ISSN 2590-9770
The Art of Discrete and Applied Mathematics 3 (2020) #P1.05
https://doi.org/10.26493/2590-9770.1290.f73
(Also available at http://adam-journal.eu)

Regular balanced Cayley maps on nonabelian metacyclic groups of odd order*

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Received 30 January 2019, accepted 2 September 2019, published online 27 July 2020

Abstract

In this paper, we show that nonabelian metacyclic groups of odd order do not have regular balanced Cayley maps.

Keywords: Regular balanced Cayley map, metacyclic group.

Math. Subj. Class. (2020): 05C25, 05C30

1 Introduction

Groups are often studied in terms of their action on the elements of a set or on particular objects within a structure. The aim of this article is to study metacyclic groups of odd order acting on maps. A *Cayley graph* $\Gamma = \mathrm{Cay}(G, X)$ will be a graph based on a group G and a generating set $X = \{x_1, x_2, \dots, x_k\}$ which does not contain 1_G, is closed under the operation of taking inverses. In this paper, we call X a *Cayley subset* of G. The vertices of the Cayley graph Γ are the elements of G, and two vertices g and h are joined by an edge if and only if $g = hx_i$ for some $x_i \in X$. The ordered pairs (h, hx) for $h \in G$ and $x \in X$ are called the darts of Γ. For a cyclic permutation ρ of the set X, the Cayley map $\mathcal{M} = \mathrm{CM}(G, X, \rho)$ is the 2-cell embedding of the Cayley graph $\mathrm{Cay}(G, X)$

*The authors want to thank the reviewer and editors for their patience on revising our paper and giving us valuable suggestions.

†Corresponding author. Supported by NSFC (No. 11671276, 11671347, 61771019, 11771258), NSFS (No. ZR2017MA022) and J16LI02.

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in some orientable surface such that the local rotation of darts emanating from every vertex is induced by the same cyclic permutation ρ of X.

An (orientation preserving) *automorphism* of a Cayley map $\mathcal{M}$ is a permutation of the set of darts of $\mathcal{M}$ which preserves the incidence relation of the vertices, edges, faces, and the orientation of the map. The full automorphism group of $\mathcal{M}$, denoted by $\mathrm{Aut}(\mathcal{M})$, is the group of all such automorphisms of $\mathcal{M}$ under the operation of composition. This group always acts semi-regularly on the set of darts of $\mathcal{M}$, that is, the stabilizer in $\mathrm{Aut}(\mathcal{M})$ of each dart of $\mathcal{M}$ is trivial. If the action of $\mathrm{Aut}(\mathcal{M})$ on the darts of $\mathcal{M}$ is transitive (and therefore regular), we say that the Cayley map $\mathcal{M}$ is a regular Cayley map. As the left regular multiplication action of the underlying group G lifts naturally into the full automorphism group of any Cayley map $\mathrm{CM}(G, X, \rho)$, Cayley maps are a very good source of regular maps. There are many papers on the topic of regular Cayley maps, we refer the readers to [2], [5] and [6] and the references therein. Furthermore, A Cayley map $\mathrm{CM}(G, X, \rho)$ is called *balanced* if $\rho(x)^{-1} = \rho(x^{-1})$ for every $x \in X$. In [6], the authors showed that a Cayley map $\mathrm{CM}(G, X, \rho)$ is regular and balanced if and only if there exists a group automorphism σ such that $\sigma|_X = \rho$, where $\sigma|_X$ denotes the restricted action of σ on X. Therefore, determining all the regular balanced Cayley maps of a group is equivalent to determining all the orbits of its automorphisms that can be Cayley subsets.

In [2], it was shown that all odd order abelian groups possess at least one regular balanced Cayley map [2]. Wang and Feng [7] classified all regular balanced Cayley maps for cyclic, dihedral and generalized quaternion groups. In [4], the author proved the non-existence of regular balanced Cayley maps with semi-dihedral groups. In [8], Yuan, Wang and Qu proved that a nonabelian metacyclic p-group for an odd prime number p does not have regular balanced Cayley maps. This was the first work on regular balanced Cayley map of nonabelian groups of odd order. We will take a step further and show that non-abelian metacyclic groups of odd order do not have regular balanced Cayley maps.

