ISSN 2590-9770
The Art of Discrete and Applied Mathematics 3 (2020) #P2.10
https://doi.org/10.26493/2590-9770.1295.25b
(Also available at http://adam-journal.eu)

Strongly regular graphs with parameters $(37, 18, 8, 9)$ having nontrivial automorphisms*

Dean Crnković, Marija Maksimović
Department of Mathematics, University of Rijeka
Radmile Matejčić 2, 51000 Rijeka, Croatia

Received 26 March 2019, accepted 20 September 2019, published online 21 August 2020

Abstract

All strongly regular graphs having at most 36 vertices have been enumerated. Hence, the first open case is enumeration of the SRGs with parameters $(37, 18, 8, 9)$. In this paper we show that there are exactly forty SRGs with parameters $(37, 18, 8, 9)$ having nontrivial automorphisms. Comparing the constructed graphs with previously known SRGs with these parameters we conclude that six of the SRGs with parameters $(37, 18, 8, 9)$ constructed in this paper are new, and that up to isomorphism there are at least 6766 strongly regular graphs with parameters $(37, 18, 8, 9)$.

Keywords: Strongly regular graph, automorphism group, orbit matrix.

Math. Subj. Class. (2020): 05E30, 20D45

1 Introduction

One of the main problems in the theory of strongly regular graphs (SRGs) is constructing and classifying SRGs with given parameters. A frequently used method of constructing combinatorial structures is a construction with a prescribed automorphism group using orbit matrices. While orbit matrices of block designs have been used for such a construction of designs since 1980s, orbit matrices of strongly regular graphs have not been introduced until 2011 (see [2]). Using orbit matrices we construct all strongly regular graphs with parameters $(37, 18, 8, 9)$ having nontrivial automorphisms. In that way we have constructed forty SRGs with parameters $(37, 18, 8, 9)$, and six of them are new. Thereby we proved that there are exactly forty SRGs with parameters $(37, 18, 8, 9)$ having nontrivial automorphisms, and at least 6766 strongly regular graphs with these parameters.

*This work has been fully supported by Croatian Science Foundation under the project 6732.
E-mail addresses: deanc@math.uniri.hr (Dean Crnković), mmaksimovic@math.uniri.hr (Marija Maksimović)

The paper is organized as follows: after a brief description of the terminology and some background results in Section 2, in Section 3 we describe the concept of orbit matrices. In Section 4 we apply the method of constructing SRGs using orbit matrices to construct all strongly regular graphs with parameters $(37, 18, 8, 9)$ having nontrivial automorphisms.

2 Background and terminology

We assume that the reader is familiar with basic notions from the theory of finite groups. For basic definitions and properties of strongly regular graphs we refer the reader to [3, 9, 14].

A graph is regular if all its vertices have the same valency. A simple regular graph $\Gamma = (\mathcal{V}, \mathcal{E})$ is strongly regular with parameters (v, k, λ, μ) if it has $|\mathcal{V}| = v$ vertices, valency k, and if any two adjacent vertices are together adjacent to λ vertices, while any two nonadjacent vertices are together adjacent to μ vertices. A strongly regular graph with parameters (v, k, λ, μ) is usually denoted by SRG(v, k, λ, μ). An automorphism of a strongly regular graph Γ is a permutation of vertices of Γ, such that every two vertices are adjacent if and only if their images are adjacent.

Let $\Gamma_1 = (\mathcal{V}, \mathcal{E}_1)$ and $\Gamma_2 = (\mathcal{V}, \mathcal{E}_2)$ be strongly regular graphs and $G \leq Aut(\Gamma_1) \cap Aut(\Gamma_2)$. An isomorphism $\alpha : \Gamma_1 \to \Gamma_2$ is called a G-isomorphism if there exists an automorphism $\tau : G \to G$ such that for each $x, y \in \mathcal{V}$ and each $g \in G$ the following holds:

$$(\tau g).(\alpha x) = \alpha y \Leftrightarrow g.x = y.$$

Strongly regular graphs having at most 36 vertices have been enumerated, so SRGs with parameters $(37, 18, 8, 9)$ are the first open case that still have to be classified (see [4]). It is known that there exists at least 6760 SRGs$(37, 18, 8, 9)$, which are obtained as the descendants of the 191 regular two-graphs on 38 vertices constructed in [11]. The adjacency matrices of these 6760 SRGs$(37, 18, 8, 9)$ can be found at [12]. In this paper we classify SRGs$(37, 18, 8, 9)$ having nontrivial automorphisms, showing that there are at least 6766 strongly regular graphs with parameters $(37, 18, 8, 9)$.

