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My journey into noncommutative lattices and their theory

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Abstract

This paper describes the motivations leading to a renewed interest in the study of noncommutative lattices, and especially skew lattices, beginning with the initial work of the author. Not only are primary concepts and results recalled, but recognition is given to the individuals involved and their particular contributions. It is the written version of a talk given at the NCS2018 workshop in May, 2018 in Portorož, Slovenia.

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I started thinking about skew lattices in 1983, while visiting Case Western Reserve University as a guest of Charles Wells. My connection with Charles was a common interest in the cohomology of monoids. I had published a paper that presented a new type of cohomology for monoids in the *Memoirs of the American Mathematical Society* in 1975 and Charles had published a follow-up paper in the *Semigroup Forum* in 1978 that connected my work to a general approach to cohomology theories due to Jonathan Beck. In my office at Case-Western I was studying the Wells-Beck approach for specific classes of monoids. In the case where the underlying monoid was a semilattice, I was led to consider bands whose maximal semilattice image was the given semilattice. Now, every band that arose within the Wells-Beck confines was *regular* in that it satisfied the identity, $xyxzx = xyzx$. This led me to look at the occurrence of regular bands in other mathematical contexts, and in particular, to their occurrence *and behavior* as multiplicative subsemigroups of a ring. This in turn led me straight to skew lattices.

Suppose first that we are given a multiplicative semilattice of idempotents L in a ring R. (L is thus closed and commutative under multiplication.) It is well known that L will generate a lattice L' of idempotents with the meet and join given by

$$x \wedge y = xy \quad \text{and} \quad x \vee y = x + y - xy \text{ (the \textit{quadratic join})}.$$

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Indeed if we include 0 in L and throw in the relative complement $x \setminus y = x - xy$, L generates a generalized Boolean algebra of idempotents. It will be fully Boolean if a top element is generated from L, and in particular if R has an identity 1 that is thus generated.

The obvious question: what occurs for a multiplicative band S of idempotents in a ring R, be S regular or otherwise? Well the following occur:

(1) In general, these two operations need not generate a larger class of idempotents that is closed under both operations … even if S is known to be regular.

(2) But, if S is known to be *left regular* ($xyx = xy$) or *right regular* ($xyx = yx$), then S generates a set of idempotents S' that is closed under both operations above.

(3) The resulting algebra $(S', \wedge, \vee)$ is a *skew lattice* in that $\wedge$ and $\vee$ are associative, idempotent binary operations that together satisfy the absorption identities:

$$x \wedge (x \vee y) = x = (y \vee x) \wedge x \quad \text{and} \quad x \vee (x \wedge y) = x = (y \wedge x) \vee x.$$

Given that $\wedge$ and $\vee$ are associative and idempotent, these identities are equivalent to the *basic dualities*:

$$u \wedge v = u \text{ iff } u \vee v = v \quad \text{and} \quad u \wedge v = v \text{ iff } u \vee v = u.$$

(4) If S is left regular, then so is $(S', \wedge)$ with $(S', \vee)$ being right regular. Dual remarks hold when S is right regular.

(5) Conversely, given any skew lattice $(S, \wedge, \vee)$ both reducts $(S, \wedge)$ and $(S, \vee)$ are regular with one operation being left regular iff the other is right regular.

Skew lattices in general had a number of other discernable properties:

(1) A *natural partial order*: $x \leq y \Leftrightarrow x \wedge y = x = y \wedge x \Leftrightarrow x \vee y = y = y \vee x$.

(2) A *natural quasiorder*: $x \preceq y \Leftrightarrow x \wedge y \wedge x = x \Leftrightarrow y \vee x \vee y = y$.

(3) A *natural congruence* $\mathcal{D}$:

$$\begin{aligned} x\, \mathcal{D}\, y \quad & \text{iff} \quad x \wedge y \wedge x = x \;\&\; y \wedge x \wedge y = y \\ & \text{iff} \quad x \vee y \vee x = x \;\&\; y \vee x \vee y = y. \end{aligned}$$

(4) **Clifford-McLean Theorem:** *given a skew lattice* $(S, \wedge, \vee)$*,* $(S/\mathcal{D}, \wedge, \vee)$ *is its maximal lattice image and the* $\mathcal{D}$*-classes are its maximal* anti-commutative *subalgebras in that:*

$$x \wedge y = y \wedge x \Leftrightarrow x = y \Leftrightarrow x \vee y = y \vee x$$

holds.

Also $x \wedge y = y \vee x$ holds in every $\mathcal{D}$-class. Thus there are two basic subvarieties of skew lattices:

- Lattices (everybody commutes).
- *Anti-lattices*, also called *rectangular* skew lattices (no nontrivial commutation).

The Clifford-McLean Theorem thus states that *every skew lattice is a lattice of anti-lattices*. See Figure 1 below.

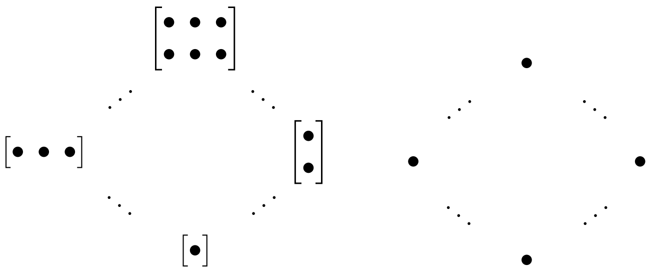

Figure 1: A lattice of anti-lattices.

Here are two more basic subvarieties:

- *Left-handed skew lattices:* $x \wedge y \wedge x = x \wedge y$ & $x \vee y \vee x = y \vee x$. $(S, \wedge)$ is left regular and thus $(S, \vee)$ is right regular.
- *Right-handed skew lattices:* $x \wedge y \wedge x = y \wedge x$ & $x \vee y \vee x = x \vee y$. $(S, \wedge)$ is right regular and thus $(S, \vee)$ is left regular. Their intersection is, of course, the variety of lattices.

