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On the Terwilliger algebra of a certain family of bipartite distance-regular graphs with $\Delta_2 = 0$

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Abstract

Let Γ denote a bipartite distance-regular graph with diameter $D \geq 4$ and valency $k \geq 3$. Let X denote the vertex set of Γ, and let A_i $(0 \leq i \leq D)$ denote the distance matrices of Γ. We abbreviate $A := A_1$. For $x \in X$ and for $0 \leq i \leq D$, let $\Gamma_i(x)$ denote the set of vertices in X that are distance i from vertex x.

Fix $x \in X$ and let $T = T(x)$ denote the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by $A, E_0^*, E_1^*, \ldots, E_D^*$, where for $0 \leq i \leq D$, E_i^* represents the projection onto the ith subconstituent of Γ with respect to x. We refer to T as the *Terwilliger algebra* of Γ with respect to x. By the *endpoint* of an irreducible T-module W we mean $\min\{i \mid E_i^* W \neq 0\}$.

In this paper we assume Γ has the property that for $2 \leq i \leq D-1$, there exist complex scalars α_i, β_i such that for all $y, z \in X$ with $\partial(x,y) = 2$, $\partial(x,z) = i$, $\partial(y,z) = i$, we have $\alpha_i + \beta_i |\Gamma_1(x) \cap \Gamma_1(y) \cap \Gamma_{i-1}(z)| = |\Gamma_{i-1}(x) \cap \Gamma_{i-1}(y) \cap \Gamma_1(z)|$.

We study the structure of irreducible T-modules of endpoint 2. Let W denote an irreducible T-module with endpoint 2, and let v denote a nonzero vector in $E_2^* W$. We show that $W = \mathrm{span}\big(\{E_i^* A_{i-2} E_2^* v \mid 2 \leq i \leq D\} \cup \{E_i^* A_{i+2} E_2^* v \mid 2 \leq i \leq D-2\}\big)$.

It turns out that, except for a particular family of bipartite distance-regular graphs with $D = 5$, this result is already known in the literature. Assume now that Γ is a member of this particular family of graphs. We show that if Γ is not almost 2-homogeneous, then up to isomorphism there exists exactly one irreducible T-module with endpoint 2 and it is not thin. We give a basis for this T-module.

Keywords: Distance-regular graphs, Terwilliger algebra, irreducible modules.

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1 Introduction

Throughout this introduction let Γ denote a bipartite distance-regular graph with diameter $D \geq 4$, valency $k \geq 3$ and path-length function ∂. Let X denote the vertex set of Γ. For $x \in X$ and $0 \leq i \leq D$, let $\Gamma_i(x)$ denote the set of vertices in X that are distance i from vertex x, and let $T = T(x)$ denote the Terwilliger algebra of Γ with respect to x (see Section 2 for formal definitions).

It is known that there exists a unique irreducible T-module with endpoint 0, and this module is thin [8, Proposition 8.4]. Moreover, Curtin showed that up to isomorphism Γ has exactly one irreducible T-module with endpoint 1, and this module is thin [4, Corollary 7.7].

We now discuss the irreducible T-modules of endpoint 2. It turns out that the structure of these modules is particularly nice if we assume that Γ has the following combinatorial property: for $2 \leq i \leq D-1$, there exist complex scalars α_i, β_i such that for all $y, z \in X$ with $\partial(x, y) = 2,\ \partial(x, z) = i,\ \partial(y, z) = i$, we have

$$\alpha_i + \beta_i |\Gamma_1(x) \cap \Gamma_1(y) \cap \Gamma_{i-1}(z)| = |\Gamma_{i-1}(x) \cap \Gamma_{i-1}(y) \cap \Gamma_1(z)|.$$

Irreducible modules of endpoint 2 of these graphs were studied extensively, see [10, 11, 12, 13, 15]. We are motivated by the fact that the above equation holds if Γ is Q-polynomial.

Assume that Γ has the above mentioned combinatorial property. We show that if W is an irreducible T-module with endpoint 2 and v is a nonzero vector in $E_2^* W$, then

$$W = \text{span}\big(\{E_i^* A_{i-2} E_2^* v \mid 2 \leq i \leq D\} \cup \{E_i^* A_{i+2} E_2^* v \mid 2 \leq i \leq D-2\}\big).$$

Except for a particular family of bipartite distance-regular graphs with $D = 5$, this result is already known in the literature. To define this particular family we introduce a certain parameter Δ_2 in terms of the intersection numbers of Γ by $\Delta_2 = (k-2)(c_3 - 1) - (c_2 - 1)p^2_{22}$. It turns out that $\Delta_2 \geq 0$ and that $\Delta_2 = 0$ implies $c_2 \in \{1, 2\}$ or $D \leq 5$. The above mentioned family of bipartite distance-regular graphs with $D = 5$ is exactly the family of such graphs with $\Delta_2 = 0$. Assume now that Γ is such a graph. We show that if Γ is not almost 2-homogeneous, then up to isomorphism there exists exactly one irreducible T-module with endpoint 2, and this module is not thin. We give a basis for this T-module. If Γ is almost 2-homogeneous, then the structure of irreducible T-modules with endpoint 2 is described in [7].

2 Preliminaries

In this section we review some definitions and basic results concerning distance-regular graphs. See the book of A. E. Brouwer, A. M. Cohen and A. Neumaier [2] for more background information.

