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# Product irregularity strength of graphs with small clique cover number*

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**Abstract**

For a graph $X$ without isolated vertices and without isolated edges, a *product-irregular labelling* $\omega : E(X) \to \{1, 2, \ldots, s\}$, first defined by Anholcer in 2009, is a labelling of the edges of $X$ such that for any two distinct vertices $u$ and $v$ of $X$ the product of labels of the edges incident with $u$ is different from the product of labels of the edges incident with $v$. The minimal $s$ for which there exists a product irregular labeling is called *the product irregularity strength* of $X$ and is denoted by $ps(X)$. *Clique cover number* of a graph is the minimum number of cliques that partition its vertex-set. In this paper we prove that connected graphs with clique cover number 2 or 3 have the product-irregularity strength equal to 3, with some small exceptions.

*Keywords: Product irregularity strength, clique-cover number.*

*Math. Subj. Class.: 05C15, 05C70, 05C78*

## 1 Introduction

Throughout this paper let $X$ be a simple graph, that is, a graph without loops or multiple edges, without isolated vertices and without isolated edges. Let $V(X)$ and $E(X)$ denote the vertex set and the edge set of $X$, respectively. Let $\omega : E(X) \to \{1, 2, \ldots, s\}$ be an integer labelling of the edges of $X$. Then the *product degree* $pd_X(v)$ of a vertex $v \in V(X)$ in the graph $X$ with respect to the labelling $\omega$ is defined by

$$pd_X(v) = \prod_{v \in e} \omega(e).$$

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If the graph $X$ is clear from the context, then we will simply use $pd(v)$. A labelling $\omega$ is said to be *product-irregular*, if any two distinct vertices $u$ and $v$ of $X$ have different corresponding product degrees, that is, $pd_X(u) \neq pd_X(v)$ for any $u$ and $v$ in $V(X)$ $(u \neq v)$. The *product irregularity strength* $ps(X)$ of $X$ is the smallest positive integer $s$ for which there exists a product-irregular labelling $\omega : E(X) \to \{1, 2, \dots, s\}$.

This concept was first introduced by Anholcer in [1] as a multiplicative version of the well-studied concept of irregularity strength of graphs introduced by Chartrand et al. in [4] and studied later quite extensively (see for example [3, 7, 8, 11]). A concept similar to product-irregular labelling is the *product anti-magic labeling* of a graph, where it is required that the labeling $\omega$ is bijective (see [9, 12]). It is clear that every product anti-magic labeling is product-irregular. Another related concept is the so-called *multiplicative vertex-colouring* (see [13, 14]), where it is required that $pd(u) \neq pd(v)$ for every pair of adjacent vertices $u$ and $v$, while non-adjacent vertices can have the same product degrees. It is easy to see that every product-irregular labelling is a multiplicative vertex-colouring.

In [1] Anholcer gave upper and lower bounds on product irregularity strength of graphs. The main results in [1] are estimates for product irregularity strength of cycles, in particular it was proved that for every $n > 2$

$$ps(C_n) \geq \lceil \sqrt{2n} - \frac{1}{2} \rceil,$$

and that for every $\varepsilon > 0$ there exists $n_0$ such that for every $n \geq n_0$

$$ps(C_n) \leq \lceil (1 + \varepsilon)\sqrt{2n} \ln n \rceil.$$

Anholcer in [2] considered product irregularity strength of complete bipartite graphs and forests. Anholcer proved that for two integers $m$ and $n$ such that $2 \geq m \geq n$ it holds $ps(K_{m,n}) = 3$ if and only if $n \geq \binom{m+2}{2}$. The main result in [2] is about product irregularity strength of almost all forests $F$ such that $\triangle(F) = D$ for arbitrary integer $D \geq 3$, $n_2 = 0$, $n_0 \leq 0$ and $n_2 = 0$ of the forest $F$ with all pendant edges removed, where $n_d$ denotes the number of vertices of degree $d$. Anholcer proved that in this case $ps(F) = n_1$.

In [5], Darda and Hujdurović proved that for any graph $X$ of order at least 4 with at most one isolated vertex and without isolated edges we have $ps(X) \leq |V(X)| - 1$. Connections between product irregularity strength of graphs and multidimensional multiplication table problem was established, see [6, 10] for some results on multidimensional multiplication problem.

It is easy to see that the lower bound for the product irregularity strength of any graph is 3. In this paper we will give some sufficient conditions for a graph to have product irregularity strength equal to 3. In particular we will prove that graphs of order at least 3 with clique-cover number 2 have product irregularity strength 3 (see Corollary 3.5), where *clique cover number* of a graph is the minimum number of cliques that partition the vertex set of the graph. Moreover, we will prove that for a connected graph such that its vertex set can be partitioned into 3 cliques of sizes at least 4 then its product irregularity strength is 3 (see Corollary 4.14).

The paper is organized as follows. In section 2 we rephrase the definition of product-irregular labellings in terms of the corresponding weighted adjacency matrices and give some constructions that will be used for proving our main results. In section 3 we will determine the product irregularity strength of graphs with clique cover number 2, while in section 4 we study product irregularity strength of graphs with clique cover number 3.

## 2 Product-irregular matrices

In this section we will rephrase the definition of product irregular labelling of graphs using weighted adjacency matrices. We start with the definition of a weighted adjacency matrix.

**Definition 2.1.** Let $w$ be an integer labelling of the edges of a graph $X$ of order n with $V(X) = \{v_1, v_2, \ldots, v_n\}$. Weighted adjacency matrix of X is $n \times n$ matrix $M$ where $M_{ij} = w(\{v_i, v_j\})$ if $v_i$ and $v_j$ are adjacent and $M_{ij} = 0$ otherwise.

**Definition 2.2** (Product-irregular matrices and product degree for matrices)**.** Assume that we have weighted adjacency $n \times n$ matrix $M$ ($n \geq 2$). Then for a $k$-th row of a matrix $M$, denoted $M_k$, define $pd(M_k) := \prod_{M_{k,i} \neq 0} M_{k,i}$ to be the product of all non-zero elements of the $k$-th row. We say that $M$ is product-irregular if $\forall i, j \in \{1, 2, \ldots, n\}$ for $i \neq j$ $pd(M_i) \neq pd(M_j)$. We will work with matrices with entries $a_{ij} \in \{0, 1, 2, 3\}$ therefore to simplify reading for a row $v$ from matrix $M$ if $pd(v) = 2^a \cdot 3^b$ then we will use notation $pd(v) := (a, b)$. Also define $pd(v)[1] := a$ and $pd(v)[2] := b$.

**Observation 2.3.** A graph labelling is product-irregular if and only if the corresponding weighted adjacency matrix is product-irregular.

Let $n \geq 4$ and let $M_n(x, y, z)$ be $n \times n$ matrix such that $M_n(x, y, z) = (m_{ij})$ where

$$m_{ij} = \begin{cases} 0, & \text{if } i = j \\ x, & \text{if } j \leq n - i + 1 \text{ and } i \neq j \\ z, & \text{if } (i, j) = (k, n) \text{ or } (i, j) = (n, k) \text{ for } k = \lceil \frac{n}{2} \rceil + 1 \\ y, & \text{otherwise} \end{cases}$$

For example:

$$M_7(x, y, z) = \begin{pmatrix} 0 & x & x & x & x & x & x \\ x & 0 & x & x & x & x & y \\ x & x & 0 & x & x & y & y \\ x & x & x & 0 & y & y & y \\ x & x & x & y & 0 & y & z \\ x & x & y & y & y & 0 & y \\ x & y & y & y & z & y & 0 \end{pmatrix}. \tag{2.1}$$

We will denote with $A \oplus B$ the direct sum of matrices $A$ and $B$, that is

$$A \oplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix},$$

where 0 denotes the zero matrix of appropriate size.

### 2.1 Properties of $M_n$

Let $x_i$, $y_i$ and $z_i$ be the number of $x$, $y$ and $z$ respectively appearing in the $i$-th row of matrix $M_n(x, y, z)$. For fixed $n$ with $k$ we denote $k := \lceil \frac{n}{2} \rceil + 1$. Then the rows of the matrix $M_n(x, y, z)$ can be separated into 3 types:

1st type: $(x_k, y_k, z_k) = (\lceil \frac{n-1}{2} \rceil, \lceil \frac{n}{2} \rceil - 2, 1)$, 2nd type: $(x_i, y_i, z_i) = (n - i, i - 1, 0)$ for $i < k$ and $(x_i, y_i, z_i) = (n - i + 1, i - 2, 0)$ for $n > i > k$,

3rd type: $(x_n, y_n, z_n) = (1, n - 3, 1)$.

We denote by $m_{(i)}(M)$ a row of type $i$ for $i \in \{1, 2, 3\}$ of matrix $M$, where $M$ is matrix $M_n(x, y, z)$ (if the matrix $M_n(x, y, z)$ is clear from the context, then we will simply use $m_{(i)}$). We start by proving the following nice property of matrix $M_n$.

**Proposition 2.4.** *If $\{x, y, z\}$ is a set of distinct pairwise relatively prime integers, then $M_n(x, y, z)$ is product irregular matrix for any $n \geq 4$.*

*Proof.* Suppose contrary, that is there exist $m_i$ and $m_j$ (that are rows of matrix $M_n(x, y, z)$) for some $i \neq j$ such that $pd(m_i) = pd(m_j)$. There are 3 types of rows therefore it is enough to check the equality above not for all rows, but for all types of rows. Observe that for every $i \in \{1, 2, \dots, n\}$ the sum $x_i + y_i + z_i = n - 1$ and $pd(m_i) = pd(m_j)$ for some $i \neq j$ if and only if $x_i = x_j$, $y_i = y_j$ and $z_i = z_j$. It follows that:

1. If $pd(m_{(1)}) = pd(m_{(3)})$ then $(\lceil \frac{n-1}{2} \rceil, \lceil \frac{n}{2} \rceil - 2, 1) = (1, n - 3, 1)$, so $n = 3$ which is a contradiction.
2. Since rows of second type have value 0 at 3rd coordinate and rows of first and third types have value 1 at 3rd coordinate, then $pd(m_{(2)}) \neq pd(m_{(i)})$ for $i \in \{1, 3\}$.
3. It is clear that $(x_i, y_i, z_i) \neq (x_j, y_j, z_j)$ for $i < k$ and $k < j < n$ i.e. product degrees of different rows of type 2 are different.

We were considering different rows, that means we did not have to consider $pd(m_{(i)}) = pd(m_{(i)})$ for every $i \in \{1, 3\}$. □

We will define 3 matrices of class $M_n(x, y, z)$ for specific $x, y$ and $z$. Assign matrix $A_n := M_n(1, 2, 3)$, $B_n := M_n(2, 3, 1)$ and $C_n := M_n(3, 1, 2)$.

### 2.2 Properties of $A_n \oplus B_m$

**Lemma 2.5.** *For every $m \geq n \geq 4$, $A_n \oplus B_m$ is product irregular if $(n, m) \notin \{(4, 4), (5, 5), (6, 6)\}$.*

*Proof.* Suppose contrary, that is there exist $a_i$ and $b_j$ (that are rows of matrices $A_n$ and $B_m$ respectively) for some i and j such that $pd(a_i) = pd(b_j)$. There are 3 types of rows therefore it is enough to check all of the 9 possibilities for different types of rows:

1. If $pd(a_{(1)}) = pd(b_{(1)})$ then $(\lceil \frac{n}{2} \rceil - 2, 1) = (\lceil \frac{m-1}{2} \rceil, \lceil \frac{m}{2} \rceil - 2)$ which contradicts with $m \geq n$.
2. If $pd(a_{(1)}) = pd(b_{(2)})$ then $(\lceil \frac{n}{2} \rceil - 2, 1) = (m - j, j - 1)$ or $(\lceil \frac{n}{2} \rceil - 2, 1) = (m - j + 1, j - 2)$ which contradicts with $m \geq n \geq 4$.
3. If $pd(a_{(1)}) = pd(b_{(3)})$ then $(\lceil \frac{n}{2} \rceil - 2, 1) = (1, m - 3)$, so $(n, m) = (5, 4)$ or $(n, m) = (6, 4)$ which contradicts with $m \geq n$.
4. If $pd(a_{(2)}) = pd(b_{(1)})$ then $(i - 1, 0) = (\lceil \frac{m-1}{2} \rceil, \lceil \frac{m}{2} \rceil - 2)$ or $(i - 2, 0) = (\lceil \frac{m-1}{2} \rceil, \lceil \frac{m}{2} \rceil - 2)$, so $m = 3$ or $m = 4$, thus $m = n = 4$ which is a contradiction.

