r_n])$ is also a maniplex, called *opp*$(\mathcal{M})$. If $\mathcal{M}$ is reflexible, so is *opp*$(\mathcal{M})$, while if $\mathcal{M}$ is chiral, then *opp*$(\mathcal{M})$ is not rotary. See [16] for more on this and other operators on maniplexes.

From this it should be clear that:

Lemma 4.1. *If $\mathcal{M}$ is a reflexible one-facet maniplex whose facet is $\mathcal{F}$, then opp$(\mathcal{M})$ is a reflexible one-facet maniplex whose facet is opp$(\mathcal{F})$.*

Suppose that $\mathcal{F}$ is a map of type $\{p, q\}$ which has Petrie paths of length r, and suppose that $\mathcal{M}$ is a reflexible one-facet 4-maniplex with facet $\mathcal{F}$ and of type $\{p, q, t\}$. Then *opp*$(\mathcal{M})$ is a reflexible one-facet 4-maniplex with facet *opp*$(\mathcal{F})$ which has type $\{p, r, \frac{2t}{(2,t)}\}$.

4.2 Twists

If $\mathcal{M} = (\Omega, [r_0, r_1, r_2, r_3])$ is an orientable 4-maniplex, let $T_j(\mathcal{M})$ be $(\Omega, [s_0, s_1, s_2, s_3])$, where $s_0 = r_0, s_1 = r_1, s_2 = r_2$ and

$$s_3 f = \begin{cases} (r_0r_1)^j r_3 f & \text{if } f \in \mathcal{R} \\ (r_1r_0)^j r_3 f & \text{if } f \in \mathcal{W} \end{cases}$$

for all $f \in \Omega$. This construction first appeared in [7] and is described in more detail in [8]. In each of those papers, it is shown that each orientation-preserving symmetry of $\mathcal{M}$ is a symmetry of $\mathcal{T}_j(\mathcal{M})$, and so if $\mathcal{M}$ is rotary then $\mathcal{T}_j(\mathcal{M})$ is rotary, though if $\mathcal{M}$ is reflexible then $\mathcal{T}_j(\mathcal{M})$ might be chiral or reflexible.

An easy consequence of these results is:

Lemma 4.2. *If $\mathcal{M}$ is an orientable rotary one-facet maniplex with facet $\mathcal{F}$, then each $T_j(\mathcal{M})$ is an orientable rotary one-facet maniplex with facet $\mathcal{F}$.*

4.3 Examples: the Cube

Under this heading, we present four examples of rotary orientable one-facet maniplexes for each of which its one facet is isomorphic to the cube.

1. **Central Inversion**

   In this example, the flag f is diametrically opposite I. More precisely, if I belongs to face F, vertex v and edge e then f is the flag belonging to face F' farthest from F, the vertex v' farthest from v and the edge e' farthest from e. It follows that every flag is 3-adjacent to its antipodal flag. This gives a reflexible maniplex of type $\{4, 3, 2\}$. It has 3 faces, 4 vertices and 6 edges.

2. **Toroidal Identification**

   Imagine the cube as one cell of the tessellation of 3-space into cubes. Identify each flag with the flag translated by $(\pm 1, 0, 0), (0, \pm 1, 0)$ or $(0, 0, \pm 1)$. The resulting maniplex is the quotient space of the tessellation under the group generated by those vectors. The maniplex is reflexible and is of type $\{4, 3, 4\}$. It has 3 faces, one vertex and 3 edges.

The central inversion and toroidal identifications generalize to construct one-facet maniplex of all dimensions, of types $\{4, 3, 3, 3, \ldots, 3, 3, 2\}$ and $\{4, 3, 3, 3, \ldots, 3, 3, 4\}$, respectively.