Let $\mathbb{C}$ denote the complex number field and let X denote a nonempty finite set. Let $\text{Mat}_X(\mathbb{C})$ denote the $\mathbb{C}$-algebra consisting of all matrices whose rows and columns are indexed by X and whose entries are in $\mathbb{C}$. Let $V = \mathbb{C}^X$ denote the vector space over $\mathbb{C}$ consisting of column vectors whose coordinates are indexed by X and whose entries are in $\mathbb{C}$. We observe $\text{Mat}_X(\mathbb{C})$ acts on V by left multiplication. We call V the *standard module*. We endow V with the Hermitean inner product $\langle\, , \rangle$ that satisfies $\langle u, v\rangle = u^t \overline{v}$ for $u, v \in V$, where t denotes transpose and $^-$ denotes complex conjugation. Recall that

$$\langle u, Bv\rangle = \langle \overline{B}^t u, v\rangle \tag{2.1}$$

for $u, v \in V$ and $B \in \mathrm{Mat}_X(\mathbb{C})$. For $y \in X$ let $\hat{y}$ denote the element of V with a 1 in the y coordinate and 0 in all other coordinates. Note that

$$\{\hat{y} \mid y \in X\} \text{ is an orthonormal basis for } V.$$

Let $\Gamma = (X, \mathcal{R})$ denote a finite, undirected, connected graph, without loops or multiple edges, with vertex set X and edge set $\mathcal{R}$. Let ∂ denote the path-length distance function for Γ, and set $D := \max\{\partial(x, y) \mid x, y \in X\}$. We call D the *diameter* of Γ. For a vertex $x \in X$ and an integer i let $\Gamma_i(x)$ denote the set of vertices at distance i from x. For an integer $k \geq 0$ we say Γ is *regular with valency* k whenever $|\Gamma_1(x)| = k$ for all $x \in X$. We say Γ is *distance-regular* whenever for all integers h, i, j $(0 \leq h, i, j \leq D)$ and for all vertices $x, y \in X$ with $\partial(x, y) = h$, the number

$$p^h_{ij} = |\Gamma_i(x) \cap \Gamma_j(y)|$$

is independent of x and y. The p^h_{ij} are called the *intersection numbers* of Γ.

For the rest of this paper we assume Γ is distance-regular with diameter $D \geq 4$. Note that $p^h_{ij} = p^h_{ji}$ for $0 \leq h, i, j \leq D$. For convenience set $c_i := p^i_{1,i-1}$ $(1 \leq i \leq D)$, $a_i := p^i_{1i}$ $(0 \leq i \leq D)$, $b_i := p^i_{1,i+1}$ $(0 \leq i \leq D-1)$, $k_i := p^0_{ii}$ $(0 \leq i \leq D)$, and $c_0 = b_D = 0$. By the triangle inequality the following hold for $0 \leq h, i, j \leq D$: (i) $p^h_{ij} = 0$ if one of h, i, j is greater than the sum of the other two; (ii) $p^h_{ij} \neq 0$ if one of h, i, j equals the sum of the other two. In particular $c_i \neq 0$ for $1 \leq i \leq D$ and $b_i \neq 0$ for $0 \leq i \leq D-1$. We observe that Γ is regular with valency $k = k_1 = b_0$ and that

$$c_i + a_i + b_i = k \qquad (0 \leq i \leq D). \tag{2.2}$$

Note that $k_i = |\Gamma_i(x)|$ for $x \in X$ and $0 \leq i \leq D$. By [2, p. 127],

$$k_i = \frac{b_0 b_1 \cdots b_{i-1}}{c_1 c_2 \cdots c_i} \qquad (1 \leq i \leq D). \tag{2.3}$$

Recall Γ is *bipartite* whenever $a_i = 0$ for $0 \leq i \leq D$. Setting $a_i = 0$ in (2.2) we find

$$b_i + c_i = k \quad (0 \leq i \leq D). \tag{2.4}$$

The following formulae for the bipartite case will be useful.

Lemma 2.1 ([2, Lemma 4.1.7]). *Let Γ denote a bipartite distance-regular graph with diameter $D \geq 4$ and valency $k \geq 3$. Then*

$$p^i_{2i} = \frac{c_i(b_{i-1} - 1) + b_i(c_{i+1} - 1)}{c_2} \quad (1 \leq i \leq D-1), \qquad p^D_{2D} = \frac{k(b_{D-1} - 1)}{c_2}.$$

We recall the Bose-Mesner algebra of Γ. For $0 \leq i \leq D$ let A_i denote the matrix in $\mathrm{Mat}_X(\mathbb{C})$ with (x, y)-entry

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } \partial(x, y) = i, \\ 0 & \text{if } \partial(x, y) \neq i \end{cases} \qquad (x, y \in X). \tag{2.5}$$

For notational convenience, we define A_i to be the zero matrix for all integers $i < 0$ or $i > D$. We call A_i the ith *distance matrix* of Γ. We abbreviate $A := A_1$ and call this the *adjacency matrix* of Γ. We observe (i) $A_0 = I$; (ii) $\sum_{i=0}^{D} A_i = J$; (iii) $\overline{A_i} = A_i$ $(0 \leq i \leq D)$; (iv) $A_i^t = A_i$ $(0 \leq i \leq D)$; (v) $A_i A_j = \sum_{h=0}^{D} p^h_{ij} A_h$ $(0 \leq i, j \leq D)$, where I (resp. J) denotes the identity matrix (resp. all 1's matrix) in $\mathrm{Mat}_X(\mathbb{C})$. Using these facts we find $A_0, A_1, \ldots, A_D$ is a basis for a commutative subalgebra M of $\mathrm{Mat}_X(\mathbb{C})$. We call M the *Bose-Mesner algebra* of Γ. It turns out that A generates M [1, p. 190].

