ISSN 2590-9770
The Art of Discrete and Applied Mathematics 4 (2021) #P3.15
https://doi.org/10.26493/2590-9770.1327.9ea
(Also available at http://adam-journal.eu)

The Configurations (13_3)*

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Received 12 September 2019, accepted 6 September 2021, published online 4 December 2021

Abstract

There are 2036 configurations (13_3). Here we establish that all of them are geometric; moreover, all have rational coordinatizations in the plane. This supports Grünbaum's conjecture that a geometric (n_3) configuration always has a rational coordinatization.

THIS PAPER IS IN HONOUR OF BRANKO GRÜNBAUM.

Keywords: $(n, 3)$*-configuration, geometric configuration, anti-Pappian, rational coordinatization*

Math. Subj. Class.: *51E20, 51E30*

1 Introduction

An (n_3) *configuration* is an incidence structure consisting of n *points* and n *lines* such that each point is contained in three lines, and each line contains three points. Any two lines are allowed to intersect in at most one point. Two recent reference books on configurations are Grünbaum [4] and Pisanski-Servatius [10]. The current paper builds upon the author's previous papers [7, 8], where one-point extensions and a coordinatization algorithm are presented.

Given an (n_3) configuration, a *one-point extension* is a construction that alters it slightly so as to produce an $((n + 1)_3)$ configuration. This construction is described in [7], where the configurations that can be obtained by it are characterized. The unique Fano (7_3) configuration cannot be obtained from it, nor can the Pappus (9_3) and Desargues (10_3) configurations, but the other two (9_3) configurations and the remaining nine (10_3) configurations can be, as well as all 31 (11_3) and all 229 (12_3) configurations. There is a family of

*The author would like to thank an anonymous referee for pointing out a minor error in the construction of the anti-Pappian in the original paper, and for reference [1].

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Fano-type configurations, defined below, that are not generated by a one-point extension (see [7]). All other (n_3) configurations can be generated by it. The smallest Fano-type configuration is the Fano (7_3) configuration, and the next one is a (13_3) configuration — it will be discussed in Section 4. There is a single Fano-type (14_3) configuration — it will be discussed in Section 5.

The Fano configuration is the unique (7_3) configuration. Denote it by F. Let its points be $\{P_1, \ldots, P_7\}$ and its lines be $\{\ell_1, \ldots, \ell_7\}$. A point P_i can be deleted from F by removing P_i from all lines containing it. This leaves three lines with only two points. Denote the result by F_p, as all P_i give isomorphic results. Similarly, a line ℓ can be removed from all points containing it, leaving three points with only two lines. Denote the result by F_ℓ. Another substructure can be obtained as follows. Choose any P_i and any ℓ_j containing P_i. Remove the incidence between P_i and ℓ_j, leaving one point with only two incident lines, and one line with only two incident points. Denote the result by F', as every P_i and every ℓ_j containing P_i gives an isomorphic result.

Note that an F_p has three lines that "need another point", and F_ℓ has three points that "need another line". We can create a (13_3) configuration by combining an F_p and an F_ℓ which are disjoint by "tying them together", i.e., we add the three points of F_p that need another line to the thee lines of F_ℓ that need another point. The result is a Fano-type configuration (13_3). It is shown in Figure 5. F' can also be used as a sub-configuration to build larger configurations. Note that F' has one point that "needs another line" and one line that "needs another point", so two copies of F' can be tied together to obtain a Fano-type (14_3) configuration, shown in Figure 7. This construction is described in more detail in [7].

Definition 1.1. A *Fano-type* configuration is any (n_3) configuration that can be constructed from a collection of disjoint sub-configurations isomorphic to F_p, F_ℓ or F', by "tying them together".

There are 2036 configurations (13_3) (see [4], P.69). Of these, one is a Fano-type configuration. The remaining 2035 can all be generated by one-point extensions starting from the (12_3) configurations.

An (n_3) configuration is said to be *geometric* if it has a representation as n distinct points and n distinct straight lines in the real plane, such that the incidences in the configuration agree with the incidences in the planar drawing, and there are no additional incidences. The Fano (7_3) and Möbius-Kantor (8_3) configurations are not geometric, whereas all (9_3) configurations are geometric, and all but one (10_3) configurations are geometric. All (11_3) and all (12_3) configurations are also geometric, as shown by Sturmfels and White [11, 12]. Grünbaum has conjectured that every geometric configuration has a planar representation for which all point and line coordinates are rational. Bokowski and Sturmfels [2] and Sturmfels and White [11, 12] have shown that this is the case when $n \leq 12$. Their method is to construct polynomials representing the planar coordinates for each configuration, and then use ad-hoc methods to find rational roots. In this paper we extend this result to $n = 13$, using a coordinatization algorithm in conjunction with one-point extensions, and show that all (13_3) configurations are geometric and rational.