(5) **Kimura's Theorem:** *If S_L and S_R are the maximal left- and right-handed images of a skew lattice S, the induced commuting diagram of epimorphisms is a pullback.*

$$\begin{tikzcd} S \arrow[r] \arrow[d] & S_L \arrow[d] \\ S_R \arrow[r] & S/\mathcal{D} \end{tikzcd}$$

Thus S is isomorphic to the fibered product: $S_R \times_{S/\mathcal{D}} S_L$.

Since both subvarieties are term equivalent, to the extent that one understands one, one understands the other, and thus to a large extent skew lattices in general. Both theorems above are so-named after two similar theorems about bands and regular bands respectively.

Here are *possible* properties that *do* occur in any skew lattice of idempotents in a ring:

(1) *Symmetry*: $x \wedge y = y \wedge x$ iff $x \vee y = y \vee x$. (A very nice condition.)

(2) *Distributivity*:

$$x \wedge (y \vee z) \wedge x = (x \wedge y \wedge x) \vee (x \wedge z \wedge x), \text{ and}$$
$$x \vee (y \wedge z) \vee x = (x \vee y \vee x) \wedge (x \vee z \vee x).$$

(3) *Cancellation*:

$$x \wedge y = x \wedge z \text{ and } x \vee y = x \vee z \Longrightarrow y = z, \text{ and}$$
$$x \wedge z = y \wedge z \text{ and } x \vee z = y \vee z \Longrightarrow x = y.$$

Some general facts:

- Neither distributivity as defined nor cancellation implies the other.
- Cancellation does imply that a skew lattice is symmetric.
- Neither distributive identity implies the other.
- But for *symmetric* skew lattices, the two distributive identities *are* equivalent.
- In the *symmetric* case, every pairwise commuting subset generates a sublattice.
- A non-symmetric example exists with 3 commuting generators, that is not a lattice.
- Clearly, *maximal left (right) regular bands of idempotents in a ring form skew lattices that have all three properties.*

Another *possible* property: A band is *normal* if it is *mid-commutative*: $xyzw = xzyw$. Normal bands are easily seen to be regular. A skew lattice $(S, \wedge, \vee)$ is *normal* if its $\wedge$-reduct $(S, \wedge)$ is normal. (The $\vee - \wedge$ dual is *conormal*.) Clearly distributive skew lattices and normal skew lattices form subvarieties of skew lattices. So do symmetric skew lattices.

Some theorems:

- *A normal skew lattice S is distributive iff $S/\mathcal{D}$ is a distributive lattice.*
- *Normal, distributive skew lattices are characterized by the identity:*

$$x \wedge (y \vee z) \wedge w = (x \wedge y \wedge w) \vee (x \wedge z \wedge w).$$

- *Normal, symmetric and distributive (NSD) skew lattices are characterized by:*

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \quad \textit{and} \quad (y \vee z) \wedge w = (y \wedge w) \vee (z \wedge w).$$

- *A normal band of idempotents in a ring generates an NSD skew lattice.* (Normal skew lattices in rings need no longer be left or right-handed. An associative *cubic join* is given by $x \nabla y = x + y + yx - xyx - yxy$. In left- or right-handed contexts $x \nabla y$ reduces to the previous *quadratic join* $x + y - xy$.)
- *Maximal normal bands in a ring form skew Boolean algebras* (defined below).
- *If the idempotents of a ring are closed under multiplication, then they are normal as a band and thus form a skew Boolean algebra.*

A *skew Boolean algebra* (SBA) is an algebra $(S, \wedge, \vee, \setminus, 0)$ for which $(S, \wedge, \vee)$ is an NSD skew lattice, $\setminus$ is a binary operation and 0 is a constant such that for all x, y:

(i) $0 \wedge x = 0 = x \wedge 0$ and hence $0 \vee x = x = x \vee 0$;

(ii) $(x \wedge y \wedge x) \vee (x \setminus y) = x = (x \setminus y) \vee (x \wedge y \wedge x)$ and
$(x \wedge y \wedge x) \wedge (x \setminus y) = 0 = (x \setminus y) \wedge (x \wedge y \wedge x)$.

This brings us to a second class of motivating examples: *partial function algebras*.

If we are given sets A and B, let $\mathcal{P}(A, B)$ denote the set of all partial functions f from A to B. Special case: B is $\{1\}$. Here $\mathcal{P}(A, \{1\})$ may be identified with the power set $\mathcal{P}(A)$ of A under the map $f \to \mathrm{dom}(f)$. $\mathcal{P}(A)$ forms, of course, a typical example of a Boolean algebra.

Likewise $\mathcal{P}(A,B)$ forms a typical example of a skew Boolean algebra. One can do this in two ways: a left-handed way and a right-handed way. For both ways, $\mathcal{P}(A)$ forms the maximal Boolean algebra image. We consider the right-handed case, the left-handed version being dual. Given f, g in $\mathcal{P}(A,B)$ with respective domains F and G, set

$$f \wedge g = g|_{F\cap G}; \qquad f \vee g = f \cup g|_{G-F}; \qquad f \setminus g = f|_{F-G}; \quad \text{and} \qquad 0 = \emptyset.$$

Basic theorems:

(1) *Every left-handed skew Boolean algebra can be embedded in a left-handed partial function algebra.*

(2) *Every right-handed skew Boolean algebra can be embedded in a right-handed partial function algebra.*

(3) *Every skew Boolean algebra is the fibered product of a left-handed SBA and right-handed SBA over their common maximal generalized Boolean algebra image.*

(4) *A skew lattice can be embedded in a skew Boolean algebra iff it is normal, distributive and symmetric.*

Skew lattices in rings and partial function algebras formed the concrete bases of my first three full-length publications on skew lattices:

- "Skew lattices in rings" appeared in *Algebra Universalis* in 1989 [44];
- "Skew Boolean algebras" appeared in *Algebra Universalis* in 1990 [45];
- "Normal skew lattices" appeared in *Semigroup Forum* in 1992 [46].