3 Terwilliger algebra

Let Γ denote a distance-regular with diameter $D \geq 4$ and valency $k \geq 3$. We first recall the dual idempotents of Γ. To do this fix a vertex $x \in X$. We view x as a "base vertex". For $0 \leq i \leq D$ let $E_i^* = E_i^*(x)$ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{C})$ with (y, y)-entry

$$(E_i^*)_{yy} = \begin{cases} 1 & \text{if } \partial(x,y) = i, \\ 0 & \text{if } \partial(x,y) \neq i \end{cases} \qquad (y \in X).$$

We call E_i^* the ith *dual idempotent* of Γ with respect to x [16, p. 378]. We observe (ei) $\sum_{i=0}^{D} E_i^* = I$; (eii) $\overline{E_i^*} = E_i^*$ $(0 \leq i \leq D)$; (eiii) $E_i^{*t} = E_i^*$ $(0 \leq i \leq D)$; (eiv) $E_i^* E_j^* = \delta_{ij} E_i^*$ $(0 \leq i, j \leq D)$. By these facts $E_0^*, E_1^*, \ldots, E_D^*$ form a basis for a commutative subalgebra $M^* = M^*(x)$ of $\mathrm{Mat}_X(\mathbb{C})$. We call M^* the *dual Bose-Mesner algebra* of Γ with respect to x [16, p. 378]. For $0 \leq i \leq D$ we have

$$E_i^* V = \mathrm{span}\{\widehat{y} \mid y \in X,\ \partial(x,y) = i\},$$

so $\dim E_i^* V = k_i$. We call $E_i^* V$ the ith *subconstituent of* Γ with respect to x. Note that

$$V = E_0^* V + E_1^* V + \cdots + E_D^* V \qquad \text{(orthogonal direct sum)}. \tag{3.1}$$

Moreover E_i^* is the projection from V onto $E_i^* V$ for $0 \leq i \leq D$.

We now recall the Terwilliger algebra of Γ. Let $T = T(x)$ denote the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by M, M^*. We call T the *Terwilliger algebra of* Γ with respect to x [16, Definition 3.3]. Recall M is generated by A, so T is generated by A and the dual idempotents. We observe T has finite dimension. By construction T is closed under the conjugate-transpose map so T is semisimple [16, Lemma 3.4(i)].

By a *T-module* we mean a subspace W of V such that $BW \subseteq W$ for all $B \in T$. Let W denote a T-module. Then W is said to be *irreducible* whenever W is nonzero and W contains no T-modules other than 0 and W.

By [9, Corollary 6.2] any T-module is an orthogonal direct sum of irreducible T-modules. In particular the standard module V is an orthogonal direct sum of irreducible T-modules. Let W, W' denote T-modules. By an *isomorphism of T-modules* from W to W' we mean an isomorphism of vector spaces $\sigma : W \to W'$ such that $(\sigma B - B\sigma)W = 0$ for all $B \in T$. The T-modules W, W' are said to be *isomorphic* whenever there exists an isomorphism of T-modules from W to W'. By [4, Lemma 3.3] any two nonisomorphic irreducible T-modules are orthogonal. Let W denote an irreducible T-module. By [16, Lemma 3.4(iii)] W is an orthogonal direct sum of the nonvanishing spaces among $E_0^* W, E_1^* W, \ldots, E_D^* W$. By the *endpoint* of W we mean $\min\{i \mid 0 \leq i \leq D,\ E_i^* W \neq 0\}$. By the *diameter* of W we mean $|\{i \mid 0 \leq i \leq D,\ E_i^* W \neq 0\}| - 1$. We say W is *thin* whenever the dimension of $E_i^* W$ is at most 1 for $0 \leq i \leq D$.

The following matrices of $\mathrm{Mat}_X(\mathbb{C})$ will be useful later in the paper.

Definition 3.1. Let Γ denote a distance-regular with diameter $D \geq 4$ and valency $k \geq 3$. Fix $x \in X$ and let $E_i^* = E_i^*(x)$ $(0 \leq i \leq D)$ and $T = T(x)$. We define matrices $L = L(x)$, $R = R(x)$ by

$$L = \sum_{h=1}^{D} E_{h-1}^* A E_h^*, \qquad R = \sum_{h=0}^{D-1} E_{h+1}^* A E_h^*.$$

Note that $A = L+R$ [4, Lemma 4.4] and $L^t = R$. We call L and R the *lowering matrix* and the *raising matrix of* Γ with respect to x, respectively. Observe that L and R are contained in T.

Definition 3.2 ([7, Definition 3.2]). Let Γ denote a distance-regular with diameter $D \geq 4$ and valency $k \geq 3$. Fix $x \in X$. For $1 \leq i \leq D$ we define matrices $\Lambda_i = \Lambda_i(x)$ in $\mathrm{Mat}_X(\mathbb{C})$ by

$$(\Lambda_i)_{zy} = \begin{cases} |\Gamma_1(x) \cap \Gamma_1(y) \cap \Gamma_{i-1}(z)|, & \text{if } \partial(x,y) = 2, \partial(x,z) = \partial(y,z) = i, \\ 0, & \text{otherwise} \end{cases}$$

for $z, y \in X$.

