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On primitive geometries of rank two*

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Abstract

In this paper, we describe a new algorithm to classify primitive coset geometries of rank two for a given group G. This algorithm allows us to classify those geometries for the 12 smallest sporadic simple groups.

Keywords: Primitive coset geometries, sporadic groups, codes, designs.

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1 Introduction

Primitive coset geometries have been studied since the 1990's, first by a team led by Francis Buekenhout in Brussels [4, 5] and later on by Peter Rowley and Nayil Kilic (see for instance [8, 9, 10, 11, 12]). These geometries may be used to construct codes, designs, graphs, etc. Recently, we had the idea of using rank two primitive geometries to construct new binary codes for the McLaughlin group [13]. More precisely, we examined the binary codes obtained from the row span over $\mathbb{F}_2$ of the adjacency matrices of some strongly regular graphs which occur as subgraphs of the McLaughlin graph, namely those with parameters $(105, 32, 4, 12)$, $(120, 42, 8, 18)$ and $(253, 112, 36, 60)$. These new codes were obtained by computing incidence matrices of rank two geometries whose element-stabilizers are maximal subgroups of the McLaughlin group. In order to be able to generate these geometries, a new approach was needed. Indeed, the previous algorithms described by Dehon and Leemans [7] did not allow for a study of a group such as the Mathieu group

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M_{24} of order roughly $2.4 \cdot 10^8$. We managed to classify such geometries with our new algorithm for the 12 smallest sporadic groups, the largest one being the Rudvalis group, of order roughly $1.4 \cdot 10^{11}$. The geometries obtained are available on the following websites:

- https://data.adam-journal.eu/apg/
- http://homepages.ulb.ac.be/~dleemans/PRI/

Our idea relies on permutation representations of groups. We present in this paper an algorithm that outperforms the previous algorithms by a factor of 1000 at least. It allows to classify rank two primitive coset geometries for the five Mathieu groups, the first three Janko groups, the Higman Sims group, the McLaughlin group, the Held group and the Rudvalis group. It also allows to confirm the classification of rank two primitive geometries of M_{11} given in [5], M_{23} given in [10] and to correct some infelicities in [12] for M_{22}. We also obtained a complete classification for M_{24} while Kilic gives in [11] a partial one. The results for the Janko groups, the Higman Sims group, the McLaughlin group, the Held group and the Rudvalis groups are complete and new.

2 Terminology

The basic concepts about geometries constructed from a group and some of its subgroups are due to Tits [14] (see also [3, Chapter 3]). The following theorem shows how to construct geometries starting from groups.

Theorem 2.1 (Tits, 1956 [14]). *Let n be a positive integer. Let $I := \{1, \dots, n\}$ be a finite set and let G be a group together with a family of subgroups $(G_i)_{i\in I}$. Let X be the set consisting of all cosets G_ig, $g \in G$, $i \in I$. Let $t\colon X \to I$ be defined by $t(G_ig) = i$. Define an incidence relation $*$ on $X \times X$ by:*

$$G_ig_1 * G_jg_2 \text{ iff } G_ig_1 \cap G_jg_2 \text{ is non-empty in } G.$$

Then the 4-tuple $\Gamma := (X, *, t, I)$ is an incidence structure having a chamber. Moreover, the group G acts by right multiplication as an automorphism group on Γ. Finally, the group G is transitive on the flags of rank less than 3.

In the Theorem 2.1 above, a *chamber* is a set containing n cosets that are pairwise non-disjoint. Let G be a group and $\{G_i : i \in \{1, \dots, n\}\}$ be a set of subgroups of G. We call $\Gamma(G; (G_i)_{i\in I})$ the *coset geometry* associated to G and the subgroups $(G_i)_{i\in I}$ using the above theorem. By Theorem 2.1, we see that the group G acts on Γ as a type-preserving automorphism group.

In this paper, we only deal with geometries of rank two, that is coset geometries constructed from a group G and two subgroups G_1, G_2 of G. Let us denote such a geometry $\Gamma := \Gamma(G; (G_1, G_2))$. We readily have that Γ is *flag-transitive*, that is, G acts transitively on the pairs of non-disjoint cosets of Γ. Moreover, we want Γ to be *primitive*, meaning that G_1 and G_2 must be maximal subgroups of G. Finally, as G_1 and G_2 are maximal subgroups of G, we necessarily have $\langle G_1, G_2\rangle = G$ provided $G_1 \neq G_2$. This implies that Γ is a firm and connected geometry.