The communicating editor for all three, Boris Schein, had once published a paper on a class of noncommutative lattices in a Russian journal that was later translated into English by the AMS.

What all was I doing between 1983 and the first publication in 1989?

(1) I gave a number of talks to various groups: seminars; AMS-MAA metings.

(2) I kept polishing up things: examples; proofs, etc.

(3) I was also preoccupied with writing papers on other topics.

A third class of examples attracted my attention in my early research: *primitive* skew lattices, which consisted of exactly two $\mathcal{D}$-classes, an upper class and a lower class: $A > B$. As it turned out, a complete characterization of these primitive algebras is easily given.

Given $A > B$, A is partitioned by B-cosets in A and B is partitioned by A-cosets in B. Here, for each a in A, its *B-coset* is $B \vee a \vee B = \{b \vee a \vee b \mid b \in B\} \subseteq A$; likewise, for each b in B, it's *A-coset* is $A \wedge b \wedge A = \{a \wedge b \wedge a \mid a \in A\} \subseteq B$. Thus given a, a' in A, either $B \vee a \vee B = B \vee a' \vee B$ or both cosets are disjoint. Similar remarks hold for A-cosets in B. What is more, all cosets in A or B are mutually isomorphic. In particular, given any B-coset A_i in A and any A-coset B_j in B, an isomorphism $\varphi_{ij}\colon A_i \cong B_j$ is given by $\varphi_{ij}(a) = b$ for $a \in A$ and $b \in B$ iff $a > b$. This gives us a picture something like the one in Figure 2.

Thus, *all* cosets are mutually isomorphic with the *coset isomorphisms* determining $\wedge$ and $\vee$ *between* cosets in A and B. That is, for all a in A_i and all b in B_j:

$$a \wedge b = \varphi_{ij}(a) \wedge b \;\&\; b \wedge a = b \wedge \varphi_{ij}(a) \;\text{ and}$$
$$a \vee b = a \vee \varphi_{ij}^{-1}(b) \;\&\; b \vee a = \varphi_{ij}^{-1}(b) \vee a.$$

$$A = \begin{bmatrix} \begin{bmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{bmatrix} & \begin{bmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{bmatrix} \\ \begin{bmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{bmatrix} & \begin{bmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{bmatrix} \end{bmatrix} \qquad B = \begin{bmatrix} \begin{bmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{bmatrix} & \begin{bmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{bmatrix} & \begin{bmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{bmatrix} \end{bmatrix}$$

Figure 2: A coset decomposition showing isomorphic cosets.

Conversely, this characterization provided a general recipe for constructing primitive algebras.

In this situation, the *coset weight* $\gamma(A, B)$ is the common size of all cosets in $A \cup B$. The index $[A : B]$ of B in A is the number of B-cosets in A; dually the index $[B : A]$ of A in B is the number of A-cosets in B.

In the finite case, as with finite groups one thus has a *Lagrange-type* theorem:

Theorem 0.1. *If A and B in the primitive skew lattice $A > B$ are both finite, then:*

$$|A| = [A : B]\gamma(A, B) \quad \textit{and} \quad |B| = [B : A]\gamma(A, B).$$

Corollary 0.2. *Given $A > B$, if $|A|$ & $|B|$ are finite and coprime, then $\gamma(A, B) = 1$ and*

$$\forall a \in A, \forall b \in B\colon (a \wedge b = b \wedge a = b \;\&\; a \vee b = b \vee a = a).$$

These algebras form the basis for an analysis of skew lattice structure – or one might say, of the *architecture* of skew lattices. For instance, consider a *skew diamond* of $\mathcal{D}$-classes in a skew lattice S. Here J and M are the join and meet $\mathcal{D}$-classes of $\mathcal{D}$-classes A and B.

$$\begin{tikzcd} & J \arrow[dl, dash, dotted] \arrow[dr, dash, dotted] & \\ A \arrow[dr, dash, dotted] & & B \arrow[dl, dash, dotted] \\ & M & \end{tikzcd}$$

If both $|A|, |B| < \infty$, then also both $|J|, |M| < \infty$. Indeed, both $|J|$ and $|M|$ divide $|A||B|$.

If S is cancellative, then $|A||B| = |J||M|$ (João Pita da Costa 2012 Dissertation [54]).

As consequences we have:

(1) The union of all singleton $\mathcal{D}$-classes forms a sublattice $\mathcal{Z}_S$ of S.

(2) $\mathcal{Z}_S$ is the center of S:

$$\begin{aligned} \mathcal{Z}_S &= \{x \in S \mid x \wedge y = y \wedge x \text{ for all } y \in S\} \\ &= \{x \in S \mid x \vee y = y \vee x \text{ for all } y \in S\}. \end{aligned}$$

(3) The union of all finite $\mathcal{D}$-classes is a subalgebra.

(4) Given prime p, the union of all $\mathcal{D}$-classes of p-power size is a subalgebra.

With the exception of Pita da Costa's result, the above formed part of the content the fourth paper:

- "The geometric structure of skew lattices" appeared in *Transactions of the American Mathematical Society* in 1993 [47].

First Contact! Some time after the publication of my paper on skew Boolean algebras, I received a letter from Robert J. Bignall of Monash University in Australia. In it I discovered that a paper entitled "Boolean skew algebras" had been published in 1980 by his dissertation advisor, William Cornish [15]. (Some may be aware that Bill Cornish was one of the first to publish work in response to the extension of Stone duality to bounded distributive lattice by Hillary Priestly at Oxford.) Bob Bignall's 1976 dissertation written in South Australia was entitled *Quasiprimal Varieties and Components of Universal Algebra* [5]. It began with a chapter entitled "Quasi-Boolean skew lattices". While not term equivalent to the skew Boolean algebras I had studied, both types of algebras were quite similar in spirit. Some will find it interesting that in his dissertation Bob studied sheaf theoretic representations of these algebras. His interest in noncommutative Boolean algebras also manifested itself in his 1991 paper, "A non-commutative multiple-valued logic", that appeared in the *Proceedings of the 21st International Symposium on Multiple-Valued Logic*, sponsored by the IEEE Computer Society [6]. Indeed he continued to author further papers in this area.