When a geometric (n_3) configuration is drawn in the plane, homogeneous real coordinates are assigned to its points and lines, such that a point with coordinates $P = (x, y, z)$ is incident on a line with coordinates $L = (a, b, c)$ iff $P \cdot L = ax + by + cz = 0$. A one-point

ℓ_1	ℓ_2	ℓ_3	ℓ_4	ℓ_5	ℓ_6	ℓ_7	ℓ_8	ℓ_9	(9_3)
1	4	1	1	2	2	3	3	7	
2	5	5	6	4	6	4	5	8	Pappus
3	6	9	7	9	8	7	8	9	
1	1	2	3	1	2	4	5	4	
2	3	3	7	7	8	6	6	5	$(9_3)\#2$
4	5	6	9	8	9	8	9	7	
1	1	2	3	2	1	5	4	6	
2	3	3	4	5	6	7	7	8	$(9_3)\#3$
4	5	6	9	8	7	9	8	9	

Table 1: The three distinct configurations (9_3)

point	Pappus	$(9_3)\#2$	$(9_3)\#3$
1	(1,0,1)	(2,4,-3)	(3,6,1)
2	(0,0,1)	(-1,1,0)	(3,1,1)
3	(1,0,0)	(1,2,-3)	(1,2,-8)
4	(-1,1,1)	(1,1,-1)	(0,1,0)
5	(1,-2,-1)	(0,0,1)	(0,0,1)
6	(0,1,0)	(1,0,-1)	(2,1,-1)
7	(1,1,1)	(2,2,-3)	(-3,3,4)
8	(0,2,1)	(0,1,0)	(3,1,-4)
9	(1,-1,0)	(1,0,0)	(1,1,-8)

Table 2: Integer coordinatizations of the configurations (9_3)

extension only extends the incidence structure of an (n_3) configuration, not the coordinatization. Using the coordinatization algorithm of [8], when a geometric configuration is derived by a one-point extension from a smaller geometric configuration, the planar drawing of the smaller configuration can usually be extended to a drawing of the configuration in question.

2 The rational coordinatizations

The smallest geometric configurations are the three (9_3) configurations. One of these is the Pappus configuration. Rational coordinatizations of them can be used as starting points for the one-point extension algorithm of [8]. Incidence tables of these three configurations are given in Table 1. Here the points are $1, 2, \ldots, 9$ and the lines are $\ell_1, \ell_2, \ldots, \ell_9$. The three points on line ℓ_i are given in the column labelled ℓ_i. Several rational coordinatizations of the configurations are given in Table 2, as homogeneous integer coordinates.

Starting with these rational coordinatizations, the one-point extension and coordinatization algorithm can be applied to obtain the geometric (10_3) configurations, except for the Desargues configuration. Rational coordinatizations of it can be found, either using polynomials, or by using the tables of [2]. When the one-point extension and coordinatization algorithm is applied to the geometric (10_3) configurations, thousands of rational coordinatizations of the (11_3) configurations result. Applying the algorithm again results in thousands of rational coordinatizations of the (12_3) configurations. Applying it again produces

ℓ_1	ℓ_2	ℓ_3	ℓ_4	ℓ_5	ℓ_6	ℓ_7	ℓ_8	ℓ_9	ℓ_{10}	ℓ_{11}	ℓ_{12}	ℓ_{13}
1	2	6	3	1	8	6	5	4	2	3	5	1
2	3	12	4	7	9	8	6	5	7	9	11	8
4	10	13	7	10	12	11	9	10	13	11	12	13

Table 3: The incidences of configuration $(13_3)\#2035$

ℓ_1	ℓ_2	ℓ_3	ℓ_4	ℓ_5	ℓ_6	ℓ_7	ℓ_8	ℓ_9	ℓ_{10}	ℓ_{11}	ℓ_{12}
1	1	2	3	1	8	6	5	4	2	3	5
4	3	6	4	7	9	8	6	5	7	9	11
12	10	12	7	8	12	11	9	10	10	11	12

Table 4: The incidences of configuration $(12_3)\#15$

thousands of rational coordinatizations of 2034 of the 2036 (13_3) configurations. Two (13_3) configurations are missing. One of them is the Fano-type configuration, $(13_3)\#2036$, treated in the section 4. We denote the other missing configuration by $(13_3)\#2035$. Investigation shows that it derives as a one-point extension from only *one* configuration (12_3). Further investigation shows that the reason it was not generated by the algorithm is that the number of digits in the coordinates was too large. In section 3, the algorithm is carried out manually, using the software Maple [13] with multi-precision integers, to obtain a rational coordinatization.

It is not feasible to list a table of coordinatizations of all (13_3) configurations here, as it would run to a hundred pages. Instead, we refer to a location on the internet [5] where the coordinatizations will be posted.

As n increases, the integers in the coordinates of an (n_3) configuration begin to grow quite large, as can be seen from the tables in the following sections. Additional programming using multi-precision integer arithmetic would be required to apply the algorithm to find rational coordinatizations of the 21,399 (14_3) configurations.

3 The configuration $(13_3)\#2035$

The (13_3) configurations are numbered $(13_3)\#1, (13_3)\#2, \ldots, (13_3)\#2035$, in the order in which they were constructed by the software. There is also a Fano-type (13_3) configuration that cannot be derived as a one-point extension. It is configuration $(13_3)\#2036$.

Given a list of (n_3) configurations with rational coordinatizations, the software that generates the configurations $((n+1)_3)$ by one-point extensions also finds rational coordinatizations. The algorithm is described in [8]. This software was used to find rational coordinatizations of most of the (10_3) configurations, and all of the (11_3) and (12_3) configurations. Tables of configurations and coordinatizations can be downloaded from [5]. When the (12_3) configurations are used as input for constructing one-point extensions, configuration $(13_3)\#2035$ turns out to be quite interesting. Its incidences are shown in Table 3, where the points are $\{1, 2, \ldots, 13\}$, and the lines are $\{\ell_1, \ell_2, \ldots, \ell_{13}\}$. It has a collineation group of order eight. $(13_3)\#2035$ derives as a one-point extension from only one (12_3) configuration, namely $(12_3)\#15$, whose incidences are shown in Table 4. It has a collineation group of order 32.

i	Point i	Line ℓ_i
1	$(-26, 12, 39)$	$(-132, 13, -92)$
2	$(-69, -368, 47)$	$(48, 13, 28)$
3	$(-2, -12, 9)$	$(-2, -12, 9)$
4	$(-38, 60, 63)$	$(18, 3, 8)$
5	$(127, 0, -138)$	$(3, 0, 2)$
6	$(1, 0, -1)$	$(0, 0, 1)$
7	$(-2, 4, 3)$	$(1, 0, 1)$
8	$(0, 1, 0)$	$(0, 1, 0)$
9	$(1, 0, 0)$	$(-2760, 919, -2540)$
10	$(816, -620, -1111)$	$(1292, -113, 1012)$
11	$(3, 4, -3)$	$(0, 3, 4)$
12	$(11, 184, 0)$	$(552, -33, 508)$

Table 5: Integer point and line coordinates of $(12_3)\#15$

We will also need point and line coordinates for $(12_3)\#15$. They are given as homogeneous integer coordinates in Table 5.