We are thus interested in classifying firm, connected, flag-transitive and primitive coset geometries of rank two for a given group G. Given two coset geometries $\Gamma_1 := \Gamma(G; (G_1, G_2))$ and $\Gamma_2 := \Gamma(G; (H_1, H_2))$, we say that Γ_1 and Γ_2 are *isomorphic* provided there exists an element $g \in \mathrm{Aut}(G)$ such that $g(\{G_1, G_2\}) = \{H_1, H_2\}$. In the next section, we describe an algorithm to classify, up to isomorphism, all firm, connected, primitive geometries of rank 2 for a given group G.

Given a geometry $\Gamma := \Gamma(G; (G_1, G_2))$, its *dual* Γ^* is the geometry

$$\Gamma^* := \Gamma(G; (G_2, G_1)).$$

To a coset geometry of rank two, $\Gamma(G; (G_1, G_2))$ we can associate a graph called the *incidence graph* $\mathcal{G}$ whose vertices are the cosets of G_1 and G_2, two distinct cosets being joined by an edge provided their intersection is nonempty. This graph is bipartite and G acts transitively on the edges of $\mathcal{G}$. Following [2], the *gonality* g of Γ is half the girth of $\mathcal{G}$. The *point-diameter* d_p (resp. *line-diameter* d_l) is the largest distance from the coset G_1 (resp. G_2) to any other coset of Γ. To Γ, we associate the triple $[d_p, g, d_l]$. Finally, we recall that a *design* $S_\lambda(t, k, v)$ is a geometry of rank two with v points and lines of size k such that each set of t points is contained in exactly λ lines.

3 Algorithm

3.1 The old algorithm

The following algorithm was described in [7] (see Section 3, top-down approach) to construct residually weakly primitive geometries. It was the same algorithm that was used by Dehon in [6] to classify primitive geometries. This algorithm constructs primitive geometries of all possible ranks, not only rank two.

Sketch of the Dehon algorithm. Let G be a group for which we want to compute all firm, residually connected, flag-transitive and primitive geometries. Construct a set S containing all maximal subgroups of G and the group $W = \mathrm{Aut}(G)$ acting as a permutation group on S. This group is used to classify geometries up to isomorphism, two geometries (Γ, G) and (Γ', G) being isomorphic if there exists an element g in $\mathrm{Aut}(G)$ such that $g(\Gamma) = \Gamma'$.

Let C_1 be a set of sets containing one representative of each conjugacy class of maximal subgroups of G. Each element $\{G_1\}$ of C_1 (where G_1 is thus a maximal subgroup of G) gives a rank one geometry $\Gamma(G, \{G_1\})$.

If C_1 is nonempty, let C_2 be the empty set and do the following:

For every set $\{G_1\}$ in C_1, determine, up to isomorphism, the subgroups G_2 in S such that $\Gamma(G; G_1, G_2)$ is firm, residually connected and flag-transitive[1]. Store these pairs $\{G_1, G_2\}$ in C_2.

If C_2 is nonempty, let C_3 be the empty set and do the following: for every element $\{G_1, G_2\}$ in C_2, determine, up to isomorphism, the subgroups G_3 in S such that $\Gamma(G; G_1, G_2, G_3)$ is firm, residually connected and flag-transitive. Store these triples $\{G_1, G_2, G_3\}$ in C_3.

And so on, until the set C_{i+1} obtained by adding subgroups to elements of C_i is empty. The sets C_j, $j = 1, \ldots, i$, contain all firm, residually connected, flag-transitive and primitive geometries of rank j of G, up to isomorphism. $\Diamond$

In the algorithm above, the bottleneck is the construction of the group W at the very beginning. That task of computing all maximal subgroups of G becomes quickly tedious

[1] In the rank two case, every possible $G_2 \neq G_1$ is good, and the group W is there to make sure we pick only pairwise non-isomorphic pairs of subgroups (G_1, G_2).

and the permutation representation of G on these subgroups has a degree that is quickly too large for MAGMA to work efficiently with it.

