Ω -lattices from skew lattices

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Abstract

Some notes on construction of Ω -lattices from special types of skew lattices, using lattices of weak congruences of skew lattices, are presented.

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1 Introduction

Weak congruences on algebras, as compatible, symmetric, transitive and weakly reflexive relations [4, 5, 14], have a distinguished role in structural investigations of algebras. Namely, a collection of all weak congruences of an algebra forms an algebraic lattice under inclusion and this lattice contains the lattice of all congruence relations as a filter generated by the identity (diagonal) relation, the lattice of all subalgebras (under isomorphism) as an ideal generated by the identity relation and also lattices of all congruences on all subalgebras.

Weak congruences are connected to Ω -valued algebraic structures (Ω is a complete lattice), Ω -valued here means that elements from the algebra carrier set are valued by elements of the lattice (i.e. the carrier set of the algebra is mapped to Ω). The denotation Ω is used since our research is connected also to Ω -sets introduced 1979. in a paper by Fourman and

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Scott [6]. They used Ω -sets to model intuitionistic logic. The main connection between weak congruences is here the notion of Ω -valued equality, which is a symmetric and transitive function from A^2 to Ω , and this is a kind of weak equivalence (if considering algebraic structure it is a lattice valued weak congruence). In Fourman and Scott's research, Ω was a complete Heyting algebra, and here Ω is just a complete lattice.

Ω -sets are connected to non-classical predicate logics, and they are used in theoretical foundations of fuzzy set theory [7, 8]. Namely, a similar concept of algebras with an L -valued equality is introduced by Bělohlávek and Vychodil [3], where a complete residuated lattice L is used as a truth-values structure. By adding operations to this structure, so-called L -algebras are obtained. Notions and basics results, analogous to those in universal algebra are obtained in this framework, including a Birkhoff-like variety theorem.

Just to highlight that an L -valued equality is reflexive [3], which is not the case in our investigations, since we use a weak reflexivity. The weak reflexivity instead of the reflexivity enables a connection with weak congruences, which are here obtained as cut sets of an Ω -valued equality. Cut sets, as the inverse images (via valuating functions) of the filters in Ω are a useful tool in this setting. By cut-sets, main algebraic and set-theoretic notions and their properties are generalized from the classical notions to the lattice-valued framework.

In lattice-valued structures with a fuzzy equality [3], also identities were introduced (together with other universal-algebraic notions), in the sense of having a graded satisfiability. Our approach is that an identity holds if the corresponding lattice-theoretic formula is fulfilled (i.e. either identity holds or it does not hold). This notion was firstly introduced in [16], and then developed in [1]. Basic properties of Ω -algebras and representation theorems for Ω -algebras in general are proved in [12].

Since an identity can hold in a lattice-valued algebra, while the underlying classical algebra need not satisfy the same identity, this concept is used in the present paper to introduce Ω -lattices from skew lattices, constructing Ω -lattices using weak congruence lattices of skew lattices.

2 Preliminaries

We use basic notions and notations from universal algebra [2]. An *algebra* is denoted by $\mathcal{A} = (A, F)$, where A is a nonempty underlying set and F is a set of (fundamental) operations on A . Notions of subalgebras, subuniverses and congruences are also well known. In addition to congruences, we use *weak congruences* on $\mathcal{A}$ as symmetric, transitive and compatible relations on the algebra $\mathcal{A}$. Compatibility here implies also weak reflexivity: the property that all nullary operations - constants are in the relation to itself. A weak congruence on $\mathcal{A}$ is obviously a congruence on the subalgebra determined by its domain. The collection $\text{Con}_w(\mathcal{A})$ of all weak congruences on an algebra $\mathcal{A}$ is an algebraic lattice under inclusion [4, 5, 14].

We also deal with *terms*, *term-operations*, and *identities* in the given language as formulas $t_1 \approx t_2$, where t_1, t_2 are terms in the same language.

Recall that a *closure system* on a nonempty set X is a collection of subsets of X closed under set intersections (including $\bigcap \emptyset$ which is X). It is a complete lattice under inclusion.

We also use notions and properties of Ω -valued sets and relations. All notions necessary for comprehending this text will be introduced in the sequel, other details can be found in papers dealing with lattice-valued structures (see e.g. [1, 8, 12, 15]).

Let $(\Omega, \wedge, \vee, \leq)$ be a complete lattice with the top and the bottom elements denoted by 1 and 0, respectively.

