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An alternate proof of the monotonicity of the number of positive entries in nonnegative matrix powers*

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Abstract

Let A be a nonnegative real matrix of order n and $f(A)$ denote the number of positive entries in A. In 2018, Xie proved that if $f(A) \leq 3$ or $f(A) \geq n^2 - 2n + 2$, then the sequence $(f(A^k))_{k=1}^{\infty}$ is monotone for positive integers k. In this note we give an alternate proof of this result by counting walks in a digraph of order n.

Keywords: Digraphs, walks, monotonicity, adjacency matrix.

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1 Introduction

A matrix is *nonnegative* (respectively, *positive*) if all its entries are nonnegative (respectively, positive) real numbers. Nonnegative matrices are widely applied in science, engineering and technology, see [1] and [2]. A nonnegative square matrix A is said to be *primitive* if there exists a positive integer k such that A^k is positive. By $f(A)$ we denote the number of positive entries in A. In [4] Šidák proved that there exists a primitive matrix A of order 9 satisfying $f(A) = 18 > f(A^2) = 16$. Motivated by this observation, in [5] Xi proved that if $f(A) \leq 3$ or $f(A) \geq n^2 - 2n + 2$, then the sequence $(f(A^k))_{k=1}^{\infty}$ is monotone for positive integers k. The proof of this result relies on linear algebra approach considering A as a $0-1$ square matrix, that is, a matrix from the vector space $\mathbb{M}_n(\mathbb{R})$ whose entries are either 0 or 1. Recall, $\mathbb{M}_n(\mathbb{R})$ is the set of all square matrices of size n under the ordinary addition and scalar multiplication of matrices. Clearly, the above restriction on the entries of A is valid since the value of each positive entry in A does not

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effect $f(A^k)$ for all positive integers k. In this note we give an alternate proof of this result using counting method from graph theory.

By a *digraph* we mean a structure $G = (V, A)$, where $V(G)$ is a finite set of *vertices*, and $A(G)$ is a set of ordered pairs (u, v) of vertices $u, v \in V(G)$ called *arcs*. The *order* of the digraph G is the number of vertices in G. An *in-neighbour* of a vertex v in a digraph G is a vertex u such that $(u, v) \in A(G)$. Similarly, an *out-neighbour* of a vertex v is a vertex w such that $(v, w) \in A(G)$. The *in-degree*, respectively *out-degree*, of a vertex $v \in V(G)$ is the number of its in-neighbours, respectively out-neighbours, in G. A *walk* w of length k in G is an alternating sequence $(v_0 a_1 v_1 a_2 \ldots a_k v_k)$ of vertices and arcs in G such that $a_i = (v_{i-1}, v_i)$ for each i. If the arcs $a_1, a_2, \ldots, a_k$ of a walk w are distinct, w is called a *trail*. A *cycle* C_k of length k is a closed trial of length $k > 0$ with all vertices distinct (except the first and the last).

If a digraph G has n vertices $v_1, v_2, \ldots, v_n$, a useful way to represent it is with an $n \times n$ matrix of zeros and ones called its *adjacency matrix*, A_G. The ij-th entry of the adjacency matrix, $(A_G)_{ij}$, is 1 if there is an arc from vertex v_i to vertex v_j and 0 otherwise. That is,

$$(A_G)_{ij} = \begin{cases} 1, \text{ if } (v_i, v_j) \in A(G) \\ 0, \text{ otherwise} \end{cases}$$

The *length-k walk counting matrix* for an n-vertex digraph G is the $n \times n$ matrix C such that

$$C_{uv} := \text{the number of length-}k\text{ walks from } u \text{ to } v.$$

The main result in this note is based on the following well-known result:

Theorem 1.1 ([3]). *The length-k counting matrix of a digraph, G, is $(A_G)^k$, for all $k \in \mathbb{N}$.*

2 Main results

In the following proposition we reprove Theorem 1 and Theorem 2 from [5].