3. **The Krughoff cube (and its reverse)**

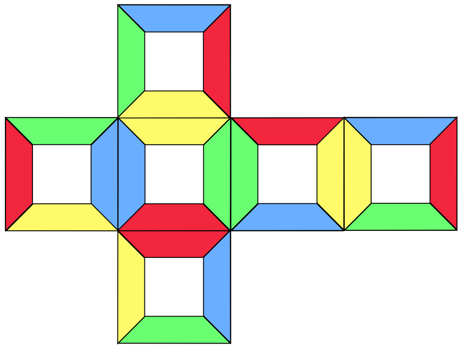

Figure 1: Krughoff's cube

Consider the cube shown in Figure 1 [12]. The edges have been colored in such a way that each of the six possible circular orderings of the four colors appears exactly once clockwise about some face. Notice that this coloring is chiral; i.e., every rotation of the cube permutes the colors, while any reflection sends edges of any one fixed color to edges of different colors.

Also note that opposite faces have the reverse circular order of colors. If we now form a 4-maniplex by glueing each face to the opposite face so that colors match, we get a chiral 4-maniplex. Each color, then, becomes a single edge. As each color appears three times, the maniplex is of type $\{4, 3, 3\}$. It has 3 faces, 2 vertices and 4 edges The mirror image of the coloring produces another maniplex, the mirror image of the first.

4. **Twists of each other** If we let $\mathcal{C}$ be the 4-maniplex made from the cube by using the central inversion, then $T_1(\mathcal{C})$ is one of the Krughoff maniplexes, $T_2(\mathcal{C})$ is the one using the toroidal identification, and $T_3(\mathcal{C})$ is the other Krughoff maniplex. In each case the corresponding involution τ is the rotation by $180°$ about the face containing $r_2 I$, followed by ρ^i for some i.

4.4 Octahedron examples

The octahedron has an edge-coloring dual to that of the cube shown above. Using that coloring, we build two mirror-image one-facet chiral maniplexes, each of them a twisted form of the maniplex made by using the central inversion.

4.5 The $f = IR^{\frac{p}{2}}$ construction for all even p

Suppose that $\mathcal{F}$ is a reflexible map whose faces have even length $2k$. Then making r_3 equal to the set of all pairs of the form $\{g, (r_0 r_1)^k g\}$ forms a 4-maniplex having just the one facet, isomorphic to $\mathcal{F}$, and is reflexible and non-orientable. Each face is 3-connected to itself after a $180°$ turn. In the case of the cube, the result is of type $\{4, 3, 4\}$, and has 6 faces, 3 edges and 2 vertices.

4.6 Non-orientable facet

When Lemma 4.1 is applied to a maniplex having an orientable facet, the result might have a non-orientble facet. Let the map $\mathcal{C}$ be the cube, and $\mathcal{M}$ be the 4-maniplex made from it by using the toroidal identification. Then $\mathcal{F} = opp(\mathcal{C})$ is a non-orientable map of type $\{4, 6\}$, and $opp(\mathcal{M})$ is of type $\{4, 6, 4\}$.

4.7 Chiral facet

Assume that a chiral 3-maniplex $(\Omega, [r_0, r_1, r_2])$ is the facet of a one-facet 4-maniplex $\mathcal{M} = (\Omega, [r_0, r_1, r_2, r_3])$. By the blue condition there exists a symmetry τ such that $\tau\rho\tau = \rho^{-1}$. This is not a common phenomenon in 3-maniplexes with few flags; however, it certainly occurs if $\mathrm{Aut}(\mathcal{M})$ is a symmetric group.

It was proven in [14, Section 3] that for $n \geq 6$, the symmetric group S_n is the symmetry group of some chiral 3-maniplex. In particular, when $n = 6$ the permutations

$$(1, 2, 3, 4, 5, 6),$$
$$(2, 6, 3)(4, 5)$$

represent the symmetry ρ and a symmetry mapping I to $r_1 r_2 I$ of a chiral 3-maniplex whose symmetry group is isomorphic to the symmetric group on six points.

By choosing as τ the symmetry acting as $(2, 6)(3, 5)$ we can construct a one-facet chiral 4-maniplex whose facet is itself chiral.

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