I also received a copy of a paper he had submitted to *Algebra Universalis*. It was about a class of algebras very much like my skew Boolean algebras except that his relative complement was different, and of course, there was a slightly different axiom system. Also, his algebras had close connections to what are called *discriminator algebras* which Stanley Burris had called "the most successful generalization of Boolean algebras to date" in his 1981 text on universal algebra [11].

Later on Bob visited me for a few days in Santa Barbara. One morning, after breakfast at a seaside restaurant, I shared some thoughts on how our two types of algebras could be merged. The means to do this was the concept of any two elements having a meet relative to the natural partial order – their *intersection* – as opposed to their skew meet (i.e. "meet" in a general noncommutative context). Bignall's difference essentially involved subtracting the intersection from one of the two given elements while mine involved subtracting their skew meet. This led us to the variety of skew Boolean algebras with intersections. As it turned out its subvariety of right-handed [or left-handed] algebras is term-equivalent to the variety of pointed discriminator algebras. Both types of algebras were the subject of our joint revision of Bob's earlier paper entitled "Skew Boolean algebras and discriminator varieties" that appeared in *Algebra Universalis* in 1995 [7]. The communicating editor was Stanley Burris.

It was during this time that Alfred Clifford passed away. A symposium in his honor was held at Tulane University where he had taught for many years. There I gave survey talk on recent developments in skew lattice theory. This talk was published as a survey article in the *Semigroup Forum* in 1996 [48]. This brought to six the number of articles I had published on skew lattices since 1989. I would not publish another until 2002.

In the meantime, much of my focus was on inverse monoids and especially the categorical foundations of symmetric inverse monoids and their duals. In particular, I coauthored a paper on dual symmetric inverse monoids with a colleague from Tasmania, Des FitzGerald, who happily was at this workshop. Our paper, "Dual symmetric inverse monoids and representation theory," appeared in 1998 in the *Journal of the Australian Mathematical Society* [31]. A main feature of inverse monoids is the fact that their elements have a natural partial order which in many cases has natural meets (or *intersections*, in our terminology). Natural meets received a good bit of attention in the papers I published during this period. My perspective on inverse monoids at this time was, no doubt, influenced by the paper coauthored with Bob Bignall. Connections between inverse monoids or inverse semigroups in general, and Boolean structures (often with intersections) has been a subject of study in recent years. For an extended exposition of these and related matters see Friedrich Wehrung's 2017 Springer monograph [60].

During this period, however, I started hearing from graduate students in Europe and Australia. One of the first was Gratiela Laslo, who was writing a dissertation on noncommutative lattices at Babes-Bolyai University in Cluj-Napoca, Romania. Her research, plus several insights from me, led to a seventh paper, co-authored with Gratiela and entitled "Green's equivalences on noncommutative lattices" that appeared in 2002 in the Szeged journal, *Acta Scientiarum Mathematicarum* [41]. Here, all involved algebras are generalizations of skew lattices with many results applying to skew lattices. The paper attempted to provide a coherent scheme consisting of four varieties into which nearly all of the previously studied types of noncommutative lattices could fit.

As it turns out, I wasn't Gratiela's only human connection to noncommutative lattices. She was also in contact with Professor Gheorghe Farcas of Petru Maior University in Targu Mures, Romania. Professor Farcas had published a number of papers on noncommutative lattices. But by the time I visited Gratiela in Targu Mures in 2005, he had been retired for some years and in ill health. Thus, regrettably, I never met him.

In the late 1990s I also started hearing from a protégé of Bob Bignall, Mathew Spinks. In an earlier paper, I had asked whether the dual pair of distributive identities that characterize distributive skew lattices are equivalent for skew lattices as they are for lattices. Spinks determined that they were not, publishing a set of four 9-element counter-examples in the *Semigroup Forum* in 2000 [58]. But there was more. Having had earlier access to his examples, I had noticed that they were non-symmetric. I asked Matthew if the two identities might be equivalent in the case of symmetric skew lattices. He initially found a computer-generated affirmative proof consisting of 757 steps. He was then able to reduce it to a 368-step proof that he published in a Monash University report: *Automated Deduction in Non-Commutative Lattice Theory* [57]. Several further reductions ensued. Finally a more standard "human" proof was obtained by Karin Cvetko-Vah and published in a short paper in the *Semigroup Forum* in 2006 [16]. Over the years Matthew and I have co-authored four papers:

- "Skew Boolean algebras derived from generalized Boolean algebras", in *Algebra Universalis* [49];
- "Cancellation in skew lattices" (with K. Cvetko-Vah and M. Kinyon), in *Order* [22];
- "Skew lattices and binary operations on functions" (with K. Cvetko-Vah), in *Journal of Applied Logic* [26];
- "Varieties of skew Boolean algebras with intersections", in *Journal of the Australian Mathematical Society* [50].

The last paper characterizes the lattice of subvarieties of this class of algebras. Spinks has also published very good papers with other individuals, including, of course, Bob Bignall.

Not all of these are about skew lattices. One of things that I appreciate about Matthew is his impressive knowledge of past and ongoing developments in universal algebra and logic. His ability to inject scholarly insights of relevance to a paper, or results of others that are critical to obtaining a proof or even a smoother proof, can make a decent paper good, and a good paper really good. It is interesting that both Matthew and Gratiela had initial connections to individuals who had engaged in serious research on noncommutative lattices.

This third graduate student was Karin Cvetko-Vah. I became aware of her in the early years of the new millennium, when she wrote and published several papers on multiplicative bands and skew lattices in rings. I recall reading her papers and discovering ideas and results that I had not considered. We first met at a Linear Algebra conference at Lake Bled in 2005. Since then Karin has written further papers about skew lattices in rings, three co-authored with me:

- "Associativity of the ∇-operation on bands in rings", in *Semigroup Forum* [23];
- "On maximal idempotent-closed subrings of $M_n(\mathbb{F})$", in *International Journal of Algebra and Computation* [24];
- "Rings whose idempotents form a multiplicative set", in *Communications in Algebra* [25].