Given a rank two coset geometry $\Gamma := \Gamma(G; (G_1, G_2))$ where G_1 and G_2 are maximal subgroups of G, the group G has a faithful primitive permutation representation on the set Ω of cosets of G_1 (and respectively on the cosets of G_2). In that permutation representation, G_2 partitions the cosets into orbits. As G_1 is maximal in G and the representation is faithful, all the conjugates of G_1 correspond to stabilizers of one coset of Ω. Hence, by fixing G_2 and taking stabilizers of representatives of the orbits of G_2 on Ω, we construct all possible rank two geometries $\Gamma(G; (H_1, H_2))$ with H_1 conjugate to G_1 and H_2 conjugate to G_2. One of them is necessarily the starting geometry.

An algorithm follows from the above paragraph: Given a group G and a maximal subgroup M_1 of G, construct the permutation representation of G on the set Ω of right cosets of M_1. This representation is primitive. For any maximal subgroup M_2 of G in that representation, compute the orbits $\mathcal{O}$ of M_2 on Ω. In each orbit $o \in \mathcal{O}$, pick one representative x. The triple $(G; (M_2, G_x))$ gives a primitive geometry of rank 2. This geometry is obviously flag-transitive and it will be firm provided $|o| \neq 1$. This gives an obvious way to produce primitive geometries of rank 2 for a given group G. Any rank two primitive geometry constructed from G can be produced in this way. Therefore, we have a technique to produce all rank two primitive geometries for G. It may happen that we generate several isomorphic copies of the same geometry. To avoid that, every time we generate a geometry with the above technique, we check whether it is isomorphic to one of the geometries obtained so far. We only keep it if it is not isomorphic to any of the previously obtained geometries.

In order to be able to apply the above algorithm, all that is needed now is that we can compute permutation representations of G on all its maximal subgroups and that we can check isomorphism between geometries that may be self-dual. Starting from the Rudvalis group, these permutation representations can become very large. This is where we decided to stop.

We implemented the Algorithm 3.1 in MAGMA [1] and determined all rank two primitive geometries for the twelve smallest sporadic simple groups. Our findings are summarized in the next section.

4 Primitive rank two geometries of sporadic groups

In this section, we summarise the classifications of primitive geometries of rank two we obtained, using the algorithm described in the previous section, for the five Mathieu groups, the first three Janko groups, the McLaughlin group, the Higman-Sims group, the Held group and the Rudvalis group. A summary of the results obtained is available in Table 1. In the last column, the computing time to classify these geometries is given. The computer used was a workstation with 4 Intel Xeon E5 processors with 6 cores each working at 2.9 GHz and 1 TB of RAM. We give the orbit tables for the groups M_{11}, M_{12}, M_{22}, M_{23} and J_1 in the following sections. We omit the tables of the remaining groups since these tables become too large.

4.1 The Mathieu group M_{11}

Table 2 gives the orbit lengths for every primitive permutation representation of M_{11}. Each column corresponds to a given primitive permutation representation and gives the ways

Algorithm 3.1 An algorithm to compute primitive geometries.

Input: $G \ldots$ a group
Output: $geos$ ... a sequence containing pairwise non-isomorphic pairs of maximal subgroups of G

```
Initialize a sequence geos that will be used to store the geometries.
Compute M, a list containing a representative of each conjugacy class of maximal sub-
groups of G.
for i := 1 to #M do
    Let M_1 be the i-th element of M.
    Construct φ: G → G/M_1, the coset action of G on the cosets of M_1.
    Let φ(M) := [φ(x) : x ∈ M].
    for j := 1 to #M do
        Let M_2 be the j-th element of φ(M).
        Compute O, the orbits of K.
        for each orbit o of size > 1 in O do
            Let x be a representative of o.
            Let G_x be the stabilizer of x in φ(G).
            if the pair {φ^{-1}(M_2), φ^{-1}(G_x)} is not isomorphic to any pair in geos then
                add the pair {φ^{-1}(M_2), φ^{-1}(G_x)} to geos
            end if
        end for
    end for
end for
return geos
```

Table 1: The 11 smallest sporadic groups and their rank two primitive geometries.