We first describe the structure of $(13_3)\#2035$, in terms of some sub-structures.

Definition 3.1. A *complete quadrilateral* in the plane is a set of four distinct lines, no three concurrent, and all six pairwise intersection points. A *complete quadrangle* in the plane is a set of four distinct points, no three collinear, and all six pairwise joining lines.

The Fano configuration (7_3) is shown in Figure 1. As is usual, one line is drawn as a circle, because the Fano configuration is non-geometric. Notice that it consists of a complete quadrangle $\{1, 2, 3, 4\}$, plus the three intersection points $\{5, 6, 7\}$ of the pairs of its six lines: $12 \cap 34$, $13 \cap 24$, $14 \cap 23$, plus a line containing the three points $\{5, 6, 7\}$.

Definition 3.2. Given a complete quadrangle $\{A, B, C, D\}$, the *diagonal points* are the three additional points determined by the intersections of the pairs of lines: $AB \cap CD$, $AC \cap BD$, $AD \cap BC$. Dually, given a complete quadrilateral $\{\ell_1, \ell_2, \ell_3, \ell_4\}$, the *diagonal lines* are the three additional lines determined by the pairs of its six intersection points.

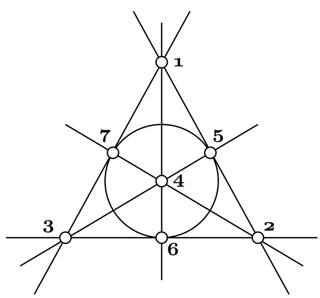

Figure 1: The Fano (7_3) configuration.

The (7_3) configuration and the non-geometric (10_3) configuration, known as the *anti-Pappian*, are both constructed from quadrangles and/or quadrilaterals. See [9, 3] for proofs that the anti-Pappian is non-geometric. It can be constructed as follows, as illustrated in Figure 2.

Construct a complete quadrangle determined by points $\{1, 2, 3, 4\}$, and a complete quadrilateral determined by lines $\{\ell_1, \ell_2, \ell_3, \ell_4\}$. We will associate point i with line ℓ_i, where $i = 1, 2, 3, 4$. The quadrangle has six lines, which can be labelled $\{\ell_5, \ell_6, \ell_7, \ell_8\, \ell_9\, \ell_{10}\}$, in some order. Each of these lines currently has just two points. The quadrilateral has six points of intersection, which are labelled $\{5, 6, 7, 8, 9, 10\}$, such that if line ℓ_i, where $5 \le i \le 10$, corresponds to points j and k, where $j, k \le 4$, then point i corresponds to lines ℓ_j and ℓ_k. Each of $\{5, 6, 7, 8, 9, 10\}$ is currently on just two lines. In order to create a (10_3) configuration out of this substructure, we now place point m on line ℓ_m, where $m = 5, 6, 7, 8, 9, 10$. The result is the anti-Pappian. Thus, it consists of a quadrangle and a quadrilateral, tied together.

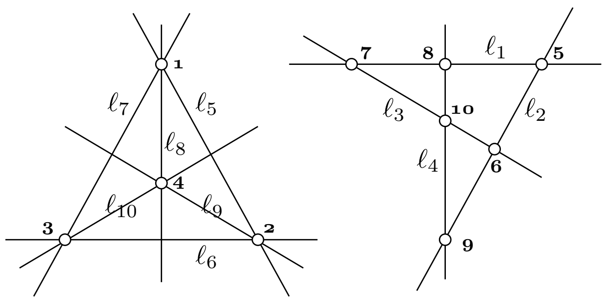

Figure 2: The anti-Pappian.

It is interesting to note that there are several ways of tying together a quadrilateral and a quadrangle. In Figure 2 of Boben, Gévay, Pisanski [1], is illustrated a Desargues configuration which can be viewed as a quadrilateral and quadrangle tied together. Six of the ten (10_3) configurations can be obtained in this way. In fact, the Desargues configuration and the anti-Pappian are very closely related. One can find distinct lines ℓ_1, ℓ_2 and distinct points $P_1 \in \ell_1$ and $P_2 \in \ell_2$ in the Desargues configuration such that, if the incidences $[P_1, \ell_1], [P_2, \ell_2]$ are changed to $[P_1, \ell_2], [P_2, \ell_1]$, the anti-Pappian results!

We now turn to configuration $(13_3)\#2035$. Consideration of Table 3 will show that it contains a complete quadrilateral $\{\ell_1, \ell_2, \ell_4, \ell_5\}$ with its six points of intersection $\{1, 2, 3, 4, 7, 10\}$. In addition, there is a complete quadrilateral $\{\ell_6, \ell_7, \ell_8, \ell_{12}\}$ with its six points of intersection $\{5, 6, 8, 9, 11, 12\}$, as illustrated in Figure 3. In addition, the first quadrilateral has two diagonal lines, namely ℓ_9 formed by joining points 4 and 10, which is extended to contain point 5, and ℓ_{10} formed by joining points 2 and 7. The second quadrilateral also has two diagonal lines, ℓ_{11} formed by joining points 9 and 11, which is extended to contain point 3, and ℓ_3 formed by joining points 6 and 12. The line shaded gray is ℓ_{13}, which intersects ℓ_3 and ℓ_{10} in point 13.