The last two were on rings whose idempotents are closed under multiplication, and thus form, with additional operations, a skew Boolean algebra.

One of the things that has helped Karin and I work well together – besides the fact that she is very bright and hard-working – is her background in operator theory and in particular, matrix theory. Her dissertation advisor, Matjaž Omladič, was connected to a research group that included Peter Fillmore, Gordon MacDonald and Heydar Radjavi who among other things, studied multiplicative bands of idempotents in matrix rings. One result: *Every multiplicative band of idempotents in a matrix ring is simultaneously triangularizable.* Consequently *every skew lattice of idempotents in a matrix ring is simultaneously triangularizable.* Nice to know when you're looking for examples! In any case, with this background it's not that surprising that Karin and I might meet up.

Karin has authored and co-authored a number of important papers on the general structure of skew lattices. Besides her connection to Spinks' distributivity result, there is, e.g., her 2011 paper "On strongly symmetric skew lattices" that appeared in *Algebra Universalis* [17]. Another important contribution was also her involvement in research on duality theory extending the work of M. H. Stone and Hillary Priestly to skew Boolean algebras and strongly distributive skew lattices. Early in 2010, I mentioned to Karin that extending Stone duality for (generalized) Boolean algebras to a duality theory for skew Boolean algebras would be a worthy project. She brought this to the attention of two colleagues in Ljubljana, Andrej Bauer and Ganna Kudryavtseva. This led to a series of publications on duality that include:

- G. Kudryavtseva, "A refinement of Stone duality to skew Boolean algebras", in *Algebra Universalis* [36];
- A. Bauer and K. Cvetko-Vah, "Stone duality for skew Boolean algebras with intersections", in *Houston Journal of Mathematics* [2];
- A. Bauer, K. Cvetko-Vah, M. Gerkhe, G. Kudryavtseva and S. J. van Gool, "A noncommutative Priestly duality", in *Topology and Applications* [3].

Karin had met Mai Gehrke at a math conference in Switzerland and Sam van Gool was Mai's student. Further studies in duality are listed below. Karin has also explored connections to Church algebras (with Antonino Salibra) [29], skew Heyting algebras [18] and noncommutative toposes (with Jens Hemelaer and Lieven Le Bruyn) [21]. Again, see the references near the end.

Clearly a major contribution has been Karin's ability to engage the interest of others in some aspect of skew lattices. Indeed many are here because of an encounter with her. Besides those already mentioned, there is her wonderful student, João Pita da Costa, who we will mention from time to time.

As for engaging the interest of others in skew lattices, Matthew has also not been idle. In the summer of 2007 he attended a conference on automated deduction at the University of New Mexico. There he met Michael Kinyon, a broadly published mathematician working in various areas of algebra and even beyond. The two began discussing skew lattices. By 2008 Michael and I started having e-conversations, initially about cancellative skew lattices. Before long Matthew and Karin joined in. Long story short, this led to a sequence of three papers that extended significantly earlier research on skew lattice architecture and other aspects of skew lattice theory. They were all co-authored by Michael and me, at least.

- The previously mentioned, "Cancellation in skew lattices" (with K. Cvetko-Vah and M. Spinks) [22];
- "Categorical skew lattices", in *Order* [32];
- "Distributivity in skew lattices" (with J. Pita da Costa), in *Semigroup Forum* [33].

The first paper was a thorough study of cancellative skew lattices. To begin, they also form a subvariety. A characterization by Michael of these algebras via a finite list of forbidden algebras was also given, along with other nice results. One was Karin's "Parallelogram Laws" for cancellative skew lattices taken from her dissertation: given $\mathcal{D}$-classes A and B and their join and meet $\mathcal{D}$-classes J and M, one has $[J : B] = [A : M]$ and $[B : J] = [M : A]$. Likewise, $[J : A] = [B : M]$ and $[A : J] = [M : B]$.

A skew lattice is *categorical* when the nonempty composition of successive coset isomorphisms is also a coset isomorphism. Distributive skew lattices are categorical and in particular, skew lattices of idempotents in rings are categorical. Categorical skew lattices were studied in the second paper. Special attention was given to *strictly categorical* skew lattices where the composition of successive coset bijections arising in any chain of $\mathcal{D}$-classes $A > B > C$ is always nonempty. They include normal skew lattices and their conormal duals, as well as all primitive skew lattices. Categorical skew lattices form a proper subvariety of skew lattices with the strictly categorical ones forming a properly smaller subvariety. Here are some as yet unanswered queries:

- Do the normal and conormal subvarieties jointly generate the strictly categorical variety?
- What subvariety does the class of primitive skew lattices generate?

Here is a nice result: *a strictly categorical skew lattice S is distributive iff its maximal lattice image $S/\mathcal{D}$ is distributive*. A nice counting theorem quoted in this paper came from João's 2012 *Algebra Universalis* publication "Coset laws for categorical skew lattices" [53]: given a strictly categorical chain $A > B > C$ of $\mathcal{D}$-classes, if A and C are finite, then B is also finite; moreover, $[C : A] = [C : B] \times [B : A]$ and dually $[A : C] = [A : B] \times [B : C]$.

Our third paper on distributive skew lattices was coauthored with João and appeared in the *Semigroup Forum* in 2015. If S is distributive then

(1) $S/\mathcal{D}$ is distributive and

(2) each $\mathcal{D}$-class chain $A > B > C$ is distributive.

Condition (2), called *linear distributivity*, turns out to be a mild generalization of being strictly categorical. Now, what about the converse? Do (1) and (2) imply S is distributive? The answer is NO in general: Spinks' examples suffice. But it is YES, if S is also symmetric. (Here is another really crisp result about distributivity occurring in the presence of symmetry.) This and other aspects of distributivity are studied.

While on the topic of skew lattice architecture, further research in this area has been carried out by Karin and/or João. The relevant published papers, all appearing since 2010, are often recognized by such phrases as "coset structure" or "coset laws" appearing in the title. Again, nice counting theorems have arisen, as we have seen.

While working on my paper with Matthew on the lattice of varieties of skew Boolean algebras with intersections, the question arose as to whether free skew Boolean algebras in general have intersections. The answer is trivially yes in the finite case, but what about the infinite case? Oddly enough, free skew Boolean algebras had never been formally studied, probably due to the fact that so much else was going on. So I emailed several individuals, asking what do free SBAs look like and do they have intersections. Someone got right on the case, Ganna (Anya) Kudryavtseva, who had been very involved in the study of duality. This led to two papers:

- "Free skew Boolean algebras", in *International Journal of Algebra and Computation* [40];
- "Free skew Boolean intersection algebras and set partitions", in *Order* [38].

The first was co-authored by Anya and me, but the second was her work. And yes, free skew Boolean algebras do have intersections. To give a glimpse of what occurs in the finite case, the free left-handed skew Boolean algebra on n generators is, to within isomorphism:

$$_L\mathbf{SBA}_n \cong \mathbf{1}^{\binom{n}{0}} \times \mathbf{2}^{\binom{n}{1}} \times \mathbf{3}_{\mathbf{L}}^{\binom{n}{2}} \times \cdots \times (\mathbf{n+1})_{\mathbf{L}}^{\binom{n}{n}}.$$

Here $(\mathbf{k+1})_{\mathbf{L}}$ is the primitive left-handed skew Boolean algebra on $\{0, 1, \ldots, k\}$ with 0 being the bottom element and $\{1, \ldots, k\}$ forming the upper $\mathcal{D}$-class. Similar decompositions in the finite case for algebras with intersection are given in Anya's paper. But instead of binomial coefficients $\binom{n}{k-1}$, the respective powers are given by Stirling numbers of the 2nd kind, $\left\{ {n+1 \atop k} \right\}$. (See [38, Theorem 28].)

Research on skew lattices and related subjects continues. We mention next a number of papers that have appeared (but not all!), loosely arranging them by topic. It is intended to give a sense of the current state of play. References are given at the end of the paper.

Further work on duality

- G. Kudryavtseva, "A dualizing object approach to noncommutative Stone duality", *Journal of the Australian Mathematical Society* [37];
- G. Kudryavtseva and M. V. Lawson, "Boolean sets, skew Boolean algebras and a non-commutative Stone duality", *Algebra Universalis* [39].

Partial function algebras

- J. Berendsen, D. N. Jansen, J. Schmaltz and F. W. Vaandrager, "The axiomatization of override and update", *Journal of Applied Logic* [4]. (This is related to the above mentioned paper by Cvetko-Vah, Leech and Spinks appearing in the same journal.)

Connections with logic, discriminator varieties and other systems

- R. J. Bignall and M. Spinks, "Propositional skew Boolean logic", in: *Proceedings 26th IEEE International Symposium on Multiple-Valued Logic*, IEEE Computer Society Press [8];
- R. J. Bignall and M. Spinks, "Implicative BCS-algebra subreducts of skew Boolean algebras", *Scientiae Mathematicae Japonicae* [9];
- J. Cirulis, "Nearlattices with an overriding operation", *Order* [14];
- K. Cvetko-Vah and A. Salibra, "The connection of skew Boolean algebras and discriminator varieties to Church algebras", *Algebra Universalis* [29];
- D. Saveliev, "Ultrafilter extensions of linearly ordered sets", *Order* [56];
- M. Spinks and R. Veroff, "Axiomatizing the skew Boolean propositional calculus", *Journal of Automated Reasoning* [59].

Cosets and skew lattice architecture

- K. Cvetko-Vah and J. Pita da Costa, "On coset laws for skew lattices in rings", *Novi Sad Journal of Mathematics* [27];
- K. Cvetko-Vah and J. Pita da Costa, "On coset laws for skew lattices", *Semigroup Forum* [20];
- J. Pita da Costa, "On the coset structure of a skew lattice", *Demonstratio Mathematica* [52];
- J. Pita da Costa, "Coset laws for categorical skew lattices", *Algebra Universalis* [53];
- J. Pita da Costa, "On the coset category of a skew lattice", *Demonstratio Mathematica* [55].

And beyond

- B. A. Alaba, M. Alamneh and Y. M. Gubena, "Skew semi-Heyting algebras", *International Journal of Computing Science and Applied Mathematics* [1];
- D. Carfi and K. Cvetko-Vah, "Skew lattices on the financial events plane", *Applied Sciences* [12];
- K. Cvetko-Vah, "On skew Heyting algebras", *Ars Mathematica Contemporanea* [18];
- K. Cvetko-Vah, "Noncommutative frames", *Journal of Algebra and Its Applications* [19];
- K. Cvetko-Vah, J. Hemelaer and L. Le Bruyn, "What is a noncommutative topos?", *Journal of Algebra and Its Applications* [21];
- K. Cvetko-Vah, M. Sadrzadeh, D. Kartsaklis, and B. Blundell, "Non-commutative logic for compositional distributional semantics", *Proceedings of the 24th Workshop on Logic, Languages, Information and Computation* [28];

- R. Koohnavard and A. Borumand Saeid, "(Skew) filters in residuated skew lattices", *Scientific Annals of Computer Science* [34];
- R. Koohnavard and A. Borumand Saeid, "(Skew) filters in residuated skew lattices II", *Honam Mathematical Journal* [35];
- L. Le Bruyn, "Covers of the arithmetic site" [43];
- Y. Zhi, X. Zhou and Q. Li, "Rough sets induced by ideals in skew lattices", *Journal of Intelligent and Fuzzy Systems* [61];
- Y. Zhi, X. Zhou and Q. Li, "Residuated skew lattices", *Information Science* [62].

(Again, these papers, and all mentioned in this article, are not intended to collectively give a comprehensive list of all publications related to skew lattices.)

Returning now to regular bands, as already indicated, regular bands and skew lattices are closely connected. I like to think that skew lattices are what regular bands can be when they grow up – just as semilattices can "grow" into lattices or even Boolean algebras. ☺ (This, of course, requires a nourishing environment, such as the multiplicative semigroup of some ring.) In any case, interest in regular bands is not limited to those studying semigroups or skew lattices. In their introductory remarks to *Cell Complexes, Poset Topology and the Representation Theory of Algebras Arising in Algebraic Combinatorics and Discrete Geometry* (to appear in the *Memoirs of the American Mathematical Society*) [51], Stuart Margolis, Franco Saliola and Benjamin Steinberg describe the relevance of left regular bands to various areas of mathematics, and mention many of the individuals involved along with selected relevant publications. For instance here are a few:

- K. S. Brown, "Semigroups, rings and Markov chains", *Journal of Theoretical Probability* [10];
- F. Chung and R. Graham, "Edge flipping in graphs", *Advances in Applied Mathematics* [13];
- P. Diaconis, "From shuffling cards to walking around the building: an introduction to modern Markov chain theory", *Proceedings of the International Congress of Mathematicians* [30];
- F. W. Lawvere, "Qualitative distinctions between some toposes of generalized graphs", *Categories in Computer Science and Logic, Proceedings of the AMS-IMS-SIAM Joint Summer Research Conference* [42]. (Lawvere used the term "graphic monoid" for left regular band.)

The authors then further develop many of these connections in their monograph. In addition, the semigroup algebra $\mathcal{K}(B)$ of a finite left regular band B where $\mathcal{K}$ is a commutative ring with identity is studied along with homological aspects of its left module category. One of the things they discovered is that if the band B is finite and left regular, then $\mathcal{K}(B)$ has a *right identity*, that is, an element β such that $x\beta = x$ for all $x \in \mathcal{K}(B)$ [51, Theorem 4.2(2)]. Moreover, if this right identity is unique, then it is *the* multiplicative identity for $\mathcal{K}(B)$. Let's see why, using skew lattice theory.

First, suppose the elements of B are $b_1, b_2, \ldots, b_n$. Upon identifying each b_i with $1b_i$ in $\mathcal{K}(B)$, set $\beta = b_1 \vee b_2 \vee \cdots \vee b_n$, where $x \vee y = x + y - xy$. Then β lies in the top $\mathcal{D}$-class of the left-handed skew lattice S generated from B in $\mathcal{K}(B)$. Thus for all $b_j \in B$, $b_j \vee \beta = \beta$ since $(S, \vee)$ is right regular. Thus by duality, $b_j \wedge \beta = b_j$, that is, $b_j\beta = b_j$ in $\mathcal{K}(B)$. But if this holds for the generators (over $\mathcal{K}$) of $\mathcal{K}(B)$, then $x\beta = x$ for all $x \in \mathcal{K}(B)$. Thus β is a right identity for $\mathcal{K}(B)$. It is important to note that $b_n < \beta$ in the natural ordering of the idempotents in $\mathcal{K}(B)$. Thus β behaves like a 2-sided identity for at least b_n. Thus also $\mathcal{K}(B)$ has an identity, namely β, if all outcomes obtained by permuting the factors of $b_1 \vee b_2 \vee \cdots \vee b_n$ agree, since β then behaves like a 2-sided identity for all the b_i which collectively generate $\mathcal{K}(B)$. Conversely and trivially, if $\mathcal{K}(B)$ has a multiplicative identity, then it can only be β, no matter in what order it is assembled. Several comments:

(1) The above β expands to the noncommutative inclusion-exclusion expression:

$$\sum b_j - \sum_{i<j} b_i b_j + \sum_{i<j<k} b_i b_j b_k - \cdots + (-1)^{n+1} b_1 b_2 b_3 \cdots b_n.$$

(2) If $\{g_1, g_2, \ldots, g_m\}$ is a set of generators for B, then $\gamma = g_1 \vee g_2 \vee \cdots \vee g_m$ must be a right identity for $\mathcal{K}(B)$. γ will be a 2-sided identity if all outcomes obtained by permuting the factors of $g_1 \vee g_2 \vee \cdots \vee g_m$ agree.

(3) A further refinement: if we choose the set $\{m_1, m_2, \ldots, m_k\}$ of all elements in B that are maximal relative to the natural partial ordering of B ($e \geq f$ iff $ef = f = fe$) and repeat the process to get $\mu = m_1 \vee m_2 \vee \cdots \vee m_k$, then μ is also a right-identity that is a 2-sided identity if all outcomes obtained by permuting the factors of $m_1 \vee m_2 \vee \cdots \vee m_k$ agree. (This set of m_is is a subset of any set of generators of B.)

(4) Returning to the main argument, one need only assume that B is a left regular band for which $B/\mathcal{D}$ is finite with say n $\mathcal{D}$-classes. In this case $b_1, b_2, \ldots, b_n$ is a cross-section of elements, one chosen from each $\mathcal{D}$-class. The $\mathcal{D}$-class of $\beta = b_1 \vee b_2 \vee \cdots \vee b_n$ must be the maximal $\mathcal{D}$-class in the generated skew lattice S, due to the Clifford-McLean Theorem. Since S is left-handed, again β is a right identity for all elements in B and thus all elements in $\mathcal{K}(B)$.

In their monograph the authors characterize those left regular bands B for which an identity exists in *all* cases of $\mathcal{K}(B)$, i.e., for any commutative ring $\mathcal{K}$ with identity. Clearly, if identities always exist, this is true for $\mathbb{Z}(B)$ where $\mathbb{Z}$ is the ring of integers. But, as authors note, the converse is easily see to hold: if $\mathbb{Z}(B)$ has an identity, so must $\mathcal{K}(B)$ for any commutative ring $\mathcal{K}$ with identity. This is in their Theorem 4.15, where the authors also give a graph theoretic characterization of those B for which all $\mathcal{K}(B)$ have an identity. Applying it requires some insight into the behavior of B, as indeed do the methods of (2) and (3).