Group	Order	Degree	Number of geometries	Computing time
M_{11}	7 920	11	37	0.28 s
M_{12}	95 040	12	166	13.79 s
M_{22}	443 520	22	81	3.23 s
M_{23}	10 200 960	23	170	54.04 s
M_{24}	244 823 040	24	5074	106704 s $\sim$ 29 h
J_1	175 560	266	669	146.22 s
J_2	604 800	100	334	71.27 s
J_3	50 232 960	6156	546	5031.72 s
HS	44 352 000	100	339	282.46 s
McL	898 128 000	275	443	2412.12 s
He	4 030 387 200	2058	4074	14.35 days
Ru	145 926 144 000	4060	1511	9.2 days

the points are split into orbits by the corresponding maximal subgroups. So for instance, the entry 12 – 18 – 36 corresponding to the line labelled $M_9 : 2$ and the column labelled 66 means that on the 66-points primitive permutation representation of M_{11} (obtained by looking at the coset action of M_{11} on the cosets of a maximal subgroup isomorphic to S_5), a maximal subgroup $M_9 : 2$ has orbits of length 12, 18 and 36.

Counting the number of orbits of size at least two in the upper triangular half, we get 38 possibilities. In fact, the Mathieu group M_{11} has, up to isomorphism, 37 primitive geometries of rank two as two of them are dual of each other. This confirms the results obtained in [5]. In Table 4, we give for each of the 37 geometries Γ the designs corresponding to Γ and to its dual Γ^*. Some geometries do not appear in that table as a rank two geometry does not necessarily give a design. Entry 28^* is the well known $S_1(4, 5, 11)$, that is the Steiner system associated to the Mathieu group M_{11}.

Table 2: Orbits of primitive permutation representations of M_{11}.

	11	12	55	66	165
M_{10}	1 – 10	12	10 – 45	30 – 36	45 – 120
$L_2(11)$	11	1 – 11	55	11 – 55	55 – 110
$M_9 : 2$	2 – 9	12	1 – 18 – 36	12 – 18 – 36	$9 – 12 – 72^2$
S_5	5 – 6	2 – 10	10 – 15 – 30	1 – 15 – 20 – 30	$10 – 15 – 20 – 60^2$
$M_8 : S_3$	3 – 8	4 – 8	$3 – 4 – 24^2$	$4 – 6 – 8 – 24^2$	$1 – 8 – 12 – 24^4 – 48$

4.2 The Mathieu group M_{12}

Table 5 gives the orbit lengths for every primitive permutation representation of M_{12}. Counting the number of orbits of size at least two in the upper triangular half, we get 268 possibilities. In fact, the Mathieu group M_{12} has 166 primitive geometries of rank two as $\mathrm{Aut}(M_{12})$ conjugates several pairs in the 268 possibilities.

4.3 The Mathieu group M_{22}

Table 6 gives the orbit lengths for every primitive permutation representation of M_{22}. Our programs gave 81 geometries up to isomorphism. This corrects the number that was obtained in [12], namely 86.

4.4 The Mathieu group M_{23}

Table 7 gives the orbit lengths for every primitive permutation representation of M_{23}. Our programs gave 170 geometries up to isomorphism. This confirms the results obtained by Kilic in [10].

4.5 The first group of Janko J_1

Table 8 gives the orbit lengths for every primitive permutation representation of J_1. Our programs gave 669 geometries up to isomorphism. This is a completely new result.

Table 3: The 37 rank two primitive geometries of M_{11}.

Nr.	G_1	G_2	$G_1 \cap G_2$	$[d_p, g, d_l]$
1	$\mathrm{GL}_2(3)$	$\mathrm{GL}_2(3)$	S_3	$[5, 3, 5]$
2	$\mathrm{GL}_2(3)$	$\mathrm{GL}_2(3)$	E_4	$[4, 2, 4]$
3	$\mathrm{GL}_2(3)$	$\mathrm{GL}_2(3)$	C_2	$[4, 2, 4]$
4	$\mathrm{GL}_2(3)$	$\mathrm{GL}_2(3)$	C_2	$[5, 2, 5]$
5	$\mathrm{GL}_2(3)$	$\mathrm{GL}_2(3)$	C_2	$[3, 2, 4]$
6	$\mathrm{GL}_2(3)$	$\mathrm{GL}_2(3)$	1	$[3, 2, 3]$
7	$\mathrm{GL}_2(3)$	S_5	D_{12}	$[5, 3, 5]$
8	$\mathrm{GL}_2(3)$	S_5	D_8	$[4, 2, 4]$
9	$\mathrm{GL}_2(3)$	S_5	S_3	$[4, 2, 4]$
10	$\mathrm{GL}_2(3)$	S_5	C_2	$[3, 2, 3]$
11	$\mathrm{GL}_2(3)$	S_5	C_2	$[3, 2, 3]$
12	$\mathrm{GL}_2(3)$	$L_2(11)$	D_{12}	$[3, 2, 4]$
13	$\mathrm{GL}_2(3)$	$L_2(11)$	S_3	$[3, 2, 3]$
14	$\mathrm{GL}_2(3)$	$M_9 : 2$	$Q_8 : 2$	$[6, 3, 5]$
15	$\mathrm{GL}_2(3)$	$M_9 : 2$	D_{12}	$[4, 3, 4]$
16	$\mathrm{GL}_2(3)$	$M_9 : 2$	C_2	$[3, 2, 3]$
17	$\mathrm{GL}_2(3)$	$M_9 : 2$	C_2	$[3, 2, 3]$
18	$\mathrm{GL}_2(3)$	M_{10}	$Q_8 : 2$	$[3, 2, 4]$
19	$\mathrm{GL}_2(3)$	M_{10}	S_3	$[3, 2, 3]$
20	S_5	S_5	D_8	$[5, 2, 5]$
21	S_5	S_5	S_3	$[3, 2, 3]$
22	S_5	S_5	E_4	$[3, 2, 3]$
23	S_5	$L_2(11)$	A_5	$[3, 3, 4]$
24	S_5	$L_2(11)$	12	$[3, 2, 3]$
25	S_5	$M_9 : 2$	12	$[4, 2, 3]$
26	S_5	$M_9 : 2$	8	$[3, 2, 3]$
27	S_5	$M_9 : 2$	C_4	$[3, 2, 3]$
28	S_5	M_{10}	24	$[3, 2, 3]$
29	S_5	M_{10}	20	$[3, 2, 3]$
30	$L_2(11)$	$L_2(11)$	A_5	$[3, 2, 3]$
31	$L_2(11)$	$M_9 : 2$	D_{12}	$[2, 2, 2]$
32	$L_2(11)$	M_{10}	A_5	$[2, 2, 2]$
33	$M_9 : 2$	$M_9 : 2$	8	$[3, 2, 3]$
34	$M_9 : 2$	$M_9 : 2$	4	$[3, 2, 3]$
35	$M_9 : 2$	M_{10}	M_9	$[3, 3, 4]$
36	$M_9 : 2$	M_{10}	16	$[3, 2, 3]$
37	M_{10}	M_{10}	M_9	$[3, 2, 3]$

Table 4: Designs $S_\lambda(t,k,v)$ from the rank two primitive geometries of M_{11}.

Nr.	Design
1	$S_8(1,8,165)$
1*	$S_8(1,8,165)$
2	$S_{12}(1,12,165)$
2*	$S_{12}(1,12,165)$
3	$S_{24}(1,24,165)$
3*	$S_{24}(1,24,165)$
4	$S_{24}(1,24,165)$
4*	$S_{24}(1,24,165)$
5	$S_{24}(1,24,165)$
5*	$S_{24}(1,24,165)$
6	$S_{48}(1,48,165)$
6*	$S_{48}(1,48,165)$
7	$S_{10}(1,4,66)$
7*	$S_4(1,10,165)$
8	$S_{15}(1,6,66)$
8*	$S_6(1,15,165)$
9	$S_{20}(1,8,66)$
9*	$S_8(1,20,165)$
10	$S_{60}(1,24,66)$
10*	$S_{24}(1,60,165)$
11	$S_{60}(1,24,66)$
11*	$S_{24}(1,60,165)$
12	$S_3(3,4,12)$
12*	$S_4(1,55,165)$
13	$S_{42}(3,8,12)$
13*	$S_8(1,110,165)$

Nr.	Design
14	$S_3(1,9,165)$
14*	$S_9(1,3,55)$
15	$S_4(1,12,165)$
15*	$S_{12}(1,4,55)$
16	$S_{24}(1,72,165)$
16*	$S_{72}(1,24,55)$
17	$S_{24}(1,72,165)$
17*	$S_{72}(1,24,55)$
18	$S_1(3,3,11)$
18*	$S_3(1,45,165)$
19	$S_1(8,8,11)$
19*	$S_8(1,120,165)$
20	$S_{15}(1,15,66)$
20*	$S_{15}(1,15,66)$
21	$S_{20}(1,20,66)$
21*	$S_{20}(1,20,66)$
22	$S_{30}(1,30,66)$
22*	$S_{30}(1,30,66)$
23	$S_1(2,2,12)$
23*	$S_2(1,11,66)$
24	$S_1(10,10,12)$
24*	$S_{10}(1,55,66)$
25	$S_{10}(1,12,66)$
25*	$S_{12}(1,10,55)$
26	$S_{15}(1,18,66)$
26*	$S_{18}(1,15,55)$

Nr.	Design
27	$S_{30}(1,36,66)$
27*	$S_{36}(1,30,55)$
28	$S_5(1,30,66)$
28*	$S_1(4,5,11)$
29	$S_6(1,36,66)$
29*	$S_3(4,6,11)$
30	$S_1(11,11,12)$
30*	$S_1(11,11,12)$
33	$S_{18}(1,18,55)$
33*	$S_{18}(1,18,55)$
34	$S_{36}(1,36,55)$
34*	$S_{36}(1,36,55)$
35	$S_1(2,2,11)$
35*	$S_2(1,10,55)$
36	$S_1(9,9,11)$
36*	$S_9(1,45,55)$
37	$S_1(10,10,11)$
37*	$S_1(10,10,11)$

Table 5: Orbits of primitive permutation representations of M_{12}.

	12	12	66	66	144	220	220
M_{11}	1 – 11	12	66	11 – 55	144	220	55 – 165
M_{11}	12	1 – 11	11 – 55	66	144	55 – 165	220
$M_{10}:2$	12	2 – 10	1 – 20 – 45	30 – 36	144	10 – 90 – 120	40 – 180
$M_{10}:2$	2 – 10	12	30 – 36	1 – 20 – 45	144	40 – 180	10 – 90 – 120
$L_2(11)$	12	12	66	66	$1 – 11^2 – 55 – 66$	220	220
$M_9:S_3$	12	3 – 9	3 – 27 – 36	12 – 54	144	1 – 12 – 27 – 72 – 108	4 – 36 – 72 – 108
$M_9:S_3$	3 – 9	12	12 – 54	3 – 27 – 36	144	4 – 36 – 72 – 108	1 – 12 – 27 – 72 – 108
$S_5 \times 2$	12	12	$6 – 30^2$	$6 – 30^2$	$20^3 – 24 – 60$	40 – 60 – 120	40 – 60 – 120
$4^2:D_{12}$	12	12	6 – 12 – 48	6 – 12 – 48	$16^2 – 32^2 – 48$	12 – 48 – 64 – 96	12 – 48 – 64 – 96
$M_8:S_4$	4 – 8	4 – 8	4 – 6 – 24 – 32	4 – 6 – 24 – 32	48 – 96	4 – 16 – 24 – 32 – 48 – 96	4 – 16 – 24 – 32 – 48 – 96
$A_4 \times S_3$	12	12	12 – 18 – 36	12 – 18 – 36	$6^2 – 18^2 – 24 – 36^2$	$4 – 12 – 24 – 36 – 72^2$	$4 – 12 – 24 – 36 – 72^2$

	396	495	495	1320
M_{11}	396	495	165 – 330	1320
M_{11}	396	495	165 – 330	1320
$M_{10}:2$	$36 – 180^2$	45 – 90 – 360	30 – 45 – 180 – 240	240 – 360 – 720
$M_{10}:2$	$36 – 180^2$	45 – 90 – 360	30 – 45 – 180 – 240	240 – 360 – 720
$L_2(11)$	$55^3 – 66 – 165$	$55^2 – 110^2 – 165$	165 – 330	$55^2 – 165^2 – 220 – 330^2$
$M_9:S_3$	72 – 108 – 216	27 – 108 – 144 – 216	9 – 36 – 54 – 72 – 108 – 216	$24 – 72 – 144 – 216 – 432^2$
$M_9:S_3$	72 – 108 – 216	27 – 108 – 144 – 216	9 – 36 – 54 – 72 – 108 – 216	$24 – 72 – 144 – 216 – 432^2$
$S_5 \times 2$	$1 – 10^2 – 15 – 30^2 – 60^3 – 120$	$5 – 15^2 – 20^2 – 60^3 – 120^2$	$15 – 30^2 – 60 – 120^3$	$10 – 30 – 60^4 – 80 – 120^4 – 240^2$
$4^2:D_{12}$	$4 – 12^2 – 16^2 – 48^3 – 96^2$	$1 – 6 – 16 – 24 – 32^2 – 48^2 – 96^3$	$3 – 12 – 48^2 – 96^4$	$16 – 24 – 48^4 – 64^2 – 96^4 – 192^3$
$M_8:S_4$	$12 – 24^2 – 48 – 96^3$	$3 – 12 – 48^2 – 96^4$	$1 – 6 – 16 – 24 – 32^2 – 48^2 – 96^3$	$8 – 48^2 – 64 – 96^4 – 192^4$
$A_4 \times S_3$	$3 – 9 – 18^4 – 24 – 36^4 – 72^2$	$6 – 9 – 18^4 – 24^2 – 36^4 – 72^3$	$3 – 18^2 – 24 – 36^4 – 72^4$	$1 – 8 – 9 – 12 – 18^3 – 24^2 – 36^7 – 72^{13}$

Table 6: Orbits of primitive permutation representations of M_{22}.

	22	77	176	176	231
$L_3(4)$	1 – 21	21 – 56	56 – 120	56 – 120	21 – 210
$2^4 : A_6$	6 – 16	1 – 16 – 60	80 – 96	80 – 96	15 – 96 – 120
A_7	7 – 15	35 – 42	1 – 70 – 105	15 – 35 – 126	21 – 105^2
A_7	7 – 15	35 – 42	15 – 35 – 126	1 – 70 – 105	21 – 105^2
$2^4 : S_5$	2 – 20	5 – 32 – 40	16 – 80^2	16 – 80^2	1 – 30 – 40 – 160
$2^3 : L_3(2)$	8 – 14	7 – 14 – 56	8 – 56 – 112	8 – 56 – 112	7 – 28 – 84 – 112
M_{10}	10 – 12	2 – 30 – 45	20 – 36 – 120	20 – 36 – 120	30 – 36 – 45 – 120
$L_2(11)$	11^2	11^2 – 55	11 – 55 – 110	11 – 55 – 110	55^3 – 66

	330	616	672
$L_3(4)$	120 – 210	280 – 336	336^2
$2^4 : A_6$	30 – 60 – 240	16 – 240 – 360	96^2 – 480
A_7	15 – 105 – 210	70 – 126 – 420	42 – 210 – 420
A_7	15 – 105 – 210	70 – 126 – 420	42 – 210 – 420
$2^4 : S_5$	10 – 40 – 120 – 160	80 – 96 – 120 – 320	160^3 – 192
$2^3 : L_3(2)$	1 – 7 – 42 – 112 – 168	56^2 – 168 – 336	112^3 – 336
M_{10}	30^2 – 90 – 180	1 – 30 – 45 – 180 – 360	72 – 120^2 – 360
$L_2(11)$	55^3 – 165	66 – 110^2 – 330	1 – 55^2 – 66 – 165 – 330

Table 7: Orbits of primitive permutation representations of M_{23}.

	23	253	253	506
M_{22}	1 – 22	77 – 176	22 – 231	176 – 330
$L_3(4) : 2$	7 – 16	1 – 112 – 140	21 – 112 – 120	30 – 140 – 336
$2^4 : A_7$	2 – 21	21 – 112 – 120	1 – 42 – 210	56 – 210 – 240
A_8	8 – 15	15 – 70 – 168	28 – 105 – 120	1 – 15 – 210 – 280
M_{11}	11 – 12	22 – 66 – 165	55 – 66 – 132	66 – 110 – 330
$2^4 : (3 \times A_5) : 2$	3 – 20	5 – 48 – 80 – 120	3 – 30 – 60 – 160	10 – 16 – 120^2 – 240
$23 : 11$	23	253	253	253^2

	1288	1771	40320
M_{22}	616 – 672	231 – 1540	40320
$L_3(4) : 2$	112 – 336 – 840	35 – 336 – 560 – 840	40320
$2^4 : A_7$	280 – 336 – 672	21 – 210 – 420 – 1120	40320
A_8	168 – 280 – 840	35 – 56 – 420^2 – 840	20160^2
M_{11}	1 – 165 – 330 – 792	165 – 220 – 330 – 396 – 660	720 – 7920^5
$2^4 : (3 \times A_5) : 2$	120 – 160 – 240 – 288 – 480	1 – 20 – 60 – 90 – 160 – 480^3	5760^7
$23 : 11$	23 – 253^5	253^7	1 – 23^4 – 253^{159}

Table 8: Orbits of primitive permutation representations of J_1.

	266	1045	1463
$L_2(11)$	$1-11-12-110-132$	$55-110-220-660$	$11-55-110-132-165-330-660$
$2^3:7:3$	$14-28-56-168$	$1-8-28-56^3-168^5$	$7-56^2-84^2-168^7$
$A_5\times 2$	$2-10-20-24-30-60-120$	$5-40^2-60^2-120^7$	$1-12-15^2-20^2-60^9-120^7$
$19:6$	$19-38^2-57^3$	$19^2-38^4-57-114^7$	$19-38^2-57^{10}-114^7$
$11:10$	$2-22^2-55^2-110$	$55-110^9$	$11-22-55^6-110^{10}$
$D_6\times D_{10}$	$5-6-15-30^4-60^2$	$10-15-30^2-60^{16}$	$3-5-15^5-30^{14}-60^{16}$
$7:6$	$7-14^2-21^3-42^4$	$2-7^2-14^3-21-42^{23}$	$7-14^2-21^{10}-42^{29}$

	1540	1596	2926	4180
$L_2(11)$	$110-220^2-330^3$	$12-132^2-330^2-660$	$55-66-165-330^4-660^2$	$110-220^2-330^3-660^4$
$2^3:7:3$	$28^2-56^4-84-168^7$	$84-168^9$	$28-42-84^2-168^{16}$	$8-28^2-56^3-84-168^{23}$
$A_5\times 2$	$20-40^2-60^{10}-120^7$	$12-24-60^6-120^{10}$	$6-10-30^5-60^{14}-120^{16}$	$20-40^2-60^{10}-120^{29}$
$19:6$	$1-19-38^4-57^6-114^9$	57^4-114^{12}	$19-57^{15}-114^{18}$	$19^2-38^4-57^6-114^{32}$
$11:10$	55^4-110^{12}	$1-11-22^2-55^2-110^{13}$	$11-55^9-110^{22}$	55^4-110^{36}
$D_6\times D_{10}$	$10-30^{15}-60^{18}$	$6-30^9-60^{22}$	$1-15^5-30^{27}-60^{34}$	$10-30^{15}-60^{62}$
$7:6$	$7^2-14^4-21^6-42^{32}$	21^4-42^{36}	$7-21^{15}-42^{62}$	$1-7-14^4-21^6-42^{95}$

5 Conclusions

We have described a new algorithm that allows to compute primitive geometries of rank two for much larger groups than the Dehon algorithm [7]. It remains to extend this algorithm to be able to construct geometries of higher ranks as the Dehon algorithm permitted. Since our first aim was to study codes arising from these rank two geometries, we did not try to extend our algorithm and we leave it as an interesting topic for future work.

The problems found in [12] for M_{22} are very likely due to an incorrect determination of non-isomorphic pairs of subgroups.

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