4 The scalars Δ_i and γ_i

Let Γ denote a distance-regular graph with diameter $D \geq 4$ and valency $k \geq 3$. From now on we assume that Γ is bipartite. In this section we introduce certain scalars Δ_i and γ_i $(2 \leq i \leq D-1)$ which we find useful.

Definition 4.1. Let Γ denote a distance-regular with diameter $D \geq 4$ and valency $k \geq 3$. Then for $2 \leq i \leq D-1$ we define

$$\Delta_i = (b_{i-1} - 1)(c_{i+1} - 1) - (c_2 - 1)p^i_{2i}$$

and

$$\gamma_i = \frac{c_i(b_{i-1} - 1)}{p^i_{2i}}$$

(observe that $p^i_{2i} > 0$ by [3, Lemma 11]).

By [3, Theorem 12] we have $\Delta_i \geq 0$ for $2 \leq i \leq D-1$. Moreover, the scalars Δ_i and γ_i are related as follows.

Lemma 4.2 ([3, Theorem 13]). *Let* Γ *denote a distance-regular with diameter* $D \geq 4$ *and valency* $k \geq 3$ *and fix an integer* $2 \leq i \leq D-1$. *Then the following* (i),(ii) *are equivalent.*

(i) $\Delta_i = 0$.

(ii) *For all* $x, y, z \in X$ *with* $\partial(x,y) = 2$, $\partial(x,z) = i$, $\partial(y,z) = i$,

$$|\Gamma_1(x) \cap \Gamma_1(y) \cap \Gamma_{i-1}(z)| = \gamma_i.$$

If $\Delta_i = 0$ for $2 \leq i \leq D-2$, then Γ is called *almost* 2*-homogeneous*, see [7]. In this case the structure of irreducible T-modules is well understood, so we will assume that Γ is not almost 2-homogeneous. In the rest of the paper we therefore consider the following situation.

Notation 4.3. Let $\Gamma = (X, \mathcal{R})$ denote a bipartite distance-regular graph with diameter $D \geq 4$, valency $k \geq 3$ and intersection numbers b_i, c_i, which is not almost 2-homogeneous. Let A_i $(0 \leq i \leq D)$ be the distance matrices of Γ, and let V denote the standard module for Γ. We fix $x \in X$ and let $E^*_i = E^*_i(x)$ $(0 \leq i \leq D)$ and $T = T(x)$ denote the dual idempotents and the Terwilliger algebra of Γ with respect to x, respectively. We assume that for $2 \le i \le D-1$, there exist complex scalars α_i, β_i such that for all $y, z \in X$ with $\partial(x,y) = 2$, $\partial(x,z) = i$, $\partial(y,z) = i$, we have

$$\alpha_i + \beta_i |\Gamma_1(x) \cap \Gamma_1(y) \cap \Gamma_{i-1}(z)| = |\Gamma_{i-1}(x) \cap \Gamma_{i-1}(y) \cap \Gamma_1(z)|.$$

Let matrices $L = L(x)$, $R = R(x)$ and $\Lambda_i = \Lambda_i(x)$ $(1 \le i \le D)$ be as in Definitions 3.1 and 3.2. Let scalars Δ_i, γ_i $(2 \le i \le D-1)$ be as in Definition 4.1.

With reference to Notation 4.3, pick $2 \le i \le D-1$ and assume that $\Delta_i \neq 0$. By [12, Theorem 5.4] scalars α_i and β_i are uniquely determined and given by

$$\begin{aligned} \alpha_i &= \frac{c_i(c_i-1)(b_{i-1}-c_2) - c_i c_{i-1}(b_i-1)(c_2-1)}{c_2 \Delta_i}, \\ \beta_i &= \frac{c_i(c_{i+1}-c_i)(b_{i-1}-1) - b_i(c_{i+1}-1)(c_i-c_{i-1})}{c_2 \Delta_i}. \end{aligned} \tag{4.1}$$

If $\Delta_i = 0$, then scalars α_i and β_i are not uniquely determined. For example, if $\Delta_2 = 0$, then one of the possible values for α_2 and β_2 is $\alpha_2 = 0$, $\beta_2 = 1$. Note however that by Lemma 4.2 this is not the only possible solution.

5 Some products in T

With reference to Notation 4.3, in this section we compute some products of matrices of T. We start by recalling the following results.

Lemma 5.1 ([14, Lemma 6.1]). *With reference to Notation 4.3, for $0 \le h, i, j \le D$ and $y, z \in X$ the (y,z)-entry of $E_h^* A_i E_j^*$ is 1 if $\partial(x,y) = h$, $\partial(y,z) = i$, $\partial(x,z) = j$, and 0 otherwise.*

Lemma 5.2 ([14, Lemma 6.5]). *With reference to Notation 4.3, for $0 \le h, i, j, r, s \le D$ and $y, z \in X$ the (y,z)-entry of $E_h^* A_r E_i^* A_s E_j^*$ is $|\Gamma_i(x) \cap \Gamma_r(y) \cap \Gamma_s(z)|$ if $\partial(x,y) = h$, $\partial(x,z) = j$, and 0 otherwise.*

Lemma 5.3 ([7, Lemma 3.3]). *With reference to Notation 4.3, we have*

$$\Lambda_1 = E_1^* A E_2^*, \qquad \Lambda_i = E_i^* A_{i-1} E_1^* A E_2^* - c_2 E_i^* A_{i-2} E_2^* \qquad (2 \le i \le D).$$

In particular, $\Lambda_i \in T$ $(1 \le i \le D)$.