3 Orbit matrices of strongly regular graphs

Orbit matrices of block designs have been frequently used for construction of block designs, see e.g. [6, 7, 8, 10]. In this section we describe the concept of orbit matrices of SRGs, which is introduced in 2011 by Behbahani and Lam (see [2]).

Let Γ be a SRG(v, k, λ, μ) and A be its adjacency matrix. Suppose an automorphism group G of Γ partitions the set of vertices V into b orbits $O_1, \ldots, O_b$, with sizes $n_1, \ldots, n_b$, respectively. The orbits divide A into submatrices $[A_{ij}]$, where A_{ij} is the adjacency matrix of vertices in O_i versus those in O_j. We define matrices $C = [c_{ij}]$ and $R = [r_{ij}]$, $1 \leq i, j \leq b$, such that

$c_{ij} =$ column sum of A_{ij},
$r_{ij} =$ row sum of A_{ij}.

The matrix R is related to C by

$$r_{ij} n_i = c_{ij} n_j. \tag{3.1}$$

Since the adjacency matrix is symmetric, it follows that

$$R = C^T. \tag{3.2}$$

The matrix R is the row orbit matrix of the graph Γ with respect to G, and the matrix C is the column orbit matrix of the graph Γ with respect to G.

Let us assume that a group G acts as an automorphism group of a SRG(v, k, λ, μ). Behbahani and Lam showed that orbit matrices $R = [r_{ij}]$ and $R^T = C = [c_{ij}]$ satisfy the condition

$$\sum_{s=1}^{b} c_{is} r_{sj} n_s = \delta_{ij}(k-\mu)n_j + \mu n_i n_j + (\lambda - \mu) c_{ij} n_j.$$

Since $R = C^T$, it follows that

$$\sum_{s=1}^{b} \frac{n_s}{n_j} c_{is} c_{js} = \delta_{ij}(k-\mu) + \mu n_i + (\lambda - \mu) c_{ij} \tag{3.3}$$

and

$$\sum_{s=1}^{b} \frac{n_s}{n_j} r_{si} r_{sj} = \delta_{ij}(k-\mu) + \mu n_i + (\lambda - \mu) r_{ji}.$$

In order to enable a construction of SRGs with a presumed automorphism group G, each matrix with the properties of an orbit matrix will be called an orbit matrix for parameters (v, k, λ, μ) and a group G (see [1]). Therefore, we introduce the following definition of orbit matrices of strongly regular graphs (see [5]).

Definition 3.1. A $(b \times b)$-matrix $R = [r_{ij}]$ with entries satisfying conditions:

$$\sum_{j=1}^{b} r_{ij} = \sum_{i=1}^{b} \frac{n_i}{n_j} r_{ij} = k \tag{3.4}$$

$$\sum_{s=1}^{b} \frac{n_s}{n_j} r_{si} r_{sj} = \delta_{ij}(k-\mu) + \mu n_i + (\lambda - \mu) r_{ji} \tag{3.5}$$

where $0 \le r_{ij} \le n_j$, $0 \le r_{ii} \le n_i - 1$ and $\sum_{i=1}^{b} n_i = v$, is called a **row orbit matrix** for a strongly regular graph with parameters (v, k, λ, μ) and the orbit lengths distribution $(n_1, \ldots, n_b)$.

Definition 3.2. A $(b \times b)$-matrix $C = [c_{ij}]$ with entries satisfying conditions:

$$\sum_{i=1}^{b} c_{ij} = \sum_{j=1}^{b} \frac{n_j}{n_i} c_{ij} = k \tag{3.6}$$

$$\sum_{s=1}^{b} \frac{n_s}{n_j} c_{is} c_{js} = \delta_{ij}(k-\mu) + \mu n_i + (\lambda - \mu) c_{ij} \tag{3.7}$$

where $0 \le c_{ij} \le n_i$, $0 \le c_{ii} \le n_i - 1$ and $\sum_{i=1}^{b} n_i = v$, is called a **column orbit matrix** for a strongly regular graph with parameters (v, k, λ, μ) and the orbit lengths distribution $(n_1, \ldots, n_b)$.

Not every orbit matrix gives rise to strongly regular graphs while, on the other hand, a single orbit matrix may produce several nonisomorphic strongly regular graphs. For the elimination of orbit matrices that produce G-isomorphic strongly regular graphs we use the same method as for the elimination of orbit matrices of G-isomorphic designs (see for example [7]). We could use row or column orbit matrices, but since we construct matrices row by row, it is more convenient for us to use column orbit matrices.

3.1 Orbit lengths distribution

Suppose an automorphism group G of the graph Γ partitions the set of vertices V into b orbits $O_1, \ldots, O_b$, with sizes $n_1, \ldots, n_b$, respectively. It is well known that n_i divides $|G|$, for $i = 1, \ldots, b$. Further,

$$\sum_{i=1}^{b} n_i = v.$$

In this paper we will be interested in groups that act in orbits having at most two lengths, since we will consider automorphism groups of prime order. If the group G acts with d_1 orbits of length 1 and d_h orbits of length h, we will denote this distribution with $(d_1 \times 1, d_h \times h)$. When determining the orbit lengths distributions we use the following result that can be found in [1].

Theorem 3.3. *Let $s < r < k$ be the eigenvalues of a SRG(v, k, λ, μ), then*

$$\phi \leq \frac{\max(\lambda, \mu)}{k - r} v,$$

where ϕ is the number of fixed points for a nontrivial automorphism.

In the case of SRGs with parameters $(37, 18, 8, 9)$ we obtain that $\phi \leq 20$, so to find all feasible orbit length distributions $(d_1 \times 1, d_h \times h)$ we need to solve the system

$$\begin{aligned} & d_1 + h \cdot d_h = 37 \\ & d_1 \leq 20. \end{aligned}$$

4 Classification of SRGs with parameters $(37, 18, 8, 9)$ having nontrivial automorphisms

It is known that there exists at least 6760 SRGs with parameters $(37, 18, 8, 9)$ (see [11]). Spence [12] listed adjacency matrices of all of them. In Table 1 we give information on orders of the full automorphism groups of these 6760 SRGs$(37, 18, 8, 9)$. The graph having the full automorphism group of order 666 is the Paley graph obtained from the field $GF(37)$, having the full automorphism group isomorphic to $Z_{37} : Z_{18}$ (see [14]).

In this section we give the classification of strongly regular graphs with parameters $(37, 18, 8, 9)$ having nontrivial automorphisms. We show that there are exactly 6 strongly regular graphs with parameters $(37, 18, 8, 9)$ having an automorphism group of order two,

Table 1: Orders of the full automorphism groups of the known SRGs$(37, 18, 8, 9)$

$\lvert\mathrm{Aut}(\Gamma_i)\rvert$	#SRGs
1	6726
2	3
3	24
9	4
18	2
666	1

all of them isomorphic to the graphs given at [12]. Further, we show that there are exactly 37 strongly regular graphs with parameters $(37, 18, 8, 9)$ having an automorphism group of order three, 6 of them nonisomorphic to any of the graphs listed at [12]. Finally we show that there is no SRG$(37, 18, 8, 9)$ having an automorphism group Z_p, where p is prime and $3 < p < 37$, and that there is exactly one SRG$(37, 18, 8, 9)$ having the automorphism of order 37 (the Paley graph with 37 vertices). Comparing the constructed SRGs with the SRGs given at [12], we establish that six of the strongly regular graphs having a nontrivial automorphism group of prime order constructed in this paper have not been previously known.

In order to construct orbit matrices of SRGs with parameters $(37, 18, 8, 9)$ that have automorphism of prime order p, we first find all permissible distibutions $(d_1 \times 1, d_p \times p)$. Then for each distribution we find all prototypes (see [1]). Using prototypes we construct orbit matrices row by row and we eliminate mutually G-isomorphic orbit matrices during this process. In the next step we construct adjacency matrices of SRGs$(37, 18, 8, 9)$.

Table 2: Number of orbit matrices and SRGs$(37, 18, 8, 9)$ for the automorphism group Z_2

distribution	#OM	#SRGs	distribution	#OM	#SRGs
$(1 \times 1, 18 \times 2)$	24	6	$(11 \times 1, 13 \times 2)$	0	0
$(3 \times 1, 17 \times 2)$	0	0	$(13 \times 1, 12 \times 2)$	0	0
$(5 \times 1, 16 \times 2)$	6	0	$(15 \times 1, 11 \times 2)$	0	0
$(7 \times 1, 15 \times 2)$	0	0	$(17 \times 1, 10 \times 2)$	0	0
$(9 \times 1, 14 \times 2)$	0	0	$(19 \times 1, 9 \times 2)$	0	0

4.1 SRGs with parameters $(37, 18, 8, 9)$ having an automorphism group of order two

Using the program Mathematica we get all the possible orbit lengths distribution that satisfy Theorem 3.3, and using our own programs written in GAP [13] we construct all orbit matrices for the given orbit lengths distributions. In Table 2 we present the number of mutually nonisomorphic orbit matrices for Z_2 for each orbit lengths distribution. In the next step we obtain the adjacency matrices of strongly regular graphs with parameters $(37, 18, 8, 9)$. Finally, we check isomorphisms of strongly regular graphs using GAP. Thereby we prove Theorem 4.1. The number of the constructed nonisomorphic SRGs with parameters $(37, 18, 8, 9)$ are presented in Table 2. Orders of the full automorphism groups of these SRGs, also determined by using GAP, are shown in Table 3.

Table 3: SRGs with parameters $(37, 18, 8, 9)$ that have automorphisms of order 2

$\lvert \mathrm{Aut}(\Gamma_i) \rvert$	#SRGs
2	3
18	2
666	1

Theorem 4.1. *Up to isomorphism there exists exactly* 6 *strongly regular graphs with parameters* $(37, 18, 8, 9)$ *having an automorphism group of order 2.*

4.2 SRGs with parameters $(37, 18, 8, 9)$ having an automorphism group of order three

Using the program Mathematica we get all the possible orbit lengths distribution that satisfy Theorem 3.3, and using our own programs written in GAP [13] we construct all orbit matrices for given orbit lengths distributions. In Table 4 we present the number of mutually nonisomorphic orbit matrices for Z_3 for each orbit lengths distribution. In the next step we obtain the adjacency matrices of strongly regular graphs with parameters $(37, 18, 8, 9)$. Finally, we check isomorphisms of strongly regular graphs using GAP. Thereby we prove Theorem 4.2. The number of the constructed nonisomorphic SRGs with parameters $(37, 18, 8, 9)$ are presented in Table 4. Orders of the full automorphism groups of these SRGs are presented in Table 5.

Table 4: Number of orbit matrices and SRGs$(37, 18, 8, 9)$ for the automorphism group Z_3

distribution	#OM	#SRGs	distribution	#OM	#SRGs
$(1 \times 1, 12 \times 3)$	18	37	$(13 \times 1, 8 \times 3)$	0	0
$(4 \times 1, 11 \times 3)$	0	0	$(16 \times 1, 7 \times 3)$	0	0
$(7 \times 1, 10 \times 3)$	0	0	$(19 \times 1, 6 \times 3)$	0	0
$(10 \times 1, 9 \times 3)$	0	0			

Theorem 4.2. *Up to isomorphism there exists exactly* 37 *strongly regular graphs with parameters* $(37, 18, 8, 9)$ *having an automorphism group of order 3.*

Table 5: SRGs with parameters $(37, 18, 8, 9)$ that have automorphisms of order 3

$\lvert \mathrm{Aut}(\Gamma_i) \rvert$	#SRGs
3	30
9	4
18	2
666	1

4.3 SRGs $(37, 18, 8, 9)$ for Z_p, where p is a prime and $3 < p \leq 37$

We show that there is no orbit matrix for Z_p, where p is a prime and $3 < p < 37$. The results are presented in Table 6. Hence, there is no $\mathrm{SRG}(37, 18, 8, 9)$ having an automorphism group isomorphic to Z_p, where p is a prime and $3 < p < 37$. Furher, there is exactly one $\mathrm{SRG}(37, 18, 8, 9)$ admitting an automorphism group isomorphic to Z_{37}, namely the Paley graph with 37 vertices having the full automorphism group isomorphic to $Z_{37} : Z_{18}$.

Table 6: Possible distributions for Z_p, p a prime and $3 < p < 37$

distribution	#OM	distribution	#OM
$(2 \times 1, 7 \times 5)$	0	$(15 \times 1, 2 \times 11)$	0
$(7 \times 1, 6 \times 5)$	0	$(11 \times 1, 2 \times 13)$	0
$(12 \times 1, 5 \times 5)$	0	$(3 \times 1, 2 \times 17)$	0
$(17 \times 1, 4 \times 5)$	0	$(20 \times 1, 1 \times 17)$	0
$(2 \times 1, 5 \times 7)$	0	$(18 \times 1, 1 \times 19)$	0
$(9 \times 1, 4 \times 7)$	0	$(14 \times 1, 1 \times 23)$	0
$(16 \times 1, 3 \times 7)$	0	$(8 \times 1, 1 \times 29)$	0
$(4 \times 1, 3 \times 11)$	0	$(6 \times 1, 1 \times 31)$	0

We summarize the presented information in Theorem 4.3.

Theorem 4.3. *Up to isomorphism there exists at least* 6766 *strongly regular graphs with parameters* $(37, 18, 8, 9)$*. These are exactly forty SRGs*$(37, 18, 8, 9)$ *having nontrivial automorphisms, and at least* 6726 *SRGs*$(37, 18, 8, 9)$ *having the full automorphism group of order one.*

The adjacency matrices of the six newly constructed SRGs can be found at the link:

`http://www.math.uniri.hr/~mmaksimovic/srg37.txt`.

ORCID iDs

Dean Crnković https://orcid.org/0000-0002-3299-7859
Marija Maksimović https://orcid.org/0000-0002-8094-3724

References

[1] M. Behbahani, *On strongly regular graphs*, Ph.D. thesis, Concordia University, 2009, https://spectrum.library.concordia.ca/976720.

[2] M. Behbahani and C. Lam, Strongly regular graphs with non-trivial automorphisms, *Discrete Math.* **311** (2011), 132–144, doi:10.1016/j.disc.2010.10.005.

[3] T. Beth, D. Jungnickel and H. Lenz, *Design theory. Vol. I*, volume 69 of *Encyclopedia of Mathematics and its Applications*, Cambridge University Press, Cambridge, 2nd edition, 1999, doi:10.1017/cbo9780511549533.

[4] A. E. Brouwer, Parameters of strongly regular graphs, https://www.win.tue.nl/~aeb/graphs/srg/srgtab51-100.html.

[5] D. Crnković, M. Maksimović, B. G. Rodrigues and S. Rukavina, Self-orthogonal codes from the strongly regular graphs on up to 40 vertices, *Adv. Math. Commun.* **10** (2016), 555–582, doi:10.3934/amc.2016026.

[6] D. Crnković and M.-O. Pavčević, Some new symmetric designs with parameters $(64, 28, 12)$, *Discrete Math.* **237** (2001), 109–118, doi:10.1016/s0012-365x(00)00364-2.

[7] D. Crnković and S. Rukavina, Construction of block designs admitting an abelian automorphism group, *Metrika* **62** (2005), 175–183, doi:10.1007/s00184-005-0407-y.

[8] D. Crnković, S. Rukavina and M. Schmidt, A classification of all symmetric block designs of order nine with an automorphism of order six, *J. Combin. Des.* **14** (2006), 301–312, doi:10.1002/jcd.20090.

[9] C. Godsil and G. Royle, *Algebraic graph theory*, volume 207 of *Graduate Texts in Mathematics*, Springer-Verlag, New York, 2001, doi:10.1007/978-1-4613-0163-9.

[10] Z. Janko, Coset enumeration in groups and constructions of symmetric designs, in: *Combinatorics '90 (Gaeta, 1990)*, North-Holland, Amsterdam, volume 52 of *Ann. Discrete Math.*, pp. 275–277, 1992, doi:10.1016/s0167-5060(08)70919-1.

[11] B. D. McKay and E. Spence, Classification of regular two-graphs on 36 and 38 vertices, *Australas. J. Combin.* **24** (2001), 293–300.

[12] E. Spence, Strongly regular graphs on at most 64 vertices, http://www.maths.gla.ac.uk/~es/srgraphs.php.

[13] The GAP Group, Gap, 2018, https://www.gap-system.org.

[14] V. D. Tonchev, *Combinatorial configurations: designs, codes, graphs*, volume 40 of *Pitman Monographs and Surveys in Pure and Applied Mathematics*, Longman Scientific & Technical, Harlow; John Wiley & Sons, Inc., New York, 1988, translated from the Bulgarian by Robert A. Melter, https://books.google.com/books?id=J_XuAAAAMAAJ.