It is also helpful to describe the structure of $(12_3)\#15$. It consists of two quadrilaterals induced by lines $\{\ell_1, \ell_2, \ell_4, \ell_{10}\}$ and $\{\ell_6, \ell_7, \ell_8, \ell_{12}\}$ tied together through their diagonal lines. See Figure 4.

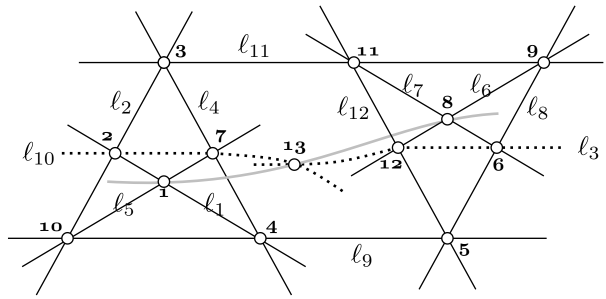

Figure 3: Configuration $(13_3)\#2035$.

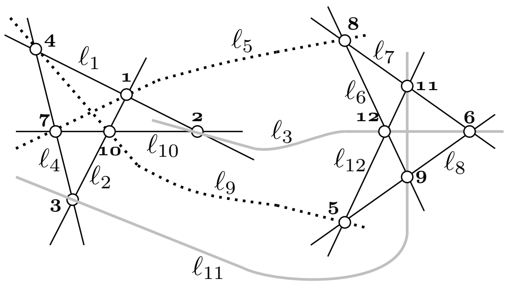

Figure 4: Configuration $(12_3)\#15$

The one-point extension in $(12_3)\#15$ adds a new point 13, and a new line ℓ_{13}, and alters lines $\ell_1, \ell_2, \ell_3, \ell_5.\ell_{10}$ slightly, so that the result is a (13_3) configuration. Refer to Tables 3 and 4. We now construct the incidence graph of the (13_3) configuration, and find three internally disjoint paths in it:

$[\ell_2, 2, \ell_{10}, 13]$
$[\ell_2, 10, \ell_{13}, 13]$
$[\ell_2, 3, \ell_{11}, 11, \ell_7, 6, \ell_3, 13]$

These paths will be used to find a coordinatization of $(13_3)\#2035$. Let the new coordinates of line ℓ_i be L_i, and the new coordinates of point i be P_i. Any point or line not lying on these paths is to have the same coordinates in $(13_3)\#2035$ as in $(12_3)\#15$, which are given in Table 5. We now set the new coordinates of ℓ_2 to be $L_2 = (x, y, z)$, being a homogeneous triple to be determined. Point 2 is incident on ℓ_1, which is not on one of the paths, so that $P_2 = L_2 \times L_1 = (-92y - 13z, 92x - 132z, 13x + 132y)$, which is a triple of linear homogeneous polynomials. Continuing like this, we find coordinates for all points and lines on the three paths as linear homogeneous triples. They are given in Table 6.

Lines $\ell_3, \ell_{10}, \ell_{13}$ must be concurrent in point 13. This gives an equation

$$p(x, y, z) = L_3 \cdot L_{10} \times L_{13} = 0$$

The cubic homogeneous polynomial $p(x, y, z)$ has enormous coefficients. We must choose

	Coordinates
L_2	(x, y, z)
P_2	$(-92y - 13z, 92x - 132z, 13x + 132y)$
P_3	$(8y - 3z, 18z - 8x, 3x - 18y)$
P_{10}	$(-2540y - 919z, -2760z + 2540y, 919x + 2760y)$
L_{11}	$(0, 3x - 18y, 8x - 18z)$
L_{10}	$(224x - 528y - 396z, -26x + 12y + 39z, 184x - 368y - 316z)$
L_{13}	$(88032x - 33120y - 107640z, -23894x + 27300y + 35841z,$ $66040x - 30480y - 82788z)$
P_{11}	$(1788x - 9144y - 594z, 4416x - 9936z, -1656x + 9936y)$
L_7	$(1656x - 9936y, 0, 1788x - 9144y - 594z)$
P_6	$(-1788x + 9144y + 594z, 0, 1656x - 9936y)$
L_3	$(-304704 + x1828224y, 18216x - 109296y,$ $-328992x + 1682496y + 109296z)$

Table 6: Homogeneous linear coordinates for the three paths

(x, y, z) so that $p(x, y, z) = 0$ and the resulting coordinates of Table 6 determine a coordinatization of $(13_3)\#2035$.

Observe that point 3 lies on lines ℓ_2, ℓ_4 and ℓ_{11}. If we choose $L_2 = (x, y, z) = L_4 = (18, 3, 8)$, then P_3 will be $(0, 0, 0)$, so that $p(18, 3, 8) = 0$. Now the equation $p(x, y, z) = 0$ is a curve in the projective plane. The tangent line at point $(x, y, z) = (18, 3, 8)$ has the equation

$$x\partial p/\partial x + y\partial p/\partial y + z\partial p/\partial z = 0$$

where the partial derivatives are evaluated at $(x, y, z) = (18, 3, 8)$. Removing a common factor from the coefficients of this equation results in

$$-2432x + 9048y + 2079z = 0$$

Solve for x in terms of y and z and substitute into $p(x, y, z)$ to obtain a cubic homogeneous polynomial $q(y, z) = 0$. Now the tangent has double contact with the curve at the point $(y, z) = (3, 8)$, so that $q(y, z)$ is divisible *twice* by $8y - 3z$. This division is easy to do. The result is

$$q(y, z) = (8y - 3z)^2(132451464y - 28581427z) = 0$$

We read off the solution (y, z), and use the previous substitution for x to obtain

$$(x, y, z) = (21956086, 28581427, 132451464)$$

These values are substituted into Table 6 to obtain point and line coordinates for $(13_3)\#2035$. The result is shown in Table 7. Here a common factor has been removed from some of the homogeneous coordinates whenever possible, in order to reduce the values of the coordinates.

These values of the coordinates ensure that all incidences of the configuration are satisfied. It is also necessary to check that these coordinates produce no unwanted incidences. This is straightforward by computer.

We summarize this calculation as a theorem.

Theorem 3.3. *Configuration* $(13_3)\#2035$ *is geometric, and has a rational coordinatization.*

i	Point i	Line ℓ_i
1	$(-26, 12, 39)$	$(-132, 13, -92)$
2	$(-2175680158, 1357990332, 3313517991)$	$(21956086, 28581427, 132451464)$
3	$(-2432, 9048, 2079)$	$(-11592, 693, -7652)$
4	$(-38, 60, 63)$	$(18, 3, 8)$
5	$(127, 0, -138)$	$(3, 0, 2)$
6	$(-1913, 0, 2898)$	$(0, 0, 1)$
7	$(-2, 4, 3)$	$(2898, 0, 1913)$
8	$(0, 1, 0)$	$(0, 1, 0)$
9	$(1, 0, 0)$	$(-2760, 919, -2540)$
10	$(-97159859998, 96058923900, 140330458527)$	$(-2295025242, -24998877, -1496684992)$
11	$(21043, -138736, -31878)$	$(0, 693, -3016)$
12	$(11, 184, 0)$	$(552, -33, 508)$
13	$(40949803542, -7065357484, -62674649163)$	$(506964732, 34572915, 327338668)$

Table 7: Integer point and line coordinates of $(13_3)\#2035$

4 The fano-type configuration (13_3)

The Fano-type configurations are constructed by tying together certain substructures of the Fano (7_3) configuration, as described in Definition 1. When an F_p and a disjoint F_ℓ are tied together, the result is a Fano-type (13_3) configuration. We denote it by $(13_3)\#2036$. It is the unique Fano-type configuration on 13 points, as it is the only (13_3) configuration not generated by a one-point extension from the (12_3) configurations. It is illustrated in Figure 5, and its incidence table is Table 8. Here the points are $\{1, 2, \ldots, 13\}$, and the lines are $\{\ell_1, \ell_2, \ldots, \ell_{13}\}$, where the three points in line ℓ_i are those in the column of ℓ_i.

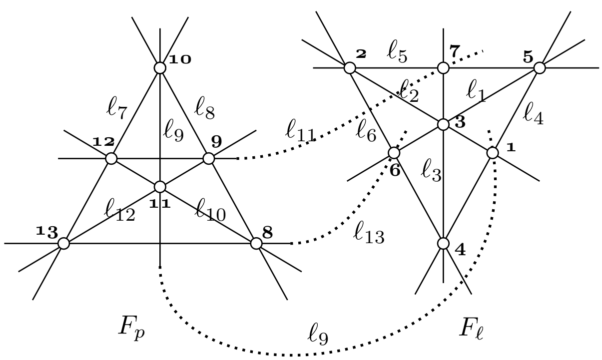

Figure 5: The Fano-type configuration $(13_3)\#2036$.

ℓ_1	ℓ_2	ℓ_3	ℓ_4	ℓ_5	ℓ_6	ℓ_7	ℓ_8	ℓ_9	ℓ_{10}	ℓ_{11}	ℓ_{12}	ℓ_{13}
3	1	3	1	2	2	10	8	1	8	7	9	6
5	2	4	4	5	4	12	9	10	11	9	11	8
6	3	7	5	7	6	13	10	11	12	12	13	13

Table 8: The incidences of the Fano-type configuration $(13_3)\#2036$

We carefully choose a subset of its points and lines, as independent as possible, so that the remaining points and lines are thereby determined. This is called a *determining set* in [6], where the term is defined precisely. Start by choosing points 2,3,4,5, and without loss of generality, assign them coordinates $P_2 = (1,0,0), P_5 = (0,1,0), P_3 = (0,0,1), P_4 = (1,1,1)$. This is possible because the real projective plane is 3-transitive on points if no three are collinear, and because the coordinates are homogeneous. These points completely determine lines $\ell_1, \ell_2, \ell_3, \ell_4, \ell_5, \ell_6$, namely $L_1 = (1,0,0), L_2 = (0,1,0), L_3 = (1,-1,0), L_4 = (1,0,-1), L_5 = (0,1,0), L_6 = (0,1,-1)$. These in turn determine points 1, 6, 7, namely $P_1 = (1,0,1), P_6 = (0,1,1), P_7 = (1,1,0)$. We construct a digraph, called a *construction sequence* for the configuration, whose vertices are the points and lines, and whose arcs indicate which points and lines determine others, e.g., points 2 and 3 uniquely determine ℓ_2. This is illustrated in Figure 6, where all edges are directed from left to right. The table of coordinates is given in Table 9.

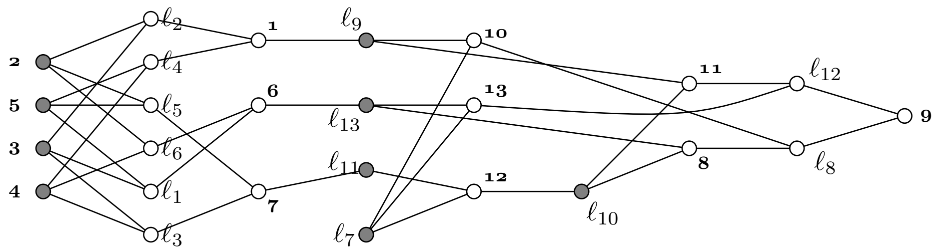

Figure 6: The construction sequence for $(13_3)\#2036$ used to find its coordinatization.

Line ℓ_9 is incident with point 1, so that $L_9 \cdot (1,0,1) = 0$. Consequently $L_9 = (u,p,-u)$ for some values p and u. But $p \neq 0$, for $p = 0$ would imply that L_9 and L_4 are equal. Without loss of generality, we can take $p = 1$. Similarly $L_{13} \cdot (0,1,1) = 0$, from which it follows that $L_{13} = (1,v,-v)$, for some value v. And $L_{11} \cdot (1,1,0) = 0$, from which it follows that $L_{13} = (w,-w,1)$, for some value w. $L_7 = (x,y,z)$ is chosen as part of the determining set. Points 10, 13, and 12 are now determined. L_{10} must be chosen so that $L_{10} \cdot P_{12} = 0$. But we already have $P_{12} \cdot L_{11} = P_{12} \cdot L_7 = 0$, where L_{11} and L_7 are linearly independent, from which it follows that $L_{10} = aL_{11} + bL_7$ for some values $a, b \neq 0$. Then $P_{11} = L_9 \times (aL_{11} + bL_7)$ and $P_8 = L_{13} \times (aL_{11} + bL_7)$ are determined.

We now calculate

$$A = L_9 \cdot P_{12} = uwy - uy + uwx - x + wz - wzu \neq 0$$

$$B = L_{11} \cdot P_{13} = wvz + wvy + wvx + wz + y - vx \neq 0$$

$P_2 = (1,0,0)$ $P_5 = (0,1,0)$ $P_3 = (0,0,1)$ $P_4 = (1,1,1)$	$L_2 = (0,1,0)$ $L_4 = (1,0,-1)$ $L_5 = (0,0,1)$ $L_6 = (0,1,-1)$ $L_1 = (1,0,0)$ $L_3 = (1,-1,0)$	$P_1 = (1,0,1)$ $P_6 = (0,1,1)$ $P_7 = (1,1,0)$	$L_9 = (u,1,-u)$ $L_{13} = (1,v,-v)$ $L_{11} = (w,-w,1)$ $L_7 = (x,y,z)$

$P_{10} = (z+uy, -ux-uz, uy-x)$ $P_{13} = (vz+vy, -vx-z, y-vx)$ $P_{12} = (-wz-y, x-wz, wy+wx)$ $L_{10} = aL_{11}+bL_7 = (aw+bx, -aw+by, a+bz)$

$P_{11} = (-a-bz+uaw-uby, uaw+ubx-ua-ubz, -uaw+uby+aw+bx)$ $P_8 = (va+vbz-vaw+vby, -vaw-vbx-a-bz, -aw+by-vaw-vbx)$

Table 9: Point and line coordinates for configuration $(13_3)\#2036$

$$C = L_{13}\cdot P_{10} = -z-uy+vux-vuz-vuy-vx \neq 0$$

$$D = L_{11}\cdot L_9\times L_{13} = -2vuw+vu+vw-uw+1 \neq 0$$

Note that $D \neq 0$, because the intersection of ℓ_9 and ℓ_{13} does not lie on ℓ_{11}, for this would imply that the Fano configuration is geometric. We then find that

$$L_9\cdot P_{13} = C, \quad L_{11}\cdot P_{10} = -A, \quad L_{13}\cdot P_{12} = -B$$

This will result in factorization and cancellation in the coordinatizing polynomial, thereby making it possible to find rational roots.

We will also need the formulas

$$L_9\times L_7 = P_{10} \quad \text{and} \quad L_{13}\times L_7 = P_{13}$$

Then

$$L_{12} = P_{11}\times P_{13} \quad \text{and} \quad L_8 = P_{10}\times P_8$$

These can both be expanded to large polynomial expressions. However, using the identity $(U\times V)\times W = (U\cdot W)V - (V\cdot W)U$, we can also write them as

$$L_{12} = P_{11}\times P_{13} = (L_9\times(aL_{11}+bL_7))\times P_{13} =$$

$$= (L_9\cdot P_{13})(aL_{11}+bL_7) - ((aL_{11}+bL_7)\cdot P_{13})L_9 = C(aL_{11}+bL_7) - aBL_9$$

and

$$L_8 = P_{10}\times P_8 = P_{10}\times(L_{13}\times(aL_{11}+bL_7)) =$$

$$= -(L_{13}\cdot P_{10})(aL_{11}+bL_7) + ((aL_{11}+bL_7)\cdot P_{10})L_{13} = -C(aL_{11}+bL_7) - aAL_{13}$$

Then

$$P_9 = L_{12}\times L_8 = -(C(aL_{11}+bL_7) - aBL_9)\times(C(aL_{11}+bL_7) + aAL_{13})$$

$$= aBC(L_9\times(aL_{11}+bL_7)) - aAC(aL_{11}+bL_7)\times L_{13} + a^2ABL_9\times L_{13}$$

$$= aBC(aL_9\times L_{11}+bP_{10}) - aAC(aL_{11}\times L_{13} - bP_{13}) + a^2ABL_9\times L_{13}$$

point	coordinates	line	coordinates
P_1	$(1,0,1)$	L_1	$(1,0,0)$
P_2	$(1,0,0)$	L_2	$(0,1,0)$
P_3	$(0,0,1)$	L_3	$(1,-1,0)$
P_4	$(1,1,1)$	L_4	$(1,0,-1)$
P_5	$(0,1,0)$	L_5	$(0,0,1)$
P_6	$(0,1,1)$	L_6	$(0,1,-1)$
P_7	$(1,1,0)$	L_7	$(2,4,3)$
P_8	$(-225,-263,-338)$	L_8	$(4958,2368,-5143)$
P_9	$(-15809212,-16351336,-22769208)$	L_9	$(2,1,-2)$
P_{10}	$(11,-10,6)$	L_{10}	$(86,-80,5)$
P_{11}	$(-155,-182,-246)$	L_{11}	$(42,-42,1)$
P_{12}	$(-130,-124,252)$	L_{12}	$(1850,5476,-5217)$
P_{13}	$(21,-9,-2)$	L_{13}	$(1,3,-3)$

Table 10: Rational point and line coordinates for configuration $(13_3)\#2036$

The missing incidence from the construction sequence is point 9 on ℓ_{11}. Thus, there will be a coordinatization of the configuration if $P_9 \cdot L_{11} = 0$. This reduces to

$$abBCP_{10} \cdot L_{11} - abACP_{13} \cdot L_{11} + a^2 ABL_{11} \cdot L_9 \times L_{13} =$$

$$= -abABC - abABC + a^2 ABD = 0$$

Cancelling aAB from the equation leaves

$$aD - 2bC = 0$$

We now look for values of a, b, x, y, z, u, v, w which satisfy this equation, and which make all point coordinates distinct, and all line coordinates distinct. In addition to $a, A, B, C, D \neq 0$, there are a number of constraints which can be written down to aid in this. For example, point $10 \notin \ell_2$ gives the condition $P_{10} \cdot L_2 = -ux - uz \neq 0$, so that $x + z \neq 0$, etc. There are other conditions like this. Experimentation then leads to a solution with $a = 2, b = 1, x = 2, y = 4, z = 3, u = 2, v = 3, w = 42$. The resulting point and line coordinates are given in Table 10. The algebraic calculations were done using the software *Maple* [13]. The results of this calculation are stated as:

Theorem 4.1. *The Fano-type configuration* $(13_3)\#2036$ *is geometric, and has a rational coordinatization.*

5 The fano-type configuration (14_3)

There is a unique Fano-type configuration with 14 points. It can be constructed as follows. Take two copies of F' (see Definition 1). Each F' has 7 points and 7 lines. Tie them together as shown in Figure 7. The result is a (14_3) configuration.

It seems that most choices of determining set and construction sequence for this configuration lead to an enormous polynomial, for which it appears to be intractable to find roots. But with a judicious choice of determining set, magic happens, and a coordinatization

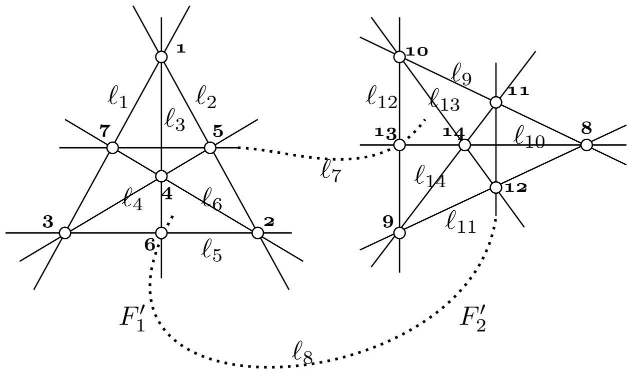

Figure 7: The Fano-type configuration on 14 points.

ℓ_1	ℓ_2	ℓ_3	ℓ_4	ℓ_5	ℓ_6	ℓ_7	ℓ_8	ℓ_9	ℓ_{10}	ℓ_{11}	ℓ_{12}	ℓ_{13}	ℓ_{14}
1	1	1	3	2	2	5	6	8	8	8	9	10	9
3	2	4	4	3	4	7	11	10	13	9	10	12	11
7	5	6	5	6	7	13	12	11	14	12	13	14	14

Table 11: The incidences of the Fano-type configuration on 14 points

can be found. Choose points 1,2,3,4 with coordinates $P_1 = (1,0,0), P_2 = (0,1,0), P_3 = (0,0,1), P_4 = (1,1,1)$. This determines lines $\ell_1, \ell_2, \ell_3, \ell_4, \ell_5, \ell_6$, which in turn determines points 5,6,7, and then line ℓ_7. At this point, choose lines $\ell_8, \ell_9, \ell_{11}$ as part of the determining set, with coordinates $L_8 = (u, v, -v), L_9 = (x, y, z), L_{11} = (a, b, c)$. And choose point 14 as part of the determining set, with coordinates $P_{14} = (p, q, r)$. All remaining coordinates are then determined. Refer to Figure 8 and Table 12.

The polynomials for P_9, P_{10} are quite large. Those for L_{12} and P_{13} are enormous. However, some factoring occurs. First note that

$$P_{11} \cdot L_{11} = -P_{12} \cdot L_9 = P_8 \cdot L_8 = (vza - ubz + vya - vcx - vbx + ucy) \neq 0$$

Calculating $P_9 \times P_{10}$, we find that it has a factor of $(vza - ubz + vya - vcx - vbx + ucy)$. As this is non-zero, we cancel it, and obtain a reduced (but still very long) expression for L_{12}. We then calculate $L_{12} \times L_{10}$ and find that it has two additional factors, namely

$$(xp + qy + rz) = P_{14} \cdot L_9 \neq 0$$

$$(ap + bq + cr) = P_{14} \cdot L_{11} \neq 0$$

which can be cancelled to obtain

$$P_{13} = [zpva + ypva - cpvx + 2cpuy - bpvx - 2bpuz + zrbv + yqvc - crvy - bqvz,$$

$$apuz + aqvy + 2aqvz - arvz - xpuc - bqvx + cquy - 2cqvx + crvx - zqub,$$

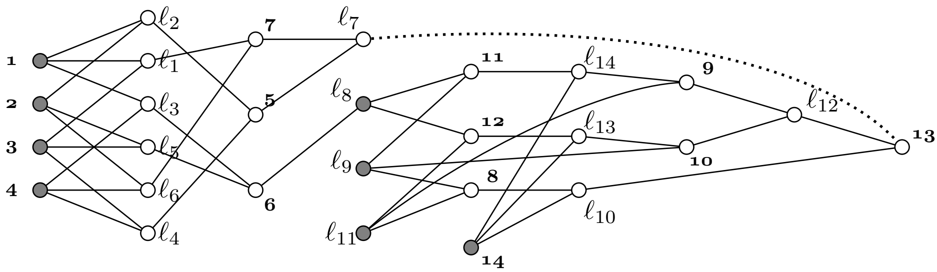

Figure 8: The construction sequence for the Fano-type configuration (14_3), used to find its coordinatization.

$P_1 = (1,0,0)$ $P_2 = (0,1,0)$ $P_3 = (0,0,1)$ $P_4 = (1,1,1)$	$L_2 = (0,0,1)$ $L_1 = (0,1,0)$ $L_3 = (0,1,-1)$ $L_5 = (1,0,0)$ $L_6 = (1,0,-1)$ $L_4 = (1,-1,0)$	$P_7 = (1,0,1)$ $P_5 = (1,1,0)$ $P_6 = (0,1,1)$	$L_7 = (1,-1,-1)$ $L_8 = (u,v,-v)$ $L_9 = (x,y,z)$ $L_{11} = (a,b,c)$

$P_{11} = (vz + vy, -vx - uz, uy - vx)$ $P_{12} = (vc + vb, -va - uc, ub - va)$ $P_8 = (yc - zb, za - xc, xb - ya)$ $P_{14} = (p,q,r)$

$L_{14} = (-rvx - ruz - quy + qvx, puy - pvx - rvz - rvy, qvz + qvy + pvx + puz)$ $L_{13} = (-rva - ruc - qub + qva, pub - pva - rvc - rvb, qvc + qvb + pva + puc)$ $L_{10} = (rza - rxc - qxb + qya, pxb - pya - tyc + rzb, qyc - qzb - pza + pxc)$

Table 12: Point and line coordinates for the Fano-type configuration (14_3)

$$-apuy + arvz + 2arvy - aqvy + xpub + bqvx - crvx - bruz - 2brvx + yruc]$$

We then calculate $P_{13} \cdot L_7$, which must be zero if the configuration is geometric. The result is

$$P_{13} \cdot L_7 = (zpva + ypva + apuy - apuz - 2arvy - 2aqvz - cpvx - bpvx + 2cpuy + xpuc$$

$$-2bpuz - xpub - cquy + 2cqvx + yqvc - bqvz - crvy - yruc + zqub + bruz + 2brvx + zrbv)$$

There are a number of additional identities that must be satisfied, e.g.,

$$P_{14} \cdot L_8 = (up + vq - vr) \neq 0$$

$$P_7 \cdot L_9 = (x + z) \neq 0$$

$$P_6 \cdot L_9 = (y + z) \neq 0$$

$$P_7 \cdot L_8 = (u + v) \neq 0$$

etc.

point	coordinates	line	coordinates
P_1	$(1,0,0)$	L_1	$(0,1,0)$
P_2	$(0,1,0)$	L_2	$(0,0,1)$
P_3	$(0,0,1)$	L_3	$(0,-1,1)$
P_4	$(1,1,1)$	L_4	$(-1,1,0)$
P_5	$(1,1,0)$	L_5	$(1,0,0)$
P_6	$(0,1,1)$	L_6	$(1,0,-1)$
P_7	$(1,0,1)$	L_7	$(1,-1,-1)$
P_8	$(-9065,2345,-105)$	L_8	$(1,2,-2)$
P_9	$(412797,-141180,50641)$	L_9	$(3,117,2354)$
P_{10}	$(336847,-174745,8256)$	L_{10}	$(24710,90335,-115815)$
P_{11}	$(4942,-2360,111)$	L_{11}	$(1,4,3)$
P_{12}	$(14,-5,2)$	L_{12}	$(1844821,3277363,-5901117)$
P_{13}	$(310,137,173)$	L_{13}	$(-74,-134,183)$
P_{14}	$(3,12,10)$	L_{14}	$(-24932,-49087,66384)$

Table 13: Integer coordinates for the Fano-type configuration (14_3)

In order to find a rational solution, we try substituting various values into the variables. Substituting $x = a = p = u = 1$, $v = 2$, $c = 3$, $b = q = 4$, we obtain

$$-39z + 21y - 13ry + 33 + 12rz + 16r = 0$$

Then trying $y = 39$ and $r = 10/3$, we obtain

$$z = 2354/3$$

We then replace (x, y, z) with $(3x, 3y, 3z)$ to obtain integer coordinates for the configuration, as shown in Table 13.

These algebraic calculations were also done using the software *Maple* [13]. It can be verified that the inner products of non-incident points and lines are all non-zero. This gives

Theorem 5.1. *The Fano-type configuration (14_3) is geometric, and has a rational coordinatization.*

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