Theorem 5.4. *With reference to Notation 4.3 the following holds for $3 \le i \le D$:*

$$L E_i^* A_{i-2} E_2^* = b_{i-1} E_{i-1}^* A_{i-3} E_2^* + (c_{i-1} - \alpha_{i-1}) E_{i-1}^* A_{i-1} E_2^* - \beta_{i-1} \Lambda_{i-1}. \tag{5.1}$$

Proof. Pick $z, y \in X$ and an integer $3 \le i \le D$. We show that (z,y)-entries of both sides of (5.1) agree. Note that by the property (eiv) of Section 3 and Lemma 5.2,

$$(L E_i^* A_{i-2} E_2^*)_{zy} = \begin{cases} |\Gamma_i(x) \cap \Gamma_{i-2}(y) \cap \Gamma_1(z)| & \text{if } \partial(x,y) = 2, \partial(x,z) = i-1, \\ 0 & \text{otherwise.} \end{cases} \tag{5.2}$$

It follows from (5.2), Lemma 5.1 and Definition 3.2 that the (z,y)-entries of both sides of (5.1) are 0 if $\partial(x,y) \neq 2$ or $\partial(x,z) \neq i-1$. Assume now $\partial(x,y) = 2$ and $\partial(x,z) = i-1$.

Observe that by the triangle inequality we have that $\partial(z, y) \in \{i-3, i-1, i+1\}$. We consider each of these three cases separately.

Case 1: $\partial(x, y) = 2$, $\partial(x, z) = i-1$ and $\partial(z, y) = i-3$. Note that in this case we have $(LE_i^* A_{i-2} E_2^*)_{zy} = b_{i-1}$ by (5.2). By Lemma 5.1 and Definition 3.2 the (z, y)-entries of both sides of (5.1) agree.

Case 2: $\partial(x, y) = 2$, $\partial(x, z) = i-1$ and $\partial(z, y) = i-1$. Observe that by (5.2) we have

$$
\begin{aligned}
(LE_i^* A_{i-2} E_2^*)_{zy} &= c_{i-1} - |\Gamma_1(z) \cap \Gamma_{i-2}(x) \cap \Gamma_{i-2}(y)| \\
&= c_{i-1} - (\alpha_{i-1} + \beta_{i-1} |\Gamma_{i-2}(z) \cap \Gamma_1(x) \cap \Gamma_1(y)|).
\end{aligned}
$$

By Lemma 5.1 and Definition 3.2 the (z, y)-entries of both sides of (5.1) agree.

Case 3: $\partial(x, y) = 2$, $\partial(x, z) = i-1$ and $\partial(z, y) = i+1$. By (5.2), Lemma 5.1 and Definition 3.2 the (z, y)-entries of both sides of (5.1) are 0. □

6 Irreducible T-modules with endpoint 2

With reference to Notation 4.3, let W denote an irreducible T-module with endpoint 2. In this section we find a spanning set for W.

Definition 6.1. With reference to Notation 4.3, let W denote an irreducible T-module with endpoint 2 and let v denote a nonzero vector in $E_2^* W$. For $0 \le i \le D$, define

$$
v_i^+ = E_i^* A_{i-2} E_2^* v, \qquad v_i^- = E_i^* A_{i+2} E_2^* v.
$$

Note that $v_2^+ = v$, $v_i^+ = 0$ if $i < 2$, and $v_i^- = 0$ if $i < 2$ or $i > D-2$.

Lemma 6.2 ([5, Corollary 9.3(i), Theorem 9.4]). *With reference to Definition 6.1, the following* (i)–(iv) *hold.*

(i) $E_i^* A_i E_2^* v = -(v_i^+ + v_i^-)$ $(2 \le i \le D)$.

(ii) $Rv_i^+ = c_{i-1} v_{i+1}^+$ $(2 \le i \le D-1)$ *and* $Rv_D^+ = 0$.

(iii) $Lv_i^- = b_{i+1} v_{i-1}^-$ $(2 \le i \le D-2)$.

(iv) $Lv_{i+1}^+ - Rv_{i-1}^- = b_i v_i^+ - c_i v_i^-$ $(1 \le i \le D-1)$.

Lemma 6.3. *With reference to Definition 6.1, the following* (i)–(iii) *hold.*

(i) $\Lambda_i v = -c_2 v_i^+$ $(2 \le i \le D)$.

(ii) $Lv_2^+ = 0$ *and*

$$
Lv_i^+ = (b_{i-1} - c_{i-1} + \alpha_{i-1} + c_2 \beta_{i-1}) v_{i-1}^+ - (c_{i-1} - \alpha_{i-1}) v_{i-1}^-
$$

for $3 \le i \le D$.

(iii)

$$
Rv_i^- = (c_2 \beta_{i+1} - c_{i+1} + \alpha_{i+1}) v_{i+1}^+ + \alpha_{i+1} v_{i+1}^-
$$

for $2 \le i \le D-2$.

Proof. (i) Immediate from Lemma 5.3 and Definition 6.1.

(ii) Note that $Lv_2^+ = 0$ as the endpoint of W is 2. To obtain the result for Lv_i^+ $(3 \le i \le D)$ apply (5.1) to v and use Definition 6.1, Lemma 6.2(i) and (i) above.