Assuming $\mathcal{K}$ is nontrivial, the map $b \mapsto 1b$ gives an easy isomorphic embedding of B into the multiplicative semigroup of $\mathcal{K}(B)$, at which location a skew lattice can be generated from the copy of B in $\mathcal{K}(B)$, if B is left or right regular, but not necessarily for all regular bands. But this simple method can be modified in the general case as follows. Given *any* regular band B with its respective maximal left and right regular images, B_L and B_R, the Kimura Theorem for regular bands initiates a chain of isomorphic embeddings from B into the multiplicative semigroup of a ring that is a product of semigroup rings:

$$B \to B_L \times B_R \to \mathcal{K}(B_L) \times \mathcal{K}(B_R).$$

In this ring, the image of B will generate a skew lattice S under the standard operations $x \wedge y = xy$ and $x \vee y = x + y - xy$. Thus, *every regular band B can be embedded in the* *reduct* $(S, \wedge)$ *of a skew lattice* $(S, \wedge, \vee)$. In this case B is embedded in a well-behaved skew lattice. When $B/\mathcal{D}$ is finite, while the relevant ring need not have a right or left identity (unless B is left or right regular), it will have a *middle identity* m such that $xmy = xy$ for all x, y in $\mathcal{K}(B_L) \times \mathcal{K}(B_R)$. In particular $xmx = x$ for all x in the generated skew lattice S. Such an m is given by *any* of the idempotents in the maximal $\mathcal{D}$-class of S (in the usual ordering of $\mathcal{D}$-classes). Thus the existence of a right, left or middle identity in the ring depends on the existence and behavior of a maximal $\mathcal{D}$-class in S. An identity occurs precisely when this class reduces to a single point.

The reader will have noticed various bits of mathematical genealogy relative to the history of noncommutative lattices. Let me say a few words about my genealogy, although in doing mine we'll "stray" into a larger arena. What follows are two genealogy sequences with the dates being when the individual received their PhD (see Figure 3). Both begin in

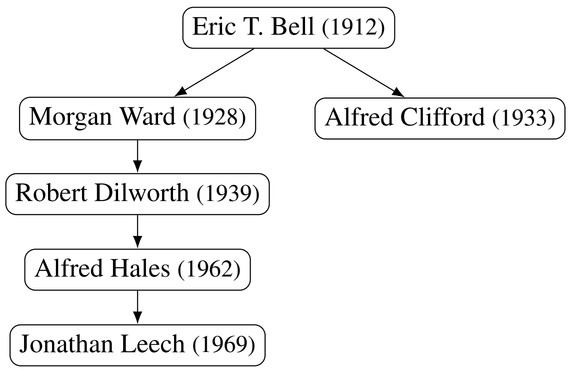

Figure 3: An advisor-student tree.

Pasadena, California with Eric Temple Bell directing dissertations at Caltech (California Institute of Technology) in the early 20th century. Indeed all you see occurs there until my advisor, Alfred Hales, received his PhD at Caltech and accepted a position at UCLA. While Bell's main interests were in number theory and related areas in algebra and analysis, with Dilworth we have arrived at a major figure in the developing theory of lattices. And while Hales may be more known for his work in combinatorics, especially Ramsey Theory (thus a co-winner of the Polya Prize), he made significant contributions to lattice theory. One surprising result, proved independently by Haim Gaifman, states that there exist countably generated complete Boolean algebras of arbitrarily high cardinality. The outside reader of my dissertation was another student of Bell, Alfred H. Clifford. By this time, he had moved to Tulane University, where he had already directed the dissertation of Naoki Kimura. But initially, after receiving his PhD he joined the Institute for Advanced Study, where he became an assistant to Hermann Weyl. Clifford was a master expositor, and in my early papers I benefitted greatly from his suggestions. (Two side-notes, courtesy of Professor Hales: Al and Alice Clifford were avid Bridge players as were the parents of Al Hales. Thus when both couples lived in Pasadena they knew each other. Also, both Alfreds attended Polytechnic School, a preparatory school in Pasadena, and though 31 years apart, in the middle grades both studied math under the nationally acclaimed teacher, Mary Ardis Schnebly.) Although they played different roles in my early career, to both Alfreds I owe a real debt of gratitude. Given another venue I would say more. But for now, to both gentlemen let me just say: Thank you! (And, of course, thank *you*, Mary Ardis Schnebly.)

As for E. T. Bell, he is known for a number of reasons. These include his study of Bell numbers that are named after him, although such a study was preceded in the notebooks of Ramanujan. (The n^{th} Bell number B_n is the number of distinct partitions of an n-element set.) Interestingly, Bell numbers appear in Anya Kudryavtseva's paper [38] where they are used to count the number of atomic $\mathcal{D}$-classes as well as the total number of atoms in a free left [right]-handed skew Boolean intersection algebra. Given n generators, these counts are respectively $B_{n+1} - 1$ and $B_{n+2} - 2B_{n+1}$. (Again, see [38, Theorem 28].)

In conclusion, in describing my journey into noncommutative lattice theory and in particular, skew lattices, I have focused not only on primary concepts and results, but also on the individuals involved in developing the current state of the subject, many of whom attended this workshop. Thankfully, I have not made this journey alone. To all of those who have been involved at its various stages, whether directly with me or not, I am grateful for your wonderful contributions. I must also thank Professors Tomaž Pisanski, Karin Cvetko-Vah and all others involved in the planning and running of the NCS2018 Workshop in Portorož and Piran – such a beautiful venue! In particular, thank you for making it possible for nearly all of my past coauthors to attend too. Your successful efforts are very much appreciated. And especially to Karin, thank you for your help in preparing the slides as well as the layout of this article. Again, it is much appreciated.

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