Proposition 2.1. *Let A be a $0-1$ matrix of order n. If $f(A) \leq 3$, then the sequence $(f(A^k))_{k=1}^{\infty}$ is monotone.*

Proof. Let G be a digraph on n vertices $v_1, v_2, \ldots, v_n$ corresponding to the adjacency matrix A, that is, there is an arc from vertex v_i to vertex v_j in G ($v_i \to v_j$) if $(A)_{ij} = 1$. We deal with four possible cases.

1. The case when $f(A) = 0$ is trivial. Since $A^k = O_n$, then $f(A^k) = 0$ for any positive integer k.
2. If $f(A) = 1$, then G contains exactly one arc $a = (v_i, v_j)$.
    - If $v_i = v_j$, then for any positive integer k there exists a unique k-walk from v_i to v_i. Therefore $(A^k)_{ii} = 1$. Moreover, since there exists no other k-walk between the vertices of G, the remaining $n^2 - 1$ entries of A^k are zeros. In this case, for any positive integer k we have $f(A^k) = 1$.
    - If $v_i \neq v_j$, then $(A)_{ij} = 1$. It is easy to see that G does not contain a walk of length $k \geq 2$, that is, for any $k \geq 2$ A^k is a zero matrix. Therefore, for any $k \geq 2$ we obtain $1 = f(A) > f(A^k) = 0$.

3. Let $f(A) = 2$, i.e., let $a_1 = (v_i, v_j)$ and $a_2 = (v_r, v_s)$ be two distinct arcs of G.
   If G contains two loops, then we consider one possible case:

   - Let $v_i = v_j \neq v_r = v_s$. For any positive integer $k \geq 1$ there exists exactly one k-walk from vertex v_i to vertex v_j and exactly one k-walk from vertex v_r to vertex v_s. It yields $f(A^k) = 2$.

   If G contains one loop, we consider the following three cases:

   - If $v_i = v_j = v_r \neq v_s$, then $f(A^k) = 2$ for any positive integer $k \geq 1$.
   - If $v_i = v_j = v_s \neq v_r$, then $f(A^k) = 2$ for any positive integer $k \geq 1$.
   - If $v_i = v_j, v_r \neq v_s, v_i \neq v_r$ and $v_i \neq v_s$, then $f(A^k) = 1$ for any positive integer $k \geq 2$.

   If G does not contain loops, then we focus on the cases when at least one of the vertices v_i, v_j, v_r and v_s has positive in-degree and positive out-degree. Otherwise, G does not contain a k-walk for $k \geq 2$.

   - If $v_i \neq v_j = v_r \neq v_s$ and $v_i \neq v_s$, then G contains exactly one 2-walk from v_i to v_s. Moreover, there is no k-walk when $k \geq 3$. Thus $2 = f(A) > 1 = f(A^2) > f(A^k) = 0$ for any positive integer $k \geq 3$.
   - If $v_i \neq v_j = v_r \neq v_s$ and $v_i = v_s$, then $f(A^k) = 2$ for any positive integer k.

4. The proof when $f(A) = 3$ follows the same reasoning as the previous cases.
   Let $a_1 = (v_i, v_j), a_2 = (v_r, v_s)$ and $a_3 = (v_p, v_t)$ be three distinct arcs of G.
   If G contains three loops, then we have:

   - Let $v_i = v_j$, $v_r = v_s$ and $v_p = v_t$. It is easy to see that $f(A^k) = 3$ for any positive integer $k \geq 1$.

   Similarly, if G contains two loops, we treat the following cases.

   - If $v_i = v_j, v_r = v_s, v_p \neq v_t$ and if there is no common vertex between the arcs a_1, a_2 and a_3, then $f(A^k) = 2$ for any positive integer $k \geq 2$.
   - If $v_i = v_j = v_p \neq v_t = v_r = v_s$, then $f(A^k) = 3$ for any positive integer $k \geq 1$.
   - If $v_i = v_j = v_p \neq v_t \neq v_r = v_s$, then $f(A^k) = 3$ for any positive integer $k \geq 1$.
   - If $v_i = v_j = v_t \neq v_p \neq v_r = v_s$, then $f(A^k) = 3$ for any positive integer $k \geq 1$.

   If G contains one loop, we obtain the following cases.

   - If $v_i = v_j, v_r = v_t \neq v_s = v_p$ and $v_i \neq v_r, v_i \neq v_s$, then $f(A^k) = 3$ for any positive integer $k \geq 1$.
   - If $v_i = v_j, v_r \neq v_s, v_p \neq v_t$ and if there is no a common vertex between the arcs a_1, a_2 and a_3, then $f(A^k) = 1$ for any positive integer $k \geq 2$.
   - If $v_i = v_j, v_r \neq v_s = v_p \neq v_t, v_r \neq v_t$ and if there is no common vertex between a_1 and a_2 and a_1 and a_3, then $f(A^2) = 2$ and $f(A^k) = 1$ for any positive integer $k \geq 3$.

- If $v_i = v_j \neq v_r = v_p \neq v_t$, $v_r \neq v_s, v_s \neq v_t, v_i \neq v_s$ and $v_i \neq v_t$, then $f(A^k) = 1$ for any positive integer $k \geq 2$.
- If $v_i = v_j, v_r \neq v_s = v_t \neq v_p, v_r \neq v_p$ and if there is no common vertex between a_1 and a_2 and a_1 and a_3, then $f(A^k) = 1$ for any positive integer $k \geq 2$.
- If $v_i = v_j = v_r \neq v_s, v_p \neq v_t$ and if there is no common vertex between a_1 and a_3 and between a_2 and a_3 , then $f(A^k) = 2$ for any positive integer $k \geq 2$.
- If $v_i = v_j = v_s \neq v_r, v_p \neq v_t$ and if there is no common vertex between a_1 and a_3 and between a_2 and a_3 , then $f(A^k) = 2$ for any positive integer $k \geq 2$.
- If $v_i = v_j = v_r \neq v_s = v_p \neq v_t$ and $v_i \neq v_t$, then $f(A^k) = 3$ for any positive integer $k \geq 1$.
- If $v_i = v_j = v_s \neq v_r = v_p \neq v_t$ and $v_i \neq v_t$, then $f(A^k) = 2$ for any positive integer $k \geq 2$.
- If $v_i = v_j = v_s \neq v_r = v_t \neq v_p$ and $v_i \neq v_p$, then $f(A^k) = 3$ for any positive integer $k \geq 1$.
- If $v_i = v_j = v_r \neq v_s = v_t \neq v_p$ and $v_i \neq v_p$, then $f(A^k) = 2$ for any positive integer $k \geq 2$.
- If $v_i \neq v_j = v_r = v_s = v_p \neq v_t$ and $v_i \neq v_t$, then $f(A^k) = 3$ for any positive integer $k \geq 1$.
- If $v_i \neq v_j = v_r = v_s = v_t \neq v_p$, then $f(A^k) = 3$ for any positive integer $k \geq 1$.
- If $v_j \neq v_i = v_r = v_s = v_p \neq v_t$, then $f(A^k) = 3$ for any positive integer $k \geq 1$.
- If $v_i \neq v_j = v_r = v_s = v_p \neq v_t$ and $v_i = v_t$, then $f(A^k) = 4$ for any positive integer $k \geq 2$.

If G does not contain loops, then each k-walk of G, $k \geq 3$, contains at least two vertices of positive in-degree and positive out-degree. Based on this observation we consider the following cases.

- If $v_i = v_s \neq v_j = v_r, v_p \neq v_t$ and if there is no common vertex between the arcs a_1 and a_3, then $f(A^k) = 2$ for any positive integers $k \geq 2$.
- If $v_i \neq v_j$, $v_r \neq v_s$, $v_p \neq v_t, v_j = v_r, v_s = v_p$ and $v_t = v_i$, then $f(A^k) = 3$ for any positive integer $k \geq 1$.
- If $v_i \neq v_j$, $v_r \neq v_s$, $v_p \neq v_t, v_j = v_r, v_s = v_p$ and $v_i \neq v_t \neq v_j$, then $f(A^2) = 2, f(A^3) = 1$ and $f(A^k) = 0$ for any positive integer $k \geq 4$.
- If $v_t \neq v_p = v_s = v_i \neq v_j = v_r$ and $v_j \neq v_t$, then $f(A^k) = 3$ for any positive integer $k \geq 1$.
- If $v_p \neq v_t = v_s = v_i \neq v_j = v_r$ and $v_j \neq v_p$, then $f(A^k) = 3$ for any positive integer $k \geq 1$. □

The following result is a reproof of Theorem 5 from [5].

Theorem 2.2. *Let A be a $0-1$ matrix of order n. If $f(A) \geq n^2-2n+2$, then the sequence $(f(A^k))_{k=1}^{\infty}$ is non-decreasing.*

Proof. Let G be a digraph on n vertices $v_1, v_2, \dots, v_n$ which corresponds to the matrix A (A is the adjacency matrix of G consisting of at most $2n-2$ zeros). According to Theorem 1.1, proving $f(A^{k+1}) \geq f(A^k)$ for every positive integer k, is equivalent to proving that the number of pairs of vertices of G for which there exists at least one $(k+1)$-walk is greater or equal than the number of pairs of vertices of G for which there exists at least one k-walk.

Let us suppose that G contains a walk of length k, i.e. let $w = (v_i, v_{i+1}, \dots, v_j)$ be a k-walk from v_i to $v_j = v_{i+k}$. Thus $(A^k)_{ij} \geq 1$. We prove the following five claims.

Claim 1: If w contains at least four distinct vertices, then there exists at least one $(k+1)$-walk from v_i to v_j. Therefore $(A^{k+1})_{ij} \geq 1$.

Let $w = (v_i, v_{i+1}, \dots, v_j)$ contain at least four distinct vertices v_i, v_t, v_s and v_j. If w contains a loop, then G contains at least one $(k+1)$-walk from v_i to v_j. Therefore we assume that $(A)_{ii} = (A)_{tt} = (A)_{ss} = (A)_{jj} = 0$. Thus $v_i \neq v_{i+1}$ and $v_{i+1} \neq v_{i+2}$. If there exists no $(k+1)$-walk from v_i to v_j, then for each vertex $v \in V(G)\backslash\{v_i, v_{i+1}\}$, G does not contain 2-walks of type (v_i, v, v_{i+1}). Otherwise we obtain $(k+1)$-walk $(v_i, v, v_{i+1}, v_{i+2}, \dots, v_j)$. This implies an existence of at least $n-2$ non-connected pairs of vertices among (v_i, v) and (v, v_{i+1}), where $v \in V(G)\backslash\{v_i, v_{i+1}\}$. Similarly, for each vertex $v \in V(G)\backslash\{v_{i+1}, v_{i+2}\}$, G does not contain 2-walks of type (v_{i+1}, v, v_{i+2}). Otherwise we obtain $(k+1)$-walk $(v_i, v_{i+1}, v, v_{i+2}, \dots, v_j)$.This implies an existence of at least $n-3$ non-connected pairs of vertices among (v_{i+1}, v) and (v, v_{i+2}), where $v \in V(G)\backslash\{v_i, v_{i+1}, v_{i+2}\}$. Since G does not contain at least four loops, we obtain at least $(n-2)+(n-3)+4 = 2n-1$ non-connected pairs of vertices in G, which is not possible.

Claim 2: If $k \geq 3$ and w contains three distinct vertices, then there exists at least one $(k+1)$-walk from v_i to v_j. Therefore $(A^{k+1})_{ij} \geq 1$.

We proceed similarly as in the previous case. Let $w = (v_i, v_{i+1}, \dots, v_j)$ contain three distinct vertices v_i, v_t and v_j. If w contains a loop, then there exists at least one $(k+1)$-walk from v_i to v_j. Therefore we suppose $(A)_{ii} = (A)_{tt} = (A)_{jj} = 0$. Clearly $v_{i+1} \neq v_i$ and $v_t \neq v_{t+1}$. Without loss of generality let $v_{i+1} = v_t$. If G does not contain a $(k+1)$-walk from v_i to v_j, then for each $v \in V(G)\backslash\{v_i, v_t, v_j\}$ there exist no walks of type (v_i, v, v_{i+1}) and (v_t, v, v_{t+1}). Otherwise we obtain the walks $(v_i, v, v_{i+1}, \dots, v_j)$ and $(v_i, v_{i+1}, \dots, v_t, v, v_{t+1}, \dots, v_j)$, both of length $k+1$. The non-existence of the walks (v_i, v, v_{i+1}) and (v_t, v, v_{t+1}) implies an existence of at least $2(n-3)$ non-connected pairs of vertices among the pairs $(v_i, v), (v, v_{i+1} = v_t), (v_t, v)$ and (v, v_{t+1}).

Let $v_{i+2} = v_i$. We suppose that the walks (v_i, v_j, v_t) and (v_t, v_j, v_i) do not exist. Otherwise we obtain $(k+1)$-walks from v_i to v_j $(v_i, v_j, v_{i+1}, v_{i+2}, \dots, v_j)$ and $(v_i, v_{i+1}, v_j, v_{i+2}, \dots, v_j)$, respectively. This yields an existence of at least two non-connected pairs among the pairs $(v_i, v_j), (v_j, v_t), (v_t, v_j)$ and (v_j, v_i). In this case G contains at least $2n-1 = 3 + 2(n-3) + 2$ non-connected pairs of vertices, which is not possible.

Let $v_{i+2} = v_j$. Similarly as in the previous case, we conclude that there exists no a walk (v_i, v_j, v_t). Otherwise we obtain the walk $(v_i, v_j, v_{i+1}, v_{i+2}, \dots, v_j)$. This yields an existence of at least one non-connected pair among the pairs (v_i, v_j) and (v_j, v_t). In this case G contains at least $2n-2$ non-connected pairs of vertices.

Since A contains at most $2n-2$ zeros, we obtain that v_t and v_j are connected to v_i. For any even $k \geq 4$ we obtain a k-walk $(v_i, v_t, v_i, v_t, \dots, v_i, v_t, v_j)$. Similarly, if $k = 5$ we obtain the walk $(v_i, v_t, v_j, v_i, v_t, v_j)$. If $k \geq 7$ is an odd number, then $k = 2s+1 = (2s-4)+5$ where $s \geq 3$. In this case we obtain a k-walk from v_i to v_j by connecting the walk $(v_i, v_t, v_i, v_t, \dots, v_t, v_i)$ of length $2s-4$ and the walk $(v_i, v_t, v_j, v_i, v_t, v_j)$ of length 5.

Claim 3: If $k = 2$ and $w = (v_i, v_t, v_j)$, then $(A^3)_{ij} \geq 1$ or the number of positive entries of A^3 at $(i,i), (i,t), (i,j), (t,i), (t,t), (t,j), (j,i), (j,t)$ and (j,j) position is greater or equal than the number of positive entries of the matrix A^2 at the same positions.

Let G does not contain 3-walk from v_i to v_j and let $v \in V(G) \backslash \{v_i, v_t, v_j\}$. If G contains walks of type (v_i, v, v_t) and (v_t, v, v_j), then there exist 3-walks (v_i, v, v_t, v_j) and (v_i, v_t, v, v_j). In this case $(A^3)_{ij} \geq 1$.

On the other hand, the non-existence of the walks (v_i, v, v_t) and (v_t, v, v_j) implies an existence of at least $2(n-3)$ non-connected pairs among the pairs $(v_i, v), (v, v_t), (v_t, v)$ and (v, v_j). Now, if v_i is connected to v_j, then v_j is not connected to v_i and v_t. Otherwise we obtain the walks (v_i, v_j, v_i, v_j) and (v_i, v_j, v_t, v_j). Since $(A)_{ji} = (A)_{jt} = 0$ the matrix A contains at least $3 + 2(n-3) + 2 = 2n-1$ zeros. This is not possible. If v_i is not connected to v_j, then A contains at least $2n-2$ zeros. Therefore v_j is connected to v_i and v_t, and v_t is connected to v_i. By counting 2-walks between the vertices v_i, v_t and v_j, we find that the matrix A^2 consists of seven positive entries and two zeros at $(i,i), (i,t), (i,j), (t,i), (t,t), (t,j), (j,i), (j,t)$ and (j,j) position. On the other hand, by counting the 3-walks between the vertices v_i, v_t and v_j we conclude that A^3 consists eight positive entries and one zero at the same positions.

Claim 4: Let $w = (v_i, v_{i+1}, \dots, v_j)$ contain two distinct vertices v_i and v_j. The number of positive entries of A^{k+1} at $(i,i), (i,j), (j,i)$ and (j,j) position is greater or equal than the number of positive entries of the matrix A^k at the same positions.

Let $k \geq 2$. If the walk w contains a loop, then it is easy to conclude that G contains a $(k+1)$-walk from v_i to v_j. In this case $(A^k)_{ij} \geq 1$ implies $(A^{k+1})_{ij} \geq 1$.

If w does not contain loops, then k is an odd number. We observe that G contains a k-walk from vertex v_j to vertex v_i, which implies $(A^k)_{ji} \geq 1$. If there exists no k-walk from v_i to v_i and if there exists no k-walk from v_j to v_j, then $(A^k)_{ii} = (A^k)_{jj} = 0$. Since $k+1$ is an even number, G contains $(k+1)$-walks from v_i to v_i and from v_j to v_j, that is, $(A^{k+1})_{ii} \geq 1$ and $(A^{k+1})_{jj} \geq 1$. Moreover, the digraph G does not contain $(k+1)$-walk from vertex v_i to vertex v_j and from vertex v_j to vertex v_i, that is, $(A^{k+1})_{ij} = (A^{k+1})_{ji} = 0$. Thus, the matrices A^k and A^{k+1} contain two zeros and two positive entries at $(i,i), (i,j), (j,i)$ and (j,j) position.

Similarly, $(A^k)_{ii} \geq 1$ implies $(A^{k+1})_{ij} \geq 1$ and $(A^k)_{jj} \geq 1$ implies $(A^{k+1})_{ji} \geq 1$.

Let $k = 1$. If v_j is connected to v_i, we have the same case as $k \geq 2$. If v_j is not connected to v_i, then there exists at least one 2-walk from v_j to v_i or from v_i to v_j. Otherwise we have at least $2n-1$ non-connected pairs of vertices in G, that is, at least $2n-1$ zeros in A, a contradiction.

Claim 5: If w contains exactly one vertex v_i, then there exists a $(k+1)$-walk from v_i to v_i. Therefore $(A^{k+1})_{ii} \geq 1$.

In this case the walk w is obtained repeating the loop $v_i \to v_i$ k-times. Thus, there exists a $(k+1)$-walk from v_i to v_i.

As a conclusion, in the four cases (whether the k-walk from vertex v_i to vertex v_j contains one, two, three or more distinct vertices), we obtain that the number of positive entries in A^{k+1} is greater or equal than the number of positive entries in A^k, that is, $f(A^{k+1}) \geq f(A^k)$. □

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