(iii) Immediately by (ii) above and Lemma 6.2(iv). □

Theorem 6.4. *With reference to Definition 6.1,*

$$W = \operatorname{span}\{v_2^+, v_3^+, \dots, v_D^+, v_2^-, v_3^-, \dots, v_{D-2}^-\}.$$

Proof. Denote $W' = \operatorname{span}\{v_2^+, v_3^+, \dots, v_D^+, v_2^-, v_3^-, \dots, v_{D-2}^-\}$ and note that $W' \subseteq W$. We now show that $W = W'$. Note that $E_i^* v_j^+ = \delta_{ij} v_j^+$ for $2 \le j \le D$ and $E_i^* v_j^- = \delta_{ij} v_j^-$ for $2 \le j \le D-2$. Therefore, W' is invariant under the action of E_i^* for $0 \le i \le D$. Observe also that W' is invariant under the action of L by Lemma 6.2(iii) and Lemma 6.3(ii), and also invariant under the action of R by Lemma 6.2(ii) and Lemma 6.3(iii). As $A = R + L$, W' is invariant under the action of A. As T is generated by A and E_i^* $(0 \le i \le D)$, this implies that W' is a T-module. Recall that W is irreducible and that W' contains a nonzero vector v. It follows that $W = W'$. □

Corollary 6.5. *With reference to Definition 6.1, we have*

$$\dim(E_{D-1}^* W) \le 1, \qquad \dim(E_D^* W) \le 1.$$

Proof. Immediately from Theorem 6.4. □

As already mentioned, the result from Theorem 6.4 is already known in the literature, except for the case $D = 5$ and $\Delta_2 = 0$, see [11, 12, 15]. In the rest of the paper we study this case in detail. If $D = 5$ and $\Delta_2 = \Delta_3 = 0$, then Γ is almost 2-homogeneous, contradicting our assumption in Notation 4.3. Therefore, we have that $\Delta_3 \neq 0$.

7 Case $\Delta_2 = 0$ and $\Delta_3 \neq 0$

With reference to Notation 4.3, in this section we study graphs with $\Delta_2 = 0$ and $\Delta_3 \neq 0$. We first have the following observation.

Lemma 7.1. *With reference to Definition 6.1, assume that $\Delta_2 = 0$ and $\Delta_3 \neq 0$. Then the following* (i), (ii) *hold.*

(i)
$$c_3 = \frac{(c_2^2 - c_2 + 1)k - c_2(c_2 + 1)}{k + c_2^2 - 3c_2}.$$

(ii)
$$\alpha_3 = 0, \qquad \beta_3 = \frac{c_2(k-2)}{k + c_2^2 - 3c_2}.$$

Proof. (i) Solve $\Delta_2 = 0$ for c_3. Note that $k + c_2^2 - 3c_2 = (c_2 - 1)(c_2 - 2) + k - 2 > 0$ as $k \ge 3$.

(ii) Use Definition 4.1, (4.1) and (i) above. □

Lemma 7.2. *With reference to Definition 6.1, assume that $\Delta_2 = 0$ and $\Delta_3 \neq 0$. Then*

$$E_2^* A_2 E_2^* v = -\frac{c_2(k-2)}{k + c_2^2 - 3c_2} v.$$

Proof. Let $\Gamma_2^2 = \Gamma_2^2(x)$ denote the graph with vertex set $\widetilde{X} = \Gamma_2(x)$ and edge set $\widetilde{R} = \{yz \,|\, y, z \in \widetilde{X}, \partial(y, z) = 2\}$. The graph Γ_2^2 has exactly k_2 vertices and it is regular with valency p^2_{22} ([6, Lemma 3.2]). Let $\widetilde{A}$ denote the adjacency matrix of Γ_2^2. The matrix $\widetilde{A}$ is symmetric with real entries. Therefore $\widetilde{A}$ is diagonalizable with all eigenvalues real. Note that eigenvalues for $E_2^* A_2 E_2^*$ and $\widetilde{A}$ are the same.

Since $\Delta_2 = 0$, we know $E_2^* A_2 E_2^*$ has exactly one distinct eigenvalue η on $E_2^* W$ by [6, Theorem 4.11, Corollary 4.13, Lemma 5.3]. Thus, every nonzero vector in $E_2^* W$ is an eigenvector for $E_2^* A_2 E_2^*$ with eigenvalue η. By [6, Lemmas 5.4, 5.5] we find $\eta = -\frac{c_2}{\gamma_2}$. The result now follows from Definition 4.1 and Lemma 7.1(i). □

Corollary 7.3. *With reference to Definition 6.1, assume that $\Delta_2 = 0$ and $\Delta_3 \neq 0$. Then*

$$v_2^- = \frac{b_2(c_2 - 1)}{k + c_2^2 - 3c_2} v_2^+.$$

Proof. By Lemma 6.2(i) and Lemma 7.2 we have

$$-v_2^+ - v_2^- = E_2^* A_2 E_2^* v = -\frac{c_2(k-2)}{k + c_2^2 - 3c_2} v_2^+.$$

The result follows. □

Corollary 7.4. *With reference to Definition 6.1, assume that $D = 5$, $\Delta_2 = 0$ and $\Delta_3 \neq 0$. Then*

$$W = \text{span}\{v_2^+, v_3^+, v_4^+, v_5^+, v_3^-\}. \tag{7.1}$$

Proof. Immediately from Theorem 6.4 and Corollary 7.3. □

Observe that by (3.1) vectors $v_2^+, v_3^+, v_4^+, v_5^+$ are linearly independent, provided they are non-zero.