5. If $pd(a_{(2)}) = pd(b_{(2)})$ then we have that in both possible cases $((i-1, 0 = (m - j, j-1))$ and $(i-2, 0) = (m-j+1, j-2))$ we get $i = m \geq n$ which is a contradiction.
6. If $pd(a_{(2)}) = pd(b_{(3)})$ then $pd(a_{(2)})[2] = 0$ and $pd(b_{(3)})[2] > 0$ which is a contradiction.
7. If $pd(a_{(3)}) = pd(b_{(1)})$ then $(n-3, 1) = (\lceil \frac{m-1}{2} \rceil, \lceil \frac{m}{2} \rceil - 2)$, so $(n, m) = (5, 5)$ or $(n, m) = (6, 6)$ which is a contradiction.
8. If $pd(a_{(3)}) = pd(b_{(2)})$ then $(n-3, 1) = (m-j, j-1)$ or $(n-3, 1) = (m-j+1, j-2)$, so $n > m$ in both cases which is a contradiction.
9. If $pd(a_{(3)}) = pd(b_{(3)})$ then $(n-3, 1) = (1, m-3)$, so $(n, m) = (4, 4)$ which is a contradiction.

This finishes the proof. □

For the next lemma we need to consider weighted adjacency matrix

$$T := \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 2 & 3 & 0 \end{pmatrix} \tag{2.2}$$

Observe that $pd(T_1) = (1, 0)$, $pd(T_2) = (0, 1)$, $pd(T_3) = (1, 1) \Rightarrow ps(K_3) = 3$.

**Lemma 2.6.** *Let $T$ be the matrix defined in (2.2). For every $n \geq 5$ $T \oplus B_n$ is product irregular.*

*Proof.* Observe that $\{pd(T_i) : i \in \{1, 2, 3\}\} \subset \{pd((A_4)_i) : i \in \{1, 2, 3, 4\}\}$ and we know from Lemma 2.5 that $\forall n \geq 5$ $A_4 \oplus B_n$ is product irregular. □

## 3 Graphs with clique-cover number 2

In this section we consider product irregularity strength of connected graphs with clique cover number two. Suppose that $G$ is a graph with clique-cover number 2, that is the vertex set of $G$ can be partitioned into two cliques $C_1$ and $C_2$, of sizes $n$ and $m$ respectively. Then it follows that $G$ has a spanning subgraph isomorphic to $K_n + K_m$, where for two graphs $H_1$ and $H_2$, $H_1 + H_2$ denotes the disjoint union of $H_1$ and $H_2$. Then by [5, Lemma 1] it follows that $3 \leq ps(G) \leq ps(K_n + K_m)$. Hence we will start by considering product irregularity strength of $K_n + K_m$.

It can be proved that any $4 \times 4$ weighted adjacency matrix $M$ (with weights $1, 2$ and $3$) is product irregular if and only if there exist row $m \in M$ such that $pd(m) = (1, 1)$. Therefore $ps(K_4 + K_4) > 3$. There are a lot of graphs of the form $K_n + K_m$ for some integers $n$ and $m$ with product irregularity strength greater than 3. But since such graphs are disconnected, we will define operation of adding an edge between components of these graphs, i.e. we will consider minimal connected graphs with clique cover number 2.

**Definition 3.1** ($+edge$)**.** Let $G_1$ and $G_2$ be two graphs with disjoint vertex sets. With $G_1 + G_2 + edge$ we denote a graph obtained by taking disjoint union of $G_1$ and $G_2$ and adding an edge between two vertices of $G_1$ and $G_2$.

**Lemma 3.2.** $\forall n \geq 4$, $ps(K_2 + K_n + edge) = 3$.

*Proof.* Consider weighted adjacency $(n+2)\times(n+2)$ matrix

$$L = \begin{pmatrix} 0 & 1 & 3 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 \\ 3 & 0 & & & & \\ \vdots & \vdots & & & & \\ 0 & 0 & & B_n & & \\ \vdots & \vdots & & & & \\ 0 & 0 & & & & \end{pmatrix} \tag{3.1}$$

where $L_{1,3} = L_{3,1} = 3$. Clearly, $L$ is weighted adjacency matrix of the graph $K_2 + K_n$. We will show that $L$ is product-irregular. Since we have that $pd((B_n)_i) = pd(L_{i+2})$ for every $i \in \{2, 3, \dots n\}$ it is enough to show that product degrees of first 3 rows of matrix $L$ are different and do not belong to the set $\{pd((B_n)_i), i \in \{2, 3, \dots, n\}\}$.

1. It is clear that those rows are different and that first two rows of $L$ are not in the set $\{pd((B_n)_i), i \in \{2, 3, \dots, n\}\}$.
2. For the row $L_3$ we have that $pd(L_3) = pd((B_n)_1) + (0, 1) = (n-1, 1)$. Therefore $pd(L_3)[1] + pd(L_3)[2] = n - 1 + 1 > n - 1 \geq pd((B_n)_j)[1] + pd((B_n)_j)[2]$ for any $j \in \{2, 3, \dots, n\}$.

This finishes the proof. □

**Corollary 3.3.** *For every $n \geq 4$, $ps(K_1 + K_n + edge) = 3$.*

*Proof.* Consider matrix $L'$ obtained from matrix $L$ from (3.1) by deleting second row and column. Clearly, $L'$ is product-irregular. □

**Theorem 3.4.** *For every positive integers $n$ and $m$ such that $n + m > 2$ we have $ps(K_n + K_m + edge) = 3$.*

*Proof.* Consider some cases that were not covered by previous Lemmas:

(i) $ps(K_5 + K_5) = 3$. For proving this fact we can take direct sum of the following weighted adjacency matrices:

$$T_5 := \begin{pmatrix} 0 & 3 & 1 & 1 & 1 \\ 3 & 0 & 1 & 3 & 2 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 3 & 1 & 0 & 2 \\ 1 & 2 & 1 & 2 & 0 \end{pmatrix} \text{ and } \tilde{T_5} := \begin{pmatrix} 0 & 2 & 2 & 2 & 1 \\ 2 & 0 & 3 & 3 & 3 \\ 2 & 3 & 0 & 2 & 3 \\ 2 & 3 & 2 & 0 & 1 \\ 1 & 3 & 3 & 1 & 0 \end{pmatrix} \tag{3.2}$$

(ii) $ps(K_6 + K_6) = 3$. For proving this fact we can take direct sum of the following weighted adjacency matrices:

$$T_6 := \begin{pmatrix} 0 & 1 & 2 & 3 & 1 & 3 \\ 1 & 0 & 1 & 3 & 1 & 1 \\ 2 & 1 & 0 & 1 & 2 & 2 \\ 3 & 3 & 1 & 0 & 1 & 1 \\ 1 & 1 & 2 & 1 & 0 & 1 \\ 3 & 1 & 2 & 1 & 1 & 0 \end{pmatrix} \text{ and } \tilde{T_6} := \begin{pmatrix} 0 & 1 & 2 & 3 & 3 & 3 \\ 1 & 0 & 2 & 3 & 3 & 2 \\ 2 & 2 & 0 & 2 & 1 & 2 \\ 3 & 3 & 2 & 0 & 3 & 1 \\ 3 & 3 & 1 & 3 & 0 & 3 \\ 3 & 2 & 2 & 1 & 3 & 0 \end{pmatrix} \tag{3.3}$$

Also consider some cases that could not be proved without adding edges between cliques.

(iii) $ps(K_4 + K_4 + edge) = 3$. For proving this fact we will consider the following product-irregular matrix:

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 2 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 3 & 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 2 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 & 2 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 & 3 & 1 & 0 \end{pmatrix} \tag{3.4}$$

(iv) $ps(K_3 + K_4 + edge) = 3$. For proving this fact we will consider the following product-irregular matrix:

$$\begin{pmatrix} 0 & 1 & 2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 3 & 0 & 0 & 0 & 0 \\ 2 & 3 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 2 & 0 & 2 & 3 \\ 0 & 0 & 0 & 2 & 2 & 0 & 1 \\ 0 & 0 & 0 & 2 & 3 & 1 & 0 \end{pmatrix} \tag{3.5}$$

Observe that this matrix is obtained from matrix (3.4) by deleting first row and column.

(v) $ps(K_3 + K_3 + edge) = 3$. For proving this fact we will consider the following product-irregular matrix:

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 3 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 2 & 0 & 3 \\ 0 & 0 & 3 & 2 & 3 & 0 \end{pmatrix} \tag{3.6}$$

(vi) $ps(K_2 + K_3 + edge) = 3$. For proving this fact we will consider the following product-irregular matrix:

$$\begin{pmatrix} 0 & 3 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 0 & 3 \\ 0 & 3 & 2 & 3 & 0 \end{pmatrix} \tag{3.7}$$

Observe that this matrix is obtained from matrix (3.6) by deleting first row and column.

(vii) $ps(K_1 + K_3 + edge) = 3$. For proving this fact we will consider the following product-irregular matrix:

$$\begin{pmatrix} 0 & 0 & 0 & 3 \\ 0 & 0 & 2 & 2 \\ 0 & 2 & 0 & 3 \\ 3 & 2 & 3 & 0 \end{pmatrix} \tag{3.8}$$

Observe that this matrix is obtained from matrix (3.7) by deleting first row and column.

We are left with some trivial cases and it is straihtforward to check that $ps(K_2 + K_2 + edge) = ps(P_4) = 3$ and $ps(K_1 + K_2 + edge) = ps(P_3) = 3$.

The proof now follows by Lemmas 2.5, 2.6 and 3.2 and Corollary 3.3. □

**Corollary 3.5.** *If $G$ is a connected graph of order at least 3 with clique-cover number 2 then $ps(G) = 3$.*

Observe that $K_1 + K_1 + edge = P_2$ is an isolated edge, for which product-irregular labelling is not defined, i.e. 2 is the lower bound of the sum $n + m$ in Theorem 3.4.

# 4 Graphs with clique-cover number 3

In this section we consider the product irregularity strength of graphs with clique-cover number 3. Observe that a graph $G$ has clique cover number 3, if and only if its complement has chromatic number equal to 3. If $G$ is a graph with clique cover number 3, then its vertex set can be partitioned into three cliques, of sizes $n$, $m$ and $l$. Then it follows that $G$ has a spanning subgraph isomorphic to $K_n + K_m + K_l$, hence we will first investigate the product irregularity strength of such graphs.

## 4.1 Properties of $A_n \oplus B_m \oplus C_l$

**Lemma 4.1.** *For every $n \geq 7$ and $m \geq 4$, $A_n \oplus C_m$ is product irregular.*

*Proof.* Suppose contrary, that is $\exists a_i$ and $c_j$ (that are rows of matrices $A_n$ and $C_m$ respectively) for some i and j such that $pd(a_i) = pd(c_j)$. We will use the same type of proof as in the Lemma 2.5.