If A is a nonempty set, then an Ω -valued function (Ω -valued set) μ on A is a map $\mu: A \rightarrow \Omega$. For $x \in A$, $\mu(x)$ is a *degree of membership* of x to μ .

For $p \in L$, a *cut set* or a *p-cut* of an Ω -valued function $\mu: A \rightarrow \Omega$ is a subset μ_p of A which is the inverse image of the principal filter $\uparrow p$ in Ω :

$$\mu_p = \mu^{-1}(\uparrow p) = \{x \in A \mid \mu(x) \geq p\}.$$

An Ω -valued (binary) relation R on A is an Ω -valued function on A^2 , i.e., it is a mapping $R: A^2 \rightarrow \Omega$. As above, for $p \in \Omega$, a *cut* R_p of R is the binary relation on A , which is the inverse image of $\uparrow p$: $R_p = R^{-1}(\uparrow p)$.

R is *symmetric* if $R(x, y) = R(y, x)$ for all $x, y \in A$, and *transitive* if $R(x, z) \wedge R(z, y) \leq R(x, y)$ for all $x, y, z \in A$.

A symmetric and transitive Ω -valued relation on A fulfills the *strictness* property:

$$R(x, y) \leq R(x, x) \wedge R(y, y). \quad (2.1)$$

The strictness is a kind of a *weak reflexivity* of R . Therefore, a symmetric and transitive Ω -valued relation on A is a *weak Ω -valued equivalence* on A .

Lemma 2.1 ([15]). *An Ω -valued binary relation R on A is a weak Ω -valued equivalence on A if and only if all cuts of R are weak equivalences (symmetric and transitive) relations on A .*

A weak Ω -valued equivalence R on A is a weak Ω -valued equality, if it satisfies the *separation property*:

$$R(x, y) = R(x, x) \quad \text{implies} \quad x = y. \quad (2.2)$$

Remark 2.2. The *separation* property is introduced in [6] by a weaker condition:

$$R(x, y) = R(x, x) = R(y, y) \quad \text{implies} \quad x = y.$$

If $\mathcal{A} = (A, F)$ is an algebra and $\mu: A \rightarrow \Omega$ an Ω -valued function on A , then μ is *compatible* with the operations in F , if for every n -ary operation $f \in F$, for all $a_1, \dots, a_n \in A$, and for every constant (nullary operation) $c \in F$

$$\bigwedge_{i=1}^n \mu(a_i) \leq \mu(f(a_1, \dots, a_n)), \quad \text{and} \quad \mu(c) = 1. \quad (2.3)$$

An Ω -valued function on A , compatible with the operations in F is also called a *fuzzy algebra*, or a *lattice valued algebra* [9, 13].

Further, an Ω -valued relation $R: A^2 \rightarrow \Omega$ on A is *compatible* with the operations in F if for every n -ary operation $f \in F$, for all $a_1, \dots, a_n, b_1, \dots, b_n \in A$, and for every constant $c \in F$

$$\bigwedge_{i=1}^n R(a_i, b_i) \leq R(f(a_1, \dots, a_n), f(b_1, \dots, b_n)), \quad \text{and} \quad R(c, c) = 1. \quad (2.4)$$

A weak Ω -valued equivalence on A which is compatible with the operations in F is called a weak Ω -valued congruence on $\mathcal{A}$.

Algebraic topics related to lattice-valued structures are here presented mostly from [1, 4, 5, 6, 8, 12, 14].

An Ω -set (originating from [6]) is a pair (A, E) , where A is a nonempty set, and E is a symmetric and transitive Ω -valued relation on A , fulfilling the separation property (2.2).

We say that E is an Ω -valued equality in (A, E) .

We also endow the Ω -set with an Ω -valued function $\mu: A \rightarrow \Omega$ defined by:

$$\mu(x) := E(x, x).$$

Throughout the text, we also say that (A, E) is a *lattice-valued set*, without particularly fixing the co-domain lattice.

Lemma 2.3 ([1]). *Every cut E_p , $p \in \Omega$, of the Ω -valued equality in an Ω -set (A, E) is an equivalence relation on the corresponding cut μ_p of μ .*

If $\mathcal{A} = (A, F)$ is an algebra, (A, E) is an Ω -set and E is compatible with the operations in F then a pair $\overline{\mathcal{A}} = (A, E)$ is an Ω -algebra. $\mathcal{A}$ is called the *underlying, basic algebra* of $\overline{\mathcal{A}}$.

Function μ defined above is then a lattice valued algebra in the sense of [9, 13] and for every $p \in \Omega$, the cut μ_p of μ is a subalgebra of $\mathcal{A}$ [1].

Proposition 2.4 ([1]). *Let $(\mathcal{A}, E)$ be an Ω -algebra. Then every cut of E is a weak congruence on $\mathcal{A}$, namely for $p \in \Omega$, E_p is a congruence on the subalgebra μ_p .*

Identities on Ω -algebras are introduced in [16] in a particular way:

Let $(\mathcal{A}, E)$ be an Ω -algebra and $u(x_1, \dots, x_n) \approx v(x_1, \dots, x_n)$, briefly $u \approx v$, be an identity in the type of $\mathcal{A}$. Then, $(\mathcal{A}, E)$ satisfies the identity $u \approx v$ (i.e., this identity holds on $(\mathcal{A}, E)$) if

$$\bigwedge_{i=1}^n \mu(a_i) \leq E(u(a_1, \dots, a_n), v(a_1, \dots, a_n)), \quad (2.5)$$

for all $a_1, \dots, a_n \in A$ and the term-operations on $\mathcal{A}$ corresponding to the terms u and v respectively.

If the Ω -algebra $(\mathcal{A}, E)$ satisfies an identity, this identity need not hold on $\mathcal{A}$, but the converse obviously holds [1].

As an example of an Ω -algebra, we introduce here a notion of an Ω -lattice. This notion will be used in the sequel.

Let Ω be a complete lattice, let $\mathcal{A} = (A, \wedge, \vee)$ be an algebra with two binary operations (without any additional conditions) and let $E: A^2 \rightarrow \Omega$ be an Ω -valued equality on A (i.e., (A, E) is an Ω -set). Let E be compatible with operations $\wedge$ and $\vee$: for all $x, y, z, t \in A$,

$$\begin{aligned} E(x, y) \wedge E(z, t) &\leq E(x \wedge z, y \wedge t) \quad \text{and} \\ E(x, y) \wedge E(z, t) &\leq E(x \vee z, y \vee t). \end{aligned}$$

An Ω -algebra $(\mathcal{A}, E)$ is an Ω -lattice, if it satisfies lattice identities:

$$\begin{aligned} \ell 1: \quad x \wedge y &\approx y \wedge x & (\text{commutativity}) \\ \ell 2: \quad x \vee y &\approx y \vee x \\ \ell 3: \quad x \wedge (y \wedge z) &\approx (x \wedge y) \wedge z & (\text{associativity}) \\ \ell 4: \quad x \vee (y \vee z) &\approx (x \vee y) \vee z \\ \ell 5: \quad (x \wedge y) \vee x &\approx x & (\text{absorption}) \\ \ell 6: \quad (x \vee y) \wedge x &\approx x \end{aligned}$$

In other words, for all $x, y, z \in A$, the following formulas are satisfied with the mapping $\mu: M \rightarrow \Omega$ defined by $\mu(x) = E(x, x)$:

$$\begin{aligned}
 L1: \quad & \mu(x) \wedge \mu(y) \leq E(x \wedge y, y \wedge x) & (\text{commutative laws}) \\
 L2: \quad & \mu(x) \wedge \mu(y) \leq E(x \vee y, y \vee x) \\
 L3: \quad & \mu(x) \wedge \mu(y) \wedge \mu(z) \leq E((x \wedge y) \wedge z, x \wedge (y \wedge z)) & (\text{associative laws}) \\
 L4: \quad & \mu(x) \wedge \mu(y) \wedge \mu(z) \leq E((x \vee y) \vee z, x \vee (y \vee z)) \\
 L5: \quad & \mu(x) \wedge \mu(y) \leq E((x \wedge y) \vee x, x) & (\text{absorption laws}) \\
 L6: \quad & \mu(x) \wedge \mu(y) \leq E((x \vee y) \wedge x, x)
 \end{aligned}$$

Remark 2.5. The operations on the lattice Ω and on the basic structure for an Ω -lattice $(A, \wedge, \vee)$ are denoted with the same symbols $\vee$ and $\wedge$. However, the confusion can not arise since one structure is the domain and the other is the co-domain of the function μ and from the context we can distinguish them.

The following propositions are true.

Theorem 2.6 ([1]). *Let $(\mathcal{A}, E)$ be an Ω -algebra, and $\mathcal{F}$ a set of identities in the language of $\mathcal{A}$. Then, $(\mathcal{A}, E)$ satisfies all identities in $\mathcal{F}$ if and only if for every $p \in \Omega$ the quotient algebra μ_p/E_p satisfies the same identities.*

Corollary 2.7. *Let $(\mathcal{A}, E)$ be an Ω -algebra with two binary operations. Then, $(\mathcal{A}, E)$ is an Ω -lattice if and only if for all $p \in \Omega$ the quotient algebras μ_p/E_p are lattices.*

Corollary 2.8. *If a diagonal relation $\Delta_A = \{(a, a) \mid a \in A\}$ is a cut of E , then each identity fulfilled by an Ω -algebra $\overline{\mathcal{A}} = (\mathcal{A}, E)$ also holds on the underlying algebra $\mathcal{A}$.*

By Corollary 2.8, we are interested in Ω -algebras which do not contain a copy of the underlying algebra among quotient substructures. An Ω -algebra $\overline{\mathcal{A}} = (\mathcal{A}, E)$ is said to be *proper* if Δ_A is not a cut of E .

Theorem 2.9 ([12]). *$\overline{\mathcal{A}} = (\mathcal{A}, E)$ is a proper Ω -algebra if and only if*

$$\text{there are } a, b \in A, a \neq b, \text{ such that } E(a, b) \geq \bigwedge \{E(x, x) \mid x \in A\}. \quad (2.6)$$

2.1 Representation

This part is mostly from [12] and it will be used in the sequel in Ω -skew lattices.

Proposition 2.10 ([12]). *The collection of the cuts of E in an Ω -algebra $\overline{\mathcal{A}} = (\mathcal{A}, E)$ is a closure system on A^2 , a subposet of the weak congruence lattice $\text{Con}_w(\mathcal{A})$ of $\mathcal{A}$.*

Theorem 2.11 ([12]). *Let $\mathcal{A}$ be an algebra and $\mathcal{R}$ a closure system in $\text{Con}_w(\mathcal{A})$ such that for all $a, b \in A, a \neq b$,*

$$(a, b) \notin \bigcap \{R \in \mathcal{R} \mid (a, a) \in R\}. \quad (2.7)$$

Then there is a complete lattice Ω and an Ω -algebra $(\mathcal{A}, E)$ with the underlying algebra $\mathcal{A}$, such that $\mathcal{R}$ consists of cuts of E .

As proved in [12], the lattice Ω in Theorem 2.11 is the starting collection $\mathcal{R}$ of weak congruences ordered by the dual of inclusion, $\supseteq$. The required Ω -algebra is $(\mathcal{A}, E)$, where the Ω -valued equality $E: A^2 \rightarrow \Omega$ is defined by:

$$E(a, b) := \bigcap \{R \in \mathcal{R} \mid (a, b) \in R\} \quad \text{for all } a, b \in A. \quad (2.8)$$

The lattice Ω and the corresponding Ω -algebra defined by means of (2.8), are said to be obtained by the *canonical construction* over the algebra $\mathcal{A}$; in this context, we say that an Ω -algebra $(\mathcal{A}, E)$, obtained by the canonical construction is a *canonical Ω -algebra*.

For a symmetric and transitive relation $R \subseteq A^2$, $\text{dom } R := \{x \in A \mid (x, x) \in R\}$. By the construction in the proof of Theorem 2.11 [12], we get:

Corollary 2.12. *Let $\mathcal{A}$ be an algebra and $\mathcal{R}$ a closure system in $\text{Con}_w(\mathcal{A})$ fulfilling condition (2.7). Let also $\mathcal{F}$ be a set of identities in the language of $\mathcal{A}$ and suppose that for every $R \in \mathcal{R}$, the algebra $\text{dom } R/R$ fulfills these identities. Then there is a complete lattice Ω and an Ω -algebra $(\mathcal{A}, E)$, such that $\mathcal{R}$ consists of cuts of E and $(\mathcal{A}, E)$ satisfies $\mathcal{F}$.*

Let $(\mathcal{A}, E1)$ and $(\mathcal{A}, E2)$ be an Ω_1 -valued algebra and an Ω_2 -valued algebra respectively. We say that the structures $(\mathcal{A}, E1)$ and $(\mathcal{A}, E2)$ are *cut-equivalent* if their collections of quotient algebras over cuts of $E1$ and $E2$ coincide, i.e., if for every $p \in \Omega_1$ there is $q \in \Omega_2$ such that $\mu 1_p / E1_p = \mu 2_q / E2_q$ and vice versa.

Theorem 2.13 ([12]). *Let $\bar{\mathcal{A}} = (\mathcal{A}, E)$ be an Ω -algebra where Ω is an arbitrary complete lattice. Then there is a lattice and a lattice-valued algebra cut-equivalent with $\bar{\mathcal{A}}$, obtained by the canonical construction over $\mathcal{A}$.*

Shortly, the Ω -algebra $(\mathcal{A}, E)$ with the collection $\{E_p \mid p \in \Omega\}$ of cuts of E is fixed. Then a new lattice $\Omega 1$ is given by $\Omega 1 = (\{E_p \mid p \in \Omega\}, \supseteq)$ and $E1: A^2 \rightarrow \Omega 1$ is defined as $E1(x, y) := \bigcap \{R \in \Omega 1 \mid (x, y) \in R\}$. Then $(\mathcal{A}, E)$ and $(\mathcal{A}, E1)$ are cut-equivalent.

2.2 Skew lattices

For basic notions related to skew lattices given in the sequel, see [10, 11].

An idempotent semigroup (associative groupoid) $(S, \cdot)$ is a *band*, i.e., it fulfills identities $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ and $x \cdot x = x$. A commutative band is a *semilattice*. A *rectangular band* is a band that satisfies $x \cdot y \cdot x = x$.

A *skew lattice* is an algebra $(S, \vee, \wedge)$ with two binary *associative* operations, satisfying the *absorption laws*:

$$\begin{aligned} x \wedge (x \vee y) &= x = (y \vee x) \wedge x, \\ x \vee (x \wedge y) &= x = (y \wedge x) \vee x. \end{aligned}$$

Duality principle clearly holds, as for lattices. Due to absorption laws, both operations are *idempotent*: $x \vee x = x$ and $x \wedge x = x$. Hence, the reduct algebras $(S, \vee)$ and $(S, \wedge)$ of a skew lattice are bands. In addition, in a skew lattice

$$\begin{aligned} x \wedge y &= x \quad \text{if and only if} \quad x \vee y = y \quad \text{and} \\ x \wedge y &= y \quad \text{if and only if} \quad x \vee y = x. \end{aligned}$$

Therefore, the natural partial order in a skew lattice S is defined as follows:

$$\begin{aligned} x \leq y \quad &\text{if and only if} \quad x \wedge y = x = y \wedge x \quad \text{or dually} \\ x \leq y \quad &\text{if and only if} \quad x \vee y = y = y \vee x. \end{aligned}$$

Clearly, lattices are skew lattices. A *rectangular skew lattice* is a skew lattice in which both band-reducts $(S, \wedge)$ and $(S, \vee)$ are rectangular bands and $x \wedge y = y \vee x$ for all $x, y \in S$.

Proposition 2.14 ([11]). *Let $(S, \vee, \wedge)$ be a skew lattice and θ a relation on S given by*

$$x \theta y \quad \text{if} \quad x \wedge y \wedge x = x \text{ and } y \wedge x \wedge y = y. \quad (2.9)$$

Then θ is a congruence relation on $(S, \vee, \wedge)$ and S/θ is the maximal lattice image of S . In addition, all congruence classes of θ are maximal rectangular skew lattices - subalgebras of S .

If $(S, \vee, \wedge)$ is a rectangular skew lattice, then clearly θ is A^2 and the factor lattice has a single class.

3 Ω -lattices from skew lattices

In this part we investigate Ω -lattices with the underlying algebras being skew lattices.

Following Theorem 2.13, we assume that all the Ω -algebras $(\mathcal{A}, E)$ are canonical, i.e., that the lattice Ω is a closure system in the weak congruence lattice $\text{Con}_w(\mathcal{A})$ ordered dually to inclusion. Therefore, for all $p \in \Omega$, we have that $E_p = p$, and by the definition (2.8),

$$E(x, y) = \bigcap \{R \in \Omega \mid (x, y) \in R\} \quad \text{for all } x, y \in A.$$

We start with a definition related to canonical Ω -algebras in general. Let $\mathcal{A}$ be an algebra and $\mathcal{J}$ a set of identities in the language of $\mathcal{A}$. We say that $\mathcal{A}$ is Ω -simple with respect to $\mathcal{J}$ if the only weak congruences in $\text{Con}_w(\mathcal{A})$ such that all the identities in $\mathcal{J}$ hold in the quotient structures over these congruences are squares. Consequently, we say that $\mathcal{A}$ is Ω -regular with respect to $\mathcal{J}$ if it is not Ω -simple with respect to $\mathcal{J}$. A straightforward consequence of this definition is:

Proposition 3.1. *If $\mathcal{A}$ is an Ω -simple algebra with respect to the set $\mathcal{J}$ of identities, then there is an Ω -algebra $(\mathcal{A}, E)$ fulfilling all the identities in $\mathcal{J}$.*

By Proposition 2.14, a skew lattice possesses a congruence with respect to which the corresponding factor algebra is the greatest factor lattice and all the cosets are non-commutative. In case of rectangular skew lattices, this congruence is the square of the skew lattice and the corresponding factor lattice has one element. We use these facts in the sequel.

Proposition 3.2. *A rectangular skew lattice is an Ω -simple algebra with respect to commutative identities for both operations.*

Corollary 3.3. *Let $\mathcal{A} = (A, \wedge, \vee)$ be a rectangular skew lattice, let Ω be the closure system on the weak congruence lattice $\text{Con}_w(\mathcal{A})$. Then, for the Ω -equality relation $E: A^2 \rightarrow \Omega$, $(\mathcal{A}, E)$ is an Ω -lattice if and only if Ω consists of squares of algebras.*

Proof. Since $\mathcal{A}$ is a rectangular skew lattice, also all the subalgebras are rectangular skew lattices. Hence, by the fact explained above, all the weak-congruences in the closure system that form Ω are squares of algebras. The proof now follows by Proposition 3.2. $\square$

Corollary 3.4. *For every complete and atomic Boolean algebra Ω , there is a rectangular skew lattice $\mathcal{A} = (A, \wedge, \vee)$, such that $(\mathcal{A}, E)$ is a canonical Ω -lattice.*

Proof. Let Ω be a complete and atomic Boolean algebra isomorphic to 2^A , where A is a set and let a rectangular skew lattice $\mathcal{A} = (A, \wedge, \vee)$ be defined by $x \wedge y = x$ and $x \vee y = y$. Then, all the weak equivalence relations are weak congruences and all the subsets are subuniverse lattices. Let $\text{Con}_w(\mathcal{A})$ be such a weak congruence lattice. Any closure system in this lattice in which all factor algebras (related to weak congruences) are commutative makes an Ω -lattice. The subset of all squares of subalgebras (which are also weak congruences) is closed under intersections. Indeed, $\emptyset$ is also a weak congruence, since there is no nullary operations in the similarity type of skew lattices. In case the family of subalgebras $\{B_i \mid i \in I\}$ has a nonempty intersection, then $\bigcap_{i \in I} B_i^2 = (\bigcap_{i \in I} B_i)^2$. By Proposition 3.2, if Ω consists of all squares, then it is an Ω -lattice. The lattice of squares of subuniverses is a sublattice of the weak congruence lattice and it is isomorphic to the subuniverse lattice which is isomorphic to the Boolean algebra Ω . $\square$

Let $\mathcal{A} = (A, \wedge, \vee)$ be a skew lattice. As mentioned above, there is a special congruence θ on $\mathcal{A}$, called *natural equivalence*, such that the following holds: the quotient algebra $\mathcal{A}/\theta$ is the maximal lattice image of $\mathcal{A}$, and all congruence classes of θ are maximal rectangular subalgebras of $\mathcal{A}$. This congruence is defined by $x \theta y$ iff both $x \vee y \vee x = x$ and $y \vee x \vee y = y$, or, dually, $x \theta y$ iff $x \wedge y \wedge x = x$ and $y \wedge x \wedge y = y$ (see [11]).

Now, if $\mathcal{B}$ is a subalgebra of $\mathcal{A}$, let us denote the restriction of θ on $\mathcal{B}$ by $\theta_{\mathcal{B}}$, that is $\theta_{\mathcal{B}} = \theta \cap \mathcal{B}^2$. Then $\theta_{\mathcal{B}}$ is a congruence on $\mathcal{B}$, and moreover, it is exactly the natural equivalence on $\mathcal{B}$ (since $\forall x, y \in \mathcal{B}: (x, y) \in \theta_{\mathcal{B}}$ iff $(x, y) \in \theta$ iff $x \vee y \vee x = x, y \vee x \vee y = y$). This means that $\mathcal{B}/\theta_{\mathcal{B}}$ is the maximal lattice image of $\mathcal{B}$, and all congruence classes of $\theta_{\mathcal{B}}$ are maximal rectangular subalgebras of $\mathcal{B}$.

It is easy to see that the set $\mathcal{R} = \{\theta_{\mathcal{B}} \mid \mathcal{B} \in \text{Sub}(\mathcal{A})\} \cup \{A^2\}$ is a closure system in $\text{Con}_w(\mathcal{A})$, therefore a complete lattice with respect to $\subseteq$ and its dual, $\supseteq$.

Indeed, let $\{\theta_{\mathcal{B}_i} \mid i \in I\}$ be a nonempty family in $\text{Sub } \mathcal{A}$. Then,

$$\bigcap_{i \in I} \theta_{\mathcal{B}_i} = \bigcap_{i \in I} (\theta \cap \mathcal{B}_i^2) = \theta \cap \bigcap_{i \in I} \mathcal{B}_i^2 = \theta_{\mathcal{B}'},$$

where

$$\mathcal{B}' = \bigcap_{i \in I} \mathcal{B}_i \in \text{Sub}(\mathcal{A}).$$

Intersection of empty family is by the definition A^2 , and it belongs to the $\mathcal{R}$ by construction. Therefore $\mathcal{R}$ is a closure system in the weak congruence lattice.

Theorem 3.5. *Let $\mathcal{A} = (A, \wedge, \vee)$ be a skew lattice and $\Omega = (\mathcal{R}, \supseteq)$ be the closure system $\mathcal{R}$ defined above, ordered by the dual of inclusion, $\supseteq$. Then, the Ω -skew lattice $\bar{\mathcal{A}}$ obtained by the canonical construction over $\mathcal{A}$, is an Ω -lattice.*

Proof. Let us notice that for each $a \in A$, $(\{a\}, \wedge, \vee)$ is a subalgebra, due to the fact that operations are idempotent. So,

$$\bigcap \{R \in \Omega \mid (a, a) \in R\} = \{(a, a)\},$$

which means that the closure system satisfies condition (2.7) from Theorem 2.11. If we define $E: A^2 \rightarrow \Omega$ by $E(a, b) = \bigcap \{R \in \Omega \mid (a, b) \in R\}$ for all $a, b \in A$, then the cuts of E are exactly the elements of Ω , that is, for each $R \in \Omega$, $E_R = R$. By construction, $\mathcal{R} = \{\theta_{\mathcal{B}} \mid \mathcal{B} \in \text{Sub}(\mathcal{A})\} \cup \{A^2\}$, so $E_{\theta_{\mathcal{B}}} = \theta_{\mathcal{B}}$, for each $\mathcal{B} \in \text{Sub}(\mathcal{A})$. Also, by

Theorem 2.11, Ω -valued relation E is an Ω -valued equality compatible with fundamental operations of $\mathcal{A}$, which means that $\overline{\mathcal{A}} = (\mathcal{A}, E)$ is an Ω -skew lattice.

Furthermore, $E_{\theta_{\mathcal{B}}}$ is a congruence on subalgebra $\mu_{\theta_{\mathcal{B}}}$ of $\mathcal{A}$, by Proposition 2.4, and also a congruence on subalgebra $\mathcal{B}$, by the construction (for all $\theta_{\mathcal{B}} \in \Omega$, that is for all $\mathcal{B} \in \text{Sub } \mathcal{A}$). This means $\mathcal{B} = \mu_{\theta_{\mathcal{B}}} = \{x \in A \mid (x, x) \in E_{\theta_{\mathcal{B}}}\}$, for all $\mathcal{B} \in \text{Sub } \mathcal{A}$.

So, quotient algebras $\mu_{\theta_{\mathcal{B}}}/E_{\theta_{\mathcal{B}}}$ are, in fact $\mathcal{B}/\theta_{\mathcal{B}}$. As mentioned earlier, for each $\mathcal{B}$, $\theta_{\mathcal{B}}$ is a natural equivalence on $\mathcal{B}$, so the quotient $\mathcal{B}/\theta_{\mathcal{B}}$ is the maximal lattice image of $\mathcal{B}$, therefore a lattice. By the Theorem 2.6, the Ω -skew lattice $\overline{\mathcal{A}} = (\mathcal{A}, E)$ is an Ω -lattice. $\square$

Remark 3.6. If $\mathcal{A}$ is a rectangular skew lattice, then the congruence θ is full, and so are all $\theta_{\mathcal{B}}$. The quotients are therefore trivial, and the obtained Ω -algebra is an Ω -lattice (as already proven). Let us notice that the Ω -lattice in this case satisfies every lattice-identity, since all the quotients do.

Remark 3.7. A skew lattice is called *primitive* if its congruence θ has exactly two blocks. If $\mathcal{A}$ is a primitive skew lattice, then its subalgebras are either rectangular or primitive. The reason for this is the following: the blocks themselves are rectangular subalgebras, and so are all subalgebras contained in either of the blocks. If a subalgebra is not contained in a block, the corresponding restriction of θ also has two blocks.

This means that the quotient algebras from the construction explained above are either trivial or two-element lattices $(\{0, 1\}, \wedge, \vee)$. The Ω -lattice, therefore, satisfies every identity satisfied by the two element lattice.

Example 3.8. We start from a three element skew lattice $\mathcal{A} = (A, \wedge, \vee)$, where $A = \{a, b, 0\}$, and the operations are defined by: $0 \wedge x = 0 = x \wedge 0$, $0 \vee x = x = x \vee 0$, $a \wedge b = b$, $b \wedge a = a$, $a \vee b = a$, $b \vee a = b$. Now, we define an Ω -lattice, where Ω is a lattice anti-isomorphic to the closure system on the weak congruence lattice of skew lattice $\mathcal{A}$. The lattice Ω is given in Figure 1. The canonical Ω -algebra $(\mathcal{A}, E)$ is defined as follows:

$$E(x, y) := \bigcap \{R \in \Omega \mid (x, y) \in R\} \quad \text{for all } x, y \in A.$$

Figure 1: The lattice Ω .

This mapping is well defined since Ω is closed under intersection, and since Ω is canonical, for every $p \in \Omega$, $E_p = p$.

Table 1: The equality E and the related function μ .

(a)				(b)			
E	0	a	b	$\mu = \begin{pmatrix} 0 & a & b \\ 0 & a & b \end{pmatrix}$			
0	0	$a0$	$b0$				
a	$a0$	a	ab				
b	$b0$	ab	b				

The equality E is given in Table 1, together with its diagonal, compatible function μ .

The cuts of μ are all subsets except $\{a, b\}$. The cuts of E are all diagonal relations on all cuts of μ , all squares of all subalgebras and $E_{ab,0}$ has two classes. Hence, the factor algebras μ_p/E_p are either one-element lattices or two element lattices. Hence, $(\mathcal{A}, E)$ is an Ω -lattice.

Example 3.9. Now we start from a rectangular skew lattice with four elements: $A = L \times R$, where $L = \{a, b\}$, $R = \{c, d\}$. The operation tables are given in Table 2(a) and 2(b). Here and in Figure 2 and Table 3 below, ordered couples are denoted ac, ad, bc, bd just for simplicity.

Table 2: The operation tables of the rectangular skew lattice.

(a)					(b)				
$\vee$	ac	ad	bc	bd	$\wedge$	ac	ad	bc	bd
ac	ac	ac	bc	bc	ac	ac	ad	ac	ad
ad	ad	ad	bd	bd	ad	ac	ad	ac	ad
bc	ac	ac	bc	bc	bc	bc	bd	bc	bd
bd	ad	ad	bd	bd	bd	bc	bd	bc	bd

Let Ω be the lattice given by the diagram in Figure 2. Ω is a lattice anti-isomorphic to the closure system on the weak congruence lattice of the skew lattice $\mathcal{A}$. Some of the subalgebras of $\mathcal{A}$ (appearing in $\text{Con}_w(\mathcal{A})$) are denoted by: $H = \{bd, ad\}$, $D = \{ac, bc\}$, $C = \{bc, bd\}$, $B = \{ac, ad\}$.

Let the Ω -equality and the related function μ be given in Table 3.

Table 3: The Ω -equality and the related function μ .

(a)					(b)				
E	ac	ad	bc	bd	$\mu = \begin{pmatrix} ac & ad & bc & bd \\ ac & ad & bc & bd \end{pmatrix}$				
ac	ac	B^2	D^2	A^2					
ad	B^2	ad	A^2	H^2					
bc	D^2	A^2	bc	C^2					
bd	A^2	H^2	C^2	bd

Figure 2: The lattice Ω .

Now we see that all the structures μ_z/E_z , $z \in \Omega$ are one-element, hence lattices. Hence, $(\mathcal{R} \times \mathcal{L}, E)$ is an Ω -lattice.

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