8 Some scalar products

With reference to Definition 6.1, assume that $D = 5$, $\Delta_2 = 0$ and $\Delta_3 \neq 0$. Our goal for the rest of this paper is to find a basis for W. In this section we compute the norms of vectors $v_3^+, v_4^+, v_5^+, v_3^-$ in terms of the intersection numbers of Γ and $\|v\|$. Note that by [10, Lemma 6.4] we have $\Delta_4 \neq 0$ as well. The assumptions of [10, Lemma 6.4] are somehow different from assumptions of Notation 4.3. However, the proof of [10, Lemma 6.4] works just fine also under assumptions of Notation 4.3.

Lemma 8.1. *With reference to Definition 6.1, assume that $\Delta_2 = 0$ and $\Delta_3 \neq 0$. Then*

$$\|v_3^+\|^2 = \frac{b_2(b_2 - c_2)}{k + c_2^2 - 3c_2} \|v\|^2.$$

In particular, if $D \geq 5$ then $v_3^+ \neq 0$.

Proof. By Lemma 6.2(ii), (2.1) and Definition 3.1 we have

$$\|v_3^+\|^2 = \langle v_3^+, v_3^+ \rangle = \langle R v_2^+, v_3^+ \rangle = \langle v_2^+, L v_3^+ \rangle.$$

The result now follows from Lemma 6.3(ii), Corollary 7.3 and since $\alpha_2 = 0$, $\beta_2 = 1$. Now assume that $v_3^+ = 0$. Observe that this implies $b_2 = c_2$. If $D \geq 5$ then by [2, Proposition 4.1.6](i),(ii) we have $c_2 \leq c_3 \leq b_2$, and so $c_2 = c_3$. But then $c_2 = 1$ by Lemma 7.1(i), and so $k = b_2 + c_2 = 2$, a contradiction. □

Lemma 8.2. *With reference to Definition 6.1, assume that $\Delta_2 = 0$ and $\Delta_3 \neq 0$. Then*

$$\langle v_3^+, v_3^- \rangle = \frac{b_2 b_4 (c_2 - 1)}{k + c_2^2 - 3c_2} \|v\|^2.$$

Proof. By Lemma 6.2(ii), (2.1) and Definition 3.1 we have

$$\langle v_3^+, v_3^- \rangle = \langle Rv_2^+, v_3^- \rangle = \langle v_2^+, Lv_3^- \rangle.$$

The result now follows from Lemma 6.2(iii) and Corollary 7.3. □

Lemma 8.3. *With reference to Definition 6.1, assume that $D = 5$, $\Delta_2 = 0$ and $\Delta_3 \neq 0$. Then*

$$\|v_4^+\|^2 = \frac{b_2((b_3 - 1)b_2 - c_3(c_2 - 1)b_4)}{c_2(k + c_2^2 - 3c_2)} \|v\|^2.$$

In particular, $v_4^+ = 0$ if and only if $c_2 \neq 1$ and $b_4 = b_2(b_3 - 1)/(c_3(c_2 - 1))$.

Proof. By Lemma 6.2(ii), (2.1) and Definition 3.1 we have

$$\langle v_4^+, v_4^+ \rangle = \frac{1}{c_2} \langle Rv_3^+, v_4^+ \rangle = \frac{1}{c_2} \langle v_3^+, Lv_4^+ \rangle.$$

The formula for $\|v_4^+\|^2$ now follows from Lemma 6.3(ii), Lemma 7.1, Lemma 8.1 and Lemma 8.2.

It is clear that $v_4^+ = 0$ if $c_2 \neq 1$ and $b_4 = b_2(b_3 - 1)/(c_3(c_2 - 1))$. Therefore assume now that $v_4^+ = 0$. It follows that $(b_3 - 1)b_2 = c_3(c_2 - 1)b_4$. If $c_2 = 1$, then also $b_3 = 1$ and $c_3 = 1$ by Lemma 7.1(i). But then $k = c_3 + b_3 = 2$, a contradiction. Therefore $c_2 \neq 1$ and the result follows. □

Lemma 8.4. *With reference to Definition 6.1, assume that $D = 5$, $\Delta_2 = 0$ and $\Delta_3 \neq 0$. Then*

$$\|v_3^-\|^2 = \left(\frac{(c_2 - 1)(c_4 - 1)b_2}{k + c_2^2 - 3c_2} + \frac{(k - 1)\Delta_3}{b_2 - 1} \right) \frac{b_2 b_4 \|v\|^2}{c_2(kc_2 - k - c_2) + b_2}.$$

Proof. By Lemma 6.2(iv), (2.1) and Definition 3.1 we have

$$c_3 \langle v_3^-, v_3^- \rangle = b_3 \langle v_3^+, v_3^- \rangle + \langle Rv_2^-, v_3^- \rangle - \langle v_4^+, Rv_3^- \rangle.$$

The result now follows from Lemmas 6.3(iii), 7.1, 8.2 and 8.3, Corollary 7.3 and (4.1). □

Corollary 8.5. *With reference to Definition 6.1, assume that $D = 5$, $\Delta_2 = 0$ and $\Delta_3 \neq 0$. Then the following* (i), (ii) *hold.*

(i) $v_3^- \neq 0$.

(ii) *v_3^+, v_3^- are linearly independent.*

Proof. (i) Note that $(c_2-1)(c_4-1)b_2/(k+c_2^2-3c_2) \geq 0$ and that $(k-1)\Delta_3/(b_2-1) > 0$ by [3, Theorem 12]. Moreover, it is easy to see that $c_2(kc_2 - k - c_2) + b_2 > 0$. The result follows.