1. If $pd(a_{(1)}) = pd(c_{(1)})$ then $(\lceil \frac{n}{2} \rceil - 2, 1) = (1, \lceil \frac{m-1}{2} \rceil)$, so $n = 5$ or $n = 6$ and $m = 2$ or $m = 3$ which is a contradiction.
2. If $pd(a_{(1)}) = pd(c_{(2)})$ then $(\lceil \frac{n}{2} \rceil - 2, 1) = (0, m-j)$ or $(\lceil \frac{n}{2} \rceil - 2, 1) = (0, m-j+1)$. In both cases $n = 3$ or $n = 4$ which is a contradiction.
3. If $pd(a_{(1)}) = pd(c_{(3)})$ then $(\lceil \frac{n}{2} \rceil - 2, 1) = (1, 1)$, so $n = 5$ or $n = 6$ which is a contradiction.
4. If $pd(a_{(2)}) = pd(c_{(1)})$ then $(i - 1, 0) = (1, \lceil \frac{m-1}{2} \rceil)$ or $(i - 2, 0) = (1, \lceil \frac{m-1}{2} \rceil)$. In both cases $m = 1$ which is a contradiction.
5. For $pd(a_{(2)}) = pd(c_{(2)})$ we have that $pd(a_{(2)})[2] = 0$ and $pd(c_{(2)})[2] > 0$ which is a contradiction.
6. If $pd(a_{(2)}) = pd(c_{(3)})$ then $(i - 1, 0) = (1, 1)$ or $(i - 2, 0) = (1, 1))$ which is, clearly, a contradiction.

7. If $pd(a_{(3)}) = pd(c_{(1)})$ then $(n-3, 1) = (1, \lceil \frac{m-1}{2} \rceil)$, so $n = 4$ which is a contradiction.
8. If $pd(a_{(3)}) = pd(c_{(2)})$ then $(n-3, 1) = (0, m-j)$ or $(n-3, 1) = (0, m-j+1)$, so $n = 3$ which is a contradiction.
9. If $pd(a_{(3)}) = pd(c_{(3)})$ then $(n-3, 1) = (1, 1)$, so $n = 4$ which is a contradiction.

This finishes the proof. □

**Lemma 4.2.** *For every $n \geq m \geq 5$, $B_n \oplus C_m$ is product irregular if $(n, m) \notin \{(5, 5), (6, 6)\}$.*

*Proof.* Suppose contrary, that is there exist $b_i$ and $c_j$ (that are rows of matrices $B_n$ and $C_m$ respectively) for some i and j such that $pd(b_i) = pd(c_j)$. We will use the same type of proof as in the Lemma 2.5.

1. If $pd(b_{(1)}) = pd(c_{(1)})$ then $(\lceil \frac{n-1}{2} \rceil, \lceil \frac{n}{2} \rceil - 2) = (1, \lceil \frac{m-1}{2} \rceil)$, so $n = 2$ or $n = 3$ which is a contradiction.
2. If $pd(b_{(1)}) = pd(c_{(2)})$ then $(\lceil \frac{n-1}{2} \rceil, \lceil \frac{n}{2} \rceil - 2) = (0, m-j)$ or $(\lceil \frac{n-1}{2} \rceil, \lceil \frac{n}{2} \rceil - 2) = (0, m-j+1)$, so $n = 1$, a contradiction.
3. If $pd(b_{(1)}) = pd(c_{(3)})$ then $(\lceil \frac{n-1}{2} \rceil, \lceil \frac{n}{2} \rceil - 2) = (1, 1)$, so $n = 2$ or $n = 3$, a contradiction.
4. If $pd(b_{(2)}) = pd(c_{(1)})$ then $(n-i, i-1) = (1, \lceil \frac{m-1}{2} \rceil)$ or $(n-i+1, i-2) = (1, \lceil \frac{m-1}{2} \rceil)$. In the first case we have that $\lceil \frac{m-1}{2} \rceil = i - 1 = n - 2$, which implies that $2n - 4 \leq m \leq 2n - 3$, so, in particular, $2n - 4 \leq m \leq n$, therefore $n \leq 4$, a contradiction. In the second case we have that $n = i$ which is a contradiction.
5. For $pd(b_{(2)}) = pd(c_{(2)})$ we have that $pd(b_{(2)})[1] > 0$ and $pd(c_{(2)})[1] = 0$ which is a contradiction.
6. If $pd(b_{(2)}) = pd(c_{(3)})$ then $(n-i, i-1) = (1, 1)$ or $(n-i+1, i-2) = (1, 1)$, so $n = 3$ which is a contradiction.
7. If $pd(b_{(3)}) = pd(c_{(1)})$ then $(1, n-3) = (1, \lceil \frac{m-1}{2} \rceil)$, so $m = 2(n-3)$ or $m = 2(n-3)+1$ which is a contradiction because for $n \geq 7$ we have that $m > n$ and for $5 \leq n < 7$ we have that $(n, m) \in \{(5, 5), (6, 6)\}$.
8. If $pd(b_{(3)}) = pd(c_{(2)})$ then $(1, n-3) = (0, m-j)$ or $(1, n-3) = (0, m-j+1)$ which is a contradiction.
9. If $pd(b_{(3)}) = pd(c_{(3)})$ then $(1, n-3) = (1, 1)$, so $n = 4$ which is a contradiction.

This finishes the proof. □

**Theorem 4.3.** *For every $n$, $m$ and $l$ such that $m \geq l \geq n \geq 7$ $A_n \oplus B_m \oplus C_l$ is product irregular.*

*Proof.* Proof follows by Lemmas 2.5, 4.1 and 4.2. □

**Corollary 4.4.** *For all positive integers $n, m$ and $l$ greater than or equal to 7 it holds that $ps(K_n + K_m + K_l) = 3$.*

**Lemma 4.5.** *For all positive integers $n$ and $m$ greater than 6 and $k \in \{4, 5, 6\}$, $ps(K_n + K_m + K_k) = 3$.*

*Proof.* Let $m \geq n$ and consider matrix $A_n \oplus B_m \oplus C_k$. From Lemmas 2.5, 4.1 and 4.2 we can conclude that this matrix is product-irregular. $\square$

**Lemma 4.6.** *For all positive integer* $n \geq 7$ $ps(K_6 + K_6 + K_n) = 3$.

*Proof.* Consider $T_6 \oplus \tilde{T_6} \oplus B_n$ which is product-irregular because for every row $b$ of matrix $B_n$ $pd(b)[1] + pd(b)[2] \geq 5$, while for every row $t$ of matrices $T_6$ and $\tilde{T_6}$ we have that $pd(t)[1] + pd(t)[2] \leq 4$. $\square$

**Lemma 4.7.** *For all positive integer* $n \geq 7$ $ps(K_5 + K_6 + K_n) = 3$.

*Proof.* Consider the following matrix:

$$M := \begin{pmatrix} 0 & 2 & 2 & 2 & 1 & 1 \\ 2 & 0 & 3 & 3 & 3 & 1 \\ 2 & 3 & 0 & 2 & 3 & 1 \\ 2 & 3 & 2 & 0 & 1 & 2 \\ 1 & 3 & 3 & 1 & 0 & 1 \\ 1 & 1 & 1 & 2 & 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 3 & 1 & 1 & 1 \\ 3 & 0 & 1 & 3 & 2 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 3 & 1 & 0 & 2 \\ 1 & 2 & 1 & 2 & 0 \end{pmatrix} \oplus B_n \tag{4.1}$$

$M$ is product-irregular because for every row $b$ of matrix $B_n$ $pd(b)[1]+pd(b)[2] \geq 5$, while for every row $v$ of first two blocks of our matrix $M$ we have that $pd(v)[1] + pd(v)[2] \leq 4$. $\square$

**Lemma 4.8.** *For all positive integers* $n \geq 6$, $ps(K_5 + K_5 + K_n) = 3$.

*Proof.* Consider weighted adjacency matrices $T_5$ and $\tilde{T_5}$ from (3.2) in the first item of the proof of Theorem 3.4:

1. $\forall n \geq 7$ we have $T_5 \oplus \tilde{T_5} \oplus B_n$ is product irregular because for every row $b$ of matrix $B_n$ $pd(b)[1] + pd(b)[2] \geq 5$.
2. For $n = 6$ we have that $T_5 \oplus \tilde{T_5} \oplus P_6$ is product-irregular, where

$$P_6 := \begin{pmatrix} 0 & 2 & 2 & 2 & 2 & 1 \\ 2 & 0 & 2 & 2 & 2 & 3 \\ 2 & 2 & 0 & 2 & 3 & 3 \\ 2 & 2 & 2 & 0 & 3 & 1 \\ 2 & 2 & 3 & 3 & 0 & 3 \\ 1 & 3 & 3 & 1 & 3 & 0 \end{pmatrix}. \tag{4.2}$$

This finishes the proof. $\square$

Consider the graph $K_4 + K_4 + K_4$. Suppose that $ps(K_4 + K_4 + K_4) = 3$. Then there exist a product-irregular adjacency matrix $K$ of the form $K = P_1 \oplus P_2 \oplus P_3$ of our graph $K_4 + K_4 + K_4$, where $P_i$ is a product-irregular adjacency matrix of a graph $K_4$ for every $i \in \{1, 2, 3\}$. Therefore, we have that for every row $v$ of matrix $K$ $pd(v)[1]+pd(v)[2] < 4$. Also, it is clear that for every row $v$ of matrix $K$ we have $pd(v)[1] < 4$ and $pd(v)[2] < 4$. But there exist only 10 different pairs of the form $(x, y)$ such that $0 \leq x, y < 4$ and $x+y < 4$, which implies that there exist two rows $v$ and $u$ of matrix $K$ such that $pd(v) = pd(u)$. Therefore, $ps(K_4 + K_4 + K_4) > 3$.

There are a lot of graphs of the form $K_n + K_m + K_k$ for some integers $n$, $m$ and $k$ with product irregularity strength greater than 3. But since such graphs are disconnected, we will define operation of adding 2 edges between components of these graphs such that the resulting graph will be connected, i.e. we will consider minimal connected graphs with clique cover number 3.

**Definition 4.9** ($+2edges$)**.** Let $+2edges$ for graphs $G_1 + G_2 + G_3$ be the operation of adding edges, i.e. applying two times $+edge$ between any 2 different pairs of different sets $V(G_1), V(G_2)$ and $V(G_3)$. We will use the following notation for that operation: $G_1 + G_2 + G_3 + 2edges$.

Now we will describe this operation using matrix language. Consider weighted adjacency matrices $A, B, C$ of sizes $n\times n$, $m\times m$ and $l\times l$ respectively. Let $T_{12}(A, B, C, i, j, w)$ be $(n+m+l)\times(n+m+l)$ matrix with all zeros except elements with coordinates $(i, n+j)$ and $(n+j, i)$ of value $w$, where $1 \le i \le n$ and $1 \le j \le m$. In a similar way we can define matrices $T_{13}(A, B, C, i, j, w)$ and $T_{23}(A, B, C, i, j, w)$ for which coordinates of non-zero elements are $(i, n + m + j)$ and $(n + m + j, i)$, where $1 \le i \le n$ and $1 \le j \le l$ and $(n + i, n + m + j)$ and $(n + m + j, n + i)$, where $1 \le i \le m$ and $1 \le j \le l$ respectively.

For example one of the weighted adjacency matrices for graph $K_n+K_m+K_l+2edges$ where the edges between cliques are between vertices $a_i$ and $b_j$ of weight $w_1$ and between vertices $b_j$ and $c_k$ of weight $w_2$ where $a_i \in V(K_n), b_j \in V(K_m)$ and $c_k \in V(K_l)$ is $A_n \oplus B_m \oplus C_l + T_{12}(A_n, B_m, C_l, i, j, w_1) + T_{23}(A_n, B_m, C_l, j, k, w_2)$.

**Definition 4.10** (In-degree and in-edges)**.** Consider graph $G := G_1 + G_2 + G_3 + 2edges$. Let $G' := G_1 + G_2 + G_3$ be a subgraph of the graph $G$. Let $g \in V(G)$ and let $d_{G'}(g)$ be the degree of the vertex $g \in V(G')$. Then define *in-degree* of vertex $g \in V(G)$ to be $d^+(g) := d(g) - d_{G'}(g)$. We say that for some $i \in \{1, 2, 3\}$ $G_i$ has $t$ *in-edges* if and only if

$$\sum_{g \in V(G_i)} d^+(g) = t.$$

For the next theorem we will define the following matrix. Let $\tilde{M}_n(x, y) := M_n(x, y, y)$ and matrices $\tilde{A}_n, \tilde{B}_n$ and $\tilde{C}_n$ to be $\tilde{M}_n(1, 2), \tilde{M}_n(2, 3)$ and $\tilde{M}_n(3, 1)$ respectively.

**Theorem 4.11.** *For all positive integers n, m and l that are greater than or equal to* 5 *we have that* $ps(K_n + K_m + K_l + 2edges) = 3$.

*Proof.* Consider some cases that were not covered by previous Lemmas:

1. For $(n, m, l) = (6, 6, 6)$ consider the following product-irregular matrix:

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 3 & 1 & 1 & 2 \\ 1 & 3 & 0 & 1 & 2 & 2 \\ 1 & 1 & 1 & 0 & 2 & 2 \\ 1 & 1 & 2 & 2 & 0 & 2 \\ 1 & 2 & 2 & 2 & 2 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 2 & 2 & 2 & 2 & 2 \\ 2 & 0 & 1 & 2 & 2 & 3 \\ 2 & 1 & 0 & 2 & 3 & 3 \\ 2 & 2 & 2 & 0 & 3 & 3 \\ 2 & 2 & 3 & 3 & 0 & 3 \\ 2 & 3 & 3 & 3 & 3 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 3 & 3 & 3 & 3 & 3 \\ 3 & 0 & 2 & 3 & 3 & 1 \\ 3 & 2 & 0 & 3 & 1 & 1 \\ 3 & 3 & 3 & 0 & 1 & 1 \\ 3 & 3 & 1 & 1 & 0 & 1 \\ 3 & 1 & 1 & 1 & 1 & 0 \end{pmatrix} \tag{4.3}$$

2. For $(n, m, l) = (5, 6, 6)$ we can consider the same matrix as in (4.3) without first row (and column), i.e. without row (and column) $v$ such that $pd(v) = (0, 0)$.

3. For $(n, m, l) = (5, 5, 5)$ we will consider $\tilde{A}_5 \oplus \tilde{B}_5 \oplus \tilde{C}_5 + 2edges$. Let $\tilde{B}_5$ to have 2 in-edges, then we have:

   (1) If $\tilde{B}_5$ has 2 in-edges from one vertex, then we can take weighted adjacency matrix $\tilde{A}_5 \oplus \tilde{B}_5 \oplus \tilde{C}_5 + T_{12}(\tilde{A}_5, \tilde{B}_5, \tilde{C}_5, 3, 3, 3) + T_{23}(\tilde{A}_5, \tilde{B}_5, \tilde{C}_5, 3, 3, 2)$ which is product-irregular.

   (2) If $\tilde{B}_5$ has 2 in-edges from different vertices then we can take weighted adjacency matrix $\tilde{A}_5 \oplus \tilde{B}_5 \oplus \tilde{C}_5 + T_{12}(\tilde{A}_5, \tilde{B}_5, \tilde{C}_5, 3, 3, 3) + T_{23}(\tilde{A}_5, \tilde{B}_5, \tilde{C}_5, 1, 3, 2)$ which is product-irregular.

The proof now follows by the above argumentation, together with Theorem 4.3 and Lemmas 4.5, 4.6, 4.7 and 4.8. □

**Lemma 4.12.** *For all positive integers $n \geq 7$ and $m \in \{5, 6\}$ we have that $ps(K_4 + K_n + K_m) = 3$.*

*Proof.* Consider three different cases for different $m$:

1. For $m = 6$ and $n \geq 8$ consider matrix $A_4 \oplus B_6 \oplus B_n$ which is product-irregular using Theorem 3.4.
2. For $m = 6$ and $n = 7$ consider matrix $A_4 \oplus B_7 \oplus \tilde{T}_6$ which is product-irregular (where $\tilde{T}_6$ is defined in (3.3)).
3. For $m = 5$ consider matrix $A_4 \oplus B_n \oplus \tilde{T}_5$ which is product-irregular (where $\tilde{T}_5$ is defined in (3.2)).

□

**Theorem 4.13.** *For all positive integers n, m and l that are greater than or equal to 4 we have that $ps(K_n + K_m + K_l + 2edges) = 3$.*

*Proof.* Consider some cases that were not covered by previous Lemmas and Theorems:

1. For $(n, m, l) = (4, 5, 6)$ consider the following product-irregular matrix:

$$A_4 \oplus \begin{pmatrix} 0 & 2 & 2 & 2 & 1 \\ 2 & 0 & 3 & 1 & 3 \\ 2 & 3 & 0 & 2 & 3 \\ 2 & 1 & 2 & 0 & 1 \\ 1 & 3 & 3 & 1 & 0 \end{pmatrix} \oplus B_6 \tag{4.4}$$

   Notice that the second block of this matrix is obtained from $\tilde{T}_5$ from (3.2) by changing the values $t_{24}$ and $t_{42}$ from 3 to 1.

2. For $(n, m, l) = (4, 6, 6)$ consider the matrix $C_4 + A_6 + \tilde{T}_6$ which is product-irregular (where $\tilde{T}_6$ is defined in (3.3)).

Consider some cases for which we will add some edges between cliques:

3. For $(n, m, l) = (4, 5, 5)$ we will consider $\tilde{A}_4 \oplus \tilde{B}_5 \oplus \tilde{C}_5 + 2edges$.

   ($\tilde{B}_5$) For the case when $d^+(\tilde{B}_5) = 2$ we have two options:

   (1) If $\tilde{B}_5$ has 2 in-edges from one vertex, then we can take weighted adjacency matrix $\tilde{A}_4 \oplus \tilde{B}_5 \oplus \tilde{C}_5 + T_{12}(\tilde{A}_4, \tilde{B}_5, \tilde{C}_5, 2, 3, 3) + T_{23}(\tilde{A}_4, \tilde{B}_5, \tilde{C}_5, 3, 3, 2)$ which is product-irregular.

(2) If $\tilde{B}_5$ has 2 in-edges from different vertices then we can take weighted adjacency matrix $\tilde{A}_5 \oplus \tilde{B}_5 \oplus \tilde{C}_5 + T_{12}(\tilde{A}_4, \tilde{B}_5, \tilde{C}_5, 3, 3, 3) + T_{23}(\tilde{A}_4, \tilde{B}_5, \tilde{C}_5, 1, 3, 2)$ which is product-irregular.

($\tilde{A}_4$) For the case when $d^+(\tilde{A}_4) = 2$ we have two options:

(1) If $\tilde{A}_4$ has 2 in-edges from one vertex then we can take weighted adjacency matrix $\tilde{A}_4 \oplus \tilde{B}_5 \oplus \tilde{C}_5 + T_{12}(\tilde{A}_4, \tilde{B}_5, \tilde{C}_5, 2, 3, 2) + T_{13}(\tilde{A}_4, \tilde{B}_5, \tilde{C}_5, 2, 3, 2)$ which is product-irregular.

(2) If $\tilde{A}_4$ has 2 in-edges from different vertices then we can take weighted adjacency matrix $\tilde{A}_4 \oplus \tilde{B}_5 \oplus \tilde{C}_5 + T_{12}(\tilde{A}_4, \tilde{B}_5, \tilde{C}_5, 2, 3, 2) + T_{13}(\tilde{A}_4, \tilde{B}_5, \tilde{C}_5, 4, 3, 2)$ which is product-irregular.

4. For $(n, m) = (4, 4)$ and $l \geq 5$ we will consider $\tilde{A}_4 \oplus \tilde{B}_l \oplus \tilde{C}_4 + 2edges$.

($\tilde{B}_l$) For the case when $d^+(\tilde{B}_l) = 2$ we have two options:

(1) If $\tilde{B}_l$ has 2 in-edges from one vertex then we can take weighted adjacency matrix $\tilde{A}_4 \oplus \tilde{B}_l \oplus \tilde{C}_4 + T_{12}(\tilde{A}_4, \tilde{B}_l, \tilde{C}_4, 2, 3, 3) + T_{23}(\tilde{A}_4, \tilde{B}_l, \tilde{C}_4, 3, 2, 2)$ which is product-irregular.

(2) If $\tilde{B}_l$ has 2 in-edges from different vertices then we can take weighted adjacency matrix $\tilde{A}_4 \oplus \tilde{B}_l \oplus \tilde{C}_4 + T_{12}(\tilde{A}_4, \tilde{B}_l, \tilde{C}_4, 2, 3, 3) + T_{23}(\tilde{A}_4, \tilde{B}_l, \tilde{C}_4, 1, 2, 2)$ which is product-irregular.

($\tilde{C}_4$) For the case when $d^+(\tilde{C}_4) = 2$ we have two options:

(1) If $\tilde{C}_4$ has 2 in-edges from one vertex then we can take weighted adjacency matrix $\tilde{A}_4 \oplus \tilde{B}_l \oplus \tilde{C}_4 + T_{13}(\tilde{A}_4, \tilde{B}_l, \tilde{C}_4, 2, 2, 3) + T_{23}(\tilde{A}_4, \tilde{B}_l, \tilde{C}_4, 3, 2, 3)$ which is product-irregular.

(2) If $\tilde{C}_4$ has 2 in-edges from different vertices then we can take weighted adjacency matrix $\tilde{A}_4 \oplus \tilde{B}_l \oplus \tilde{C}_4 + T_{13}(\tilde{A}_4, \tilde{B}_l, \tilde{C}_4, 2, 2, 3) + T_{23}(\tilde{A}_4, \tilde{B}_l, \tilde{C}_4, 3, 1, 3)$ which is product-irregular.

5. For $(n, m, l) = (4, 4, 4)$ we will consider $\tilde{A}_4 \oplus \tilde{B}_4 \oplus \tilde{C}_4 + 2edges$. Let $\tilde{C}_4$ to have 2 in-edges.

(1) If $\tilde{C}_4$ has 2 in-edges from one vertex then we can take weighted adjacency matrix $\tilde{A}_4 \oplus \tilde{B}_4 \oplus \tilde{C}_4 + T_{13}(\tilde{A}_4, \tilde{B}_4, \tilde{C}_4, 2, 2, 3) + T_{23}(\tilde{A}_4, \tilde{B}_4, \tilde{C}_4, 3, 2, 3)$ which is product-irregular.

(2) If $\tilde{C}_4$ has 2 in-edges from different vertices then we can take weighted adjacency matrix $\tilde{A}_4 \oplus \tilde{B}_4 \oplus \tilde{C}_4 + T_{13}(\tilde{A}_4, \tilde{B}_4, \tilde{C}_4, 2, 2, 3) + T_{23}(\tilde{A}_4, \tilde{B}_4, \tilde{C}_4, 3, 1, 3)$ which is product-irregular.

The proof now follows by the above argumentation, together with Theorem 4.11 and Lemma 4.12. □

**Corollary 4.14.** *If $G$ is a connected graph such that its vertex set can be partitioned into 3 cliques of sizes at least 4 then $ps(G) = 3$.*

We would like to conclude the paper with proposing the following problem for possible further research.

**Problem 4.15.** Are there only finitely many connected graphs with clique cover number 4 and product irregularity strength more than 3?

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