2 Preliminaries

We use the standard notation for group theory; see [3]. We denote by (r, s) the greatest common divisor of two positive integers r and s. By $|x|$, $|H|$, we denote the order of element x and subgroup H of a group G, respectively. By $N : H$, we denote a semidirect product of the group N by the group H. Set $[x, y] = x^{-1}y^{-1}xy$, the commutator of x and y and set $[H, K] = \langle [h, k] \mid h \in H, k \in K \rangle$, where $H, K \leq G$. For $\alpha \in \mathrm{Aut}(G)$ and $g \in G$, denote the orbit of g under $\langle \alpha \rangle$ by $g^{\langle \alpha \rangle}$.

Define $G_1 = G$, and, proceeding recursively, define $G_n = [G_{n-1}, G]$ for $1 < n \in \mathbb{Z}$. Then G is said to be *nilpotent* if $G_n = 1$ for some $1 \leq n \in \mathbb{Z}$. The following lemma is basic for nilpotent groups.

Lemma 2.1 ([3, Kapitel III.2.3])**.** *A finite group is nilpotent if and only if it is the direct product of its Sylow groups.*

Lemma 2.2 ([3, Kapitel III.7.2])**.** *Let G be a p-group, $N \trianglelefteq G$ and $|N| = p$. Then $N \leq Z(G)$.*

Define $M_{p,q}(m, r) = \langle a, b \mid a^p = 1, b^{q^m} = 1, b^{-1}ab = a^r \rangle$, where p and q are distinct prime numbers, m is a positive integer and $r \not\equiv 1 \pmod{p}$ but $r^q \equiv 1 \pmod{p}$. The following lemma is about the automorphism group of $M_{p,q}(m, r)$.

Lemma 2.3 ([8, Lemma 2.2]). *The automorphism group of $M_{p,q}(m,r)$ is*

$$\mathrm{Aut}(M_{p,q}(m,r)) = \{\sigma \mid a^\sigma = a^i,\ b^\sigma = b^j a^k, 1 \leqslant i \leqslant p-1,\ 1 \leqslant j \leqslant q^m-1, q \mid (j-1)\ \}.$$

The following lemma shows that G/N has a regular balanced Cayley map whenever G has.

Lemma 2.4 ([8, Lemma 2.5]). *Let G be a finite group and N be a nontrivial characteristic subgroup of G. Take $\alpha \in \mathrm{Aut}(G)$ and $g \in G$. If $X = g^{\langle\alpha\rangle}$ is a Cayley subset of G, then $\overline{X} = \overline{g^{\langle\alpha\rangle}} = \bar{g}^{\langle\bar{\alpha}\rangle}$ is a Cayley subset of $\overline{G} = G/N$.*

Proposition 2.5 ([8, Corollary 4.7]). *For any odd prime number p, the nonabelian metacyclic p-group does not have regular balanced Cayley maps.*

3 Regular balanced Cayley maps on nonabelian metacyclic group of odd order

A metacyclic group is an extension of a cyclic group by a cyclic group. That is, it is a group G having a cyclic normal subgroup $\langle a\rangle$ such that the quotient $G/\langle a\rangle$ is also cyclic. We now prove our main theorem.

Theorem 3.1. Nonabelian metacyclic groups of odd order do not have regular balanced Cayley maps.

Proof. Let G be a counterexample of minimal order. Then G is a metacyclic group of odd order which has regular balanced Cayley maps. Set $G = \langle a\rangle\langle b\rangle$ where $\langle a\rangle \trianglelefteq G$. We will get a contradiction through the following seven steps.

Step 1. The order of the derived subgroup G' of G is a prime number p:

Clearly, the derived subgroup G' is a subgroup of $\langle a\rangle$. Thus G' is cyclic. If the order of G' is not a prime number, then there is a proper subgroup N of G' which has prime order. Since N is characteristic in G' and G' is characteristic in G, we find that N is characteristic in G. Now consider the quotient group G/N. On one hand, by Lemma 2.4, G/N has a regular balanced Cayley map. On the other hand, from the choice of G, G/N does not have regular balanced Cayley maps; a contradiction. Thus $|G'|$ is a prime number p.

Step 2. G does not have a nontrivial normal p'-subgroup. In particular, $Z(G) = 1$ or $Z(G)$ is a p-group:

If not, let N be a nontrivial normal p'-subgroup of G. We consider G/N. Since $|G'| = p$, we know that G/N is still nonabelian. By the minimality of G, it follows that G/N does not have regular balanced Cayley maps. However, by Lemma 2.4, G/N has a regular balanced Cayley map. This is a contradiction.