(ii) Assume on the contrary that v_3^+, v_3^- are linearly dependent. Let

$$B = \begin{pmatrix} \langle v_3^+, v_3^+ \rangle & \langle v_3^+, v_3^- \rangle \\ \langle v_3^-, v_3^+ \rangle & \langle v_3^-, v_3^- \rangle \end{pmatrix}$$

and note that $\det(B) = 0$. Using Lemmas 8.1, 8.2 and 8.4 one could easily see that the only factor of $\det(B)$ which could be zero is

$$c_4k - c_2^3k + 2c_2^2k - 2c_2k + c_2^3c_4 - 2c_2^2c_4 - c_2c_4 + 2c_2^2.$$

Solving this for c_4 and then computing Δ_3 using Definition 4.1, we obtain $\Delta_3 = 0$, a contradiction. This shows that v_3^+, v_3^- are linearly independent. □

Lemma 8.6. *With reference to Definition 6.1, assume that* $D = 5$, $\Delta_2 = 0$ *and* $\Delta_3 \neq 0$. *Then*

$$\|v_5^+\|^2 = \frac{b_4 - c_4 + \alpha_4 + c_2\beta_4}{c_3}\|v_4^+\|^2.$$

In particular, $v_5^+ = 0$ *if and only if* $v_4^+ = 0$ *or* $b_4 - c_4 + \alpha_4 + c_2\beta_4 = 0$.

Proof. By Lemma 6.2(ii), (2.1) and Definition 3.1 we have

$$\langle v_5^+, v_5^+ \rangle = \frac{1}{c_3}\langle Rv_4^+, v_5^+ \rangle = \frac{1}{c_3}\langle v_4^+, Lv_5^+ \rangle.$$

The result now follows from Lemma 6.3(ii). □

9 A basis

With reference to Definition 6.1, assume that $D = 5$, $\Delta_2 = 0$ and $\Delta_3 \neq 0$. In this section we display a basis for W. We will also show that, up to isomorphism, Γ has a unique irreducible T-module with endpoint 2.

Theorem 9.1. *With reference to Definition 6.1, assume that* $D = 5$, $\Delta_2 = 0$ *and* $\Delta_3 \neq 0$. *Then the following* (i)–(iii) *hold.*

(i) *If* $v_5^+ \neq 0$, *then the following is a basis for* W:

$$v_i^+ \ (2 \leq i \leq 5), \qquad v_3^-. \tag{9.1}$$

(ii) *If* $v_4^+ \neq 0$ *and* $v_5^+ = 0$, *then the following is a basis for* W:

$$v_i^+ \ (2 \leq i \leq 4), \qquad v_3^-. \tag{9.2}$$

(iii) *If* $v_4^+ = 0$, *then the following is a basis for* W:

$$v_i^+ \ (2 \leq i \leq 3), \qquad v_3^-. \tag{9.3}$$

In particular, W *is not thin.*

Proof. Note that by (7.1), W is spanned by vectors v_i^+ $(2 \leq i \leq 5)$ and v_3^-. Vector $v_2^+ = v$ is nonzero by definition. Vectors v_3^+ and v_3^- are nonzero by Lemma 8.1 and Corollary 8.5(i), respectively. We prove part (i) of the theorem. Proofs of parts (ii) and (iii) are similar.

If $v_5^+ \neq 0$, then $v_4^+ \neq 0$ by Lemma 8.6. Vectors v_i^+ $(2 \leq i \leq 5)$ and v_3^- are linearly independent by (3.1) and Corollary 8.5(ii). This shows that (9.1) is a basis for W. As $\dim(E_2^*(W)) = 2$, W is not thin. The result follows. □

Theorem 9.2. *With reference to Definition 6.1, assume that $D = 5$, $\Delta_2 = 0$ and $\Delta_3 \neq 0$. Then Γ has, up to isomorphism, exactly one irreducible T-module with endpoint 2.*

Proof. Let U denote an irreducible T-module with endpoint 2, different from W. Fix nonzero $u \in E_2^* U$, and for $2 \leq i \leq 5$ define

$$u_i^+ = E_i^* A_{i-2} E_2^* u$$

and let $u_3^- = E_3^* A_5 E_2^* u$. It follows from the results of Section 8 and Theorem 9.1 that u_2^+, u_3^+, u_3^- are nonzero and that nonzero vectors in the set $\{u_i^+ \mid 2 \leq i \leq 5\} \cup \{u_3^-\}$ form a basis for U. Furthermore, it follows from Lemma 8.3 and Lemma 8.6 that u_4^+ (u_5^+, respectively) is nonzero if and only if v_4^+ (v_5^+, respectively) is nonzero.

Let $\sigma : W \to U$ be defined by $\sigma(v_i^+) = u_i^+$ $(2 \leq i \leq 5)$ and $\sigma(v_3^-) = u_3^-$. It follows from the comments above that σ is a vector space isomorphism from W to U. We show that σ is a T-module isomorphism. Since A generates M and $E_0^*, E_1^*, \ldots, E_5^*$ is a basis for M^*, it suffices to show that σ commutes with each of $A, E_0^*, E_1^*, \ldots, E_5^*$. Using the fact that $E_i^* E_j^* = \delta_{ij} E_i^*$ and the definition of σ we immediately find that σ commