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On 2-skeleta of hypercubes

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Abstract

It is shown that the 2-skeleton of the odd-d-dimensional hypercube can be decomposed into s_d spheres and τ_d tori, where $s_d = (d-1)2^{d-4}$ and τ_d is asymptotically in the range $(64/9)2^{d-7}$ to $(d-1)(d-3)2^{d-7}$.

Keywords: Cube decomposition, even-degree 2*-complex, generalized book.*

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1 Introduction

A *decomposition* of a graph is an edge-disjoint family of subgraphs such that each edge of the graph is in exactly one of the subgraphs. In recent decades, research on decomposition of graphs into cycles of varying lengths has been carried out for various graphs, including hypercubes.

The symbol "$\times$" denotes Cartesian product of topological spaces.

It is natural to try to extend decomposition (and other frameworks) from graphs to 2-complexes. We do that for *the* 2*-skeleton of the* d*-dimensional hypercube*: the 2-complex Q_d^2 obtained from the d-dimensional hypercube *graph* Q_d by attaching a topological 2-cell $[0,1] \times [0,1]$ to each Q_2-subgraph of Q_d in the natural way, and the decompositions are into spheres and tori.

A necessary condition to decompose a 2-complex into surfaces is that the complex be *even*: each edge belongs to a positive even number of 2-cells. But the condition isn't sufficient; e.g., a surface can intersect itself like the Klein bottle in 3-space. Note Q_d^2 is even iff $d \geq 3$ is odd.

The next section contains definitions, a precise statement of the results, and the proofs. The paper concludes with a brief discussion.

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2 Definitions, theorems, and proofs

In this section, we define complexes in a more general sense and give a product $\mathcal{K} \,\square\, \mathcal{L}$ of 2-complexes (analogous to the Cartesian product of graphs).

A 2-*cell* is any space homeomorphic to the standard unit disk in the plane. A 2-*complex* is a graph together with a non-empty family of closed 2-cells which are attached by homeomorphisms from their boundaries to some of the cycles in the graph. The *degree* of an edge is the number of 2-cells which contain it; a complex is *even* iff all its edges have positive even degree.

If $\mathcal{K}$ is a complex, we write $\mathcal{K}^{(r)}$ for the set of r-cells, $0 \le r \le 2$, where the vertices and edges, resp., are the 0- and 1-cells. The *box-product* of two 2-complexes $\mathcal{K}$ and $\mathcal{L}$ is the 2-complex $\mathcal{M} := \mathcal{K} \,\square\, \mathcal{L}$, where for $k = 0, 1, 2$

$$Y \in \mathcal{M}^{(k)} \iff Y = A \times B,\ A \in \mathcal{K}^{(i)},\ B \in \mathcal{L}^{(j)},\ i + j = k; \tag{2.1}$$

we call Y *of type* (i, j) in this case. It is easy to check that for all $d \ge 2$,

$$Q_d^2 = Q_{d-2}^2 \,\square\, Q_2^2. \tag{2.2}$$

E.g., the 2-cells of $Q_4^2 = Q_2^2 \,\square\, Q_2^2$ consist of four of type $(0, 2)$, four of type $(2, 0)$, and 16 of type $(1, 1)$. The box product of even complexes is even.

A *decomposition* of a 2-complex $\mathcal{K}$ is a set of 2-complexes whose union is $\mathcal{K}$ such that every 2-cell in $\mathcal{K}$ is in exactly one of the components.

An r-*factor* of a graph is a spanning r-regular subgraph and a factorization of a graph G is an edge-disjoint family of factors whose union is G. The following result is due to El-Zanati and Vanden Eynden [3, Theorem 7].

Theorem A. *A Let $d \ge 3$ be odd and suppose $2 \le r \le d$. Then there is a 1-factor F of Q_d such that $Q_d - F$ has a factorization into s-cycles with $s = 2^r$.*

A complex is a *sphere* or *torus* if it is homeomorphic to a sphere or torus. If a complex is isomorphic to K, we call it a K-complex.

Theorem 2.1. *For d odd ≥ 5, Q_d^2 has a decomposition into s_d spheres and t_d tori, where the spheres are Q_3^2, each torus is $C_4 \times C_\ell$ for some $\ell = 2^r$, r odd, $3 \le r \le d - 2$, and*

$$s_d = (d-1)2^{d-4} \quad \textit{and} \quad t_d = \Big(2^{d-1} - (3/2)(d-3) - 4\Big)/9. \tag{2.3}$$

Theorem 2.2. *For d odd ≥ 5, Q_d^2 has a decomposition into s_d spheres and T_d tori, where each sphere equals ∂Q_3, each torus is $C_4 \times C_4$, and*

$$T_d = (d-1)(d-3)2^{d-7}. \tag{2.4}$$

For $d = 5, 7, 9$, $s_d = 8, 48, 256$, $t_d = 1, 6, 27$, and $T_d = 2, 24, 192$, respectively.

Proof of Theorem 2.1. By Theorem A, with $r = d - 2$, Q_{d-2} can be factored into Hamiltonian cycles and a 1-factor F. We proceed by induction.

For the basis case $d = 5$, by equation (2.2), $Q_5^2 = Q_3^2 \,\square\, Q_2^2$ As Q_3^2 is a sphere, the union of all 2-cells of type $(2, 0)$ in Q_5^2 is a set of four disjoint spheres. If F is the 1-factor in Q_3, then $F \,\square\, \partial(Q_2^2)$, is the union of four disjoint cylinders formed by 16 2-cells of type

$(1,1)$, while there are eight 2-cells of type $(0,2)$ which constitute the tops and bottoms of the cylinders, giving a total of 8 spheres in the decomposition of Q_5^2. Finally, if H is the Hamiltonian cycle in $Q_3 - F$, then the 2-cells in $H \,\square\, \partial(Q_2^2)$, each of type $(1,1)$, determine a torus of the form $C_4 \times C_8$. Thus, $s_5 = 8$ and $t_5 = 1$.

Noting that $s_3 = 1$, for the induction step, we again use equation (2.2) and the above argument to see that for $d \geq 5$, $s_d = 4s_{d-2} + 2^{d-3}$ and it is straightforward to check that $s_d = (d-1)2^{d-4}$ satisfies the recursion. Indeed, for $d \geq 5$

$$4(d-3)2^{d-6} + 2 \cdot 2^{d-4} = (d-1)2^{d-4}.$$

Similarly, as $(d-3)/2$ is the number of Hamiltonian cycles in the factorization of $Q_{d-2} - F$, we find that $t_d = 4t_{d-2} + (d-3)/2$, and for d odd ≥ 5, one easily checks that

$$4\Big(2^{d-3} - (3/2)(d-5) - 4\Big)/9 + (d-3)/2 = \Big(2^{d-1} - (3/2)(d-3) - 4\Big)/9,$$

which proves the theorem as the recursively added tori are of the form $C_4 \times C_\ell$, for ℓ the number of vertices in odd hypercubes of dimensions $< d$. $\square$

For instance, writing $\mathcal{T}_k$ for $C_k \times C_4$, the 6 tori for Q_7^2 are 4 copies of $\mathcal{T}_8$ and 2 copies of $\mathcal{T}_{32}$. For Q_9^2, there are 16 copies of $\mathcal{T}_8$, 8 of $\mathcal{T}_{32}$, and 3 of $\mathcal{T}_{128}$.

Using Theorem A with $r = 2$, one proves Theorem 2.2.

3 Conclusion

The decomposition of the odd-dimensional hypercube 2-complex into spheres and tori is an example of decomposing an even complex into surfaces, as proposed in [4]. We believe that similar decompositions are possible for even 2-complexes related to complete graphs (i.e., the simplex).

Decomposition into surfaces may allow improved display for graphs and 2-complexes embeddable in hypercubes. For instance, embedding the graph Q_d in a surface requires genus $1 + (d-4)2^{d-3}$ (e.g., [5, p. 119]) and such an embedding does not include all of the 2-complex. In contrast, a set of spheres and tori with 1-dimensional intersections suffice for the complex.

The problem of finding such representations has been considered by L. De Floriani and colleagues in a series of papers, e.g., [1, 2]. Two types of singularities 0-dimensional ("pinch points") and 1-dimensional (where several disks share a common line) are shown in Figures 3 and 1, respectively, of [1]. Their work, however, concentrates on simplicial complexes, rather than the cubical complexes considered here, and they don't consider the issue of topological complexity.

Our hypercube decompositions, which are face-disjoint unions of spheres and tori, are examples of *generalized books* in the sense of Overbay [6, 7].

If decompositions include surfaces with boundary, then every 2-complex has a decomposition. Indeed, if $\mathcal{K}$ is a 2-complex, then take a genus embedding of the underlying graph, and put each 2-cell, not corresponding to a region of the embedding, onto a separate disk.

That Q_d^2 ($d \geq 5$ odd) is decomposable into closed surfaces follows from Euler's theorem using induction as above. Indeed, removing any 1-factor from Q_{d-2} leaves a $(d-3)$-regular graph, which must be decomposable into cycles. Using [3] instead gives the least and greatest numbers of tori.

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