Step 3. The subgroup $\langle a\rangle$ is a p-group:

Otherwise, suppose $\langle a\rangle = P \times Q$, where P is a p-group and Q is a non-trivial p'-group. Since Q is a characteristic subgroup of $\langle a\rangle$, it is normal in G. This contradicts Step 2.

Step 4. $G' \nleq Z(G)$:

Otherwise, assume $G' \leq Z(G)$. Then G is nilpotent. By Lemma 2.1, we have $G = P_1 \times Q_1$, where P_1 is a p-group and Q_1 is a p'-group. By Step 2, $Q_1 = 1$. Thus G is a p-group. This contradicts Proposition 2.5.

Now, we assume $\langle b\rangle = P_2 \times Q_2$, where P_2 is a p-group and Q_2 is a non-trivial p'-group. We will show $P_2 = 1$, so that $\langle b\rangle$ is a p'-group.

Step 5. The order of $\langle a\rangle$ is p:

Set $|a| = p^n$ and $Q_2 = \langle c\rangle$. By Step 1, $[a, c] = a^{ip^{n-1}}$. Hence $a^c = a^{1+ip^{n-1}}$. Notice that $(1 + ip^{n-1})^p \equiv 1 \bmod p^n$ when $n > 1$. If $n > 1$, then $a^{c^p} = a$. Hence $[c^p, a] = 1$. Since $(|c|, p) = 1$, we find that $c \in \langle c^p\rangle$. Thus $[a, c] = 1$. This gives $Q_2 \le Z(G)$, contradicting Step 2. Thus $n = 1$.

Step 6. $Z(G) = 1$ and $P_2 = 1$:

Since G is nonabelian, $\langle a\rangle \not\le Z(G)$. By Step 5, $\langle a\rangle \cap Z(G) = 1$. Thus $G/Z(G)$ is still nonabelian. By the choice of G, we find that $Z(G) = 1$.

Since $\langle a\rangle$ is a normal subgroup of $\langle a\rangle P_2$ and the order of $\langle a\rangle$ is p, from Lemma 2.2, we have $\langle a\rangle \le Z(\langle a\rangle P_2)$. It follows that $P_2 \le Z(G)$. So $P_2 = 1$.

Step 7. G does not have regular balanced Cayley maps:

We can assume $G = \langle a\rangle : \langle b\rangle$, where $\langle a\rangle \cong \mathbb{Z}_p$ and $\langle b\rangle$ is a p'-group. Since $Z(G) = 1$, we have $C_{\langle b\rangle}(a) = 1$. By the Normalizer-Centralizer (N/C) Theorem, $N_{\langle b\rangle}(a)/C_{\langle b\rangle}(a) \lesssim \mathrm{Aut}(\langle a\rangle)$, and so $\langle b\rangle \lesssim \mathrm{Aut}(\langle a\rangle)$. Take a prime factor q of $|b|$. If G has a regular balanced Cayley map, then we may take a corresponding Cayley subset X of G. Without loss of generality, we assume $b \in X$. Then there is some $\sigma \in \mathrm{Aut}(G)$ such that $b^\sigma = b^{-1}$. Let $b_1 = b^{|b|/q}$. Then $b_1^\sigma = b_1^{-1}$. Set $K = \langle a\rangle : \langle b_1\rangle$ and let σ_1 be the restricted action of σ on K. Then $\sigma_1 \in \mathrm{Aut}(K)$. However, from Lemma 2.3, the metacyclic group K of order pq does not have any automorphism that can reverse b_1, a contradiction. Hence nonabelian metacyclic groups of odd order do not have regular balanced Cayley maps. □

From Lemma 2.4 and Theorem 3.1, we get the following:

Corollary 3.2. Let H be a characteristic subgroup of G. If the quotient group G/H is isomorphic to a nonabelian metacyclic group of odd order, then G does not admit regular balanced Cayley maps. In particular, a direct product of a nonabelian metacyclic group of odd order and a 2-group does not admit regular balanced Cayley maps.

Remark 3.1. Since the only nonabelian group with odd order less than 26 is metacyclic, we know from Corollary 3.2 that the minimal odd order of a nonabelian group that admits a regular balanced Cayley map is at least 27. The only non-metacyclic and nonabelian group of order 27 is $M_3(1,1,1) = \langle a, b, c \mid a^3 = b^3 = c^3 = 1, [a, b] = c, [c, a] = [c, b] = 1\rangle$. With the help of MAGMA [1], we can easily see that $M_3(1, 1, 1)$ has regular balanced Cayley maps, and the corresponding Cayley graphs have valency $4, 6$ and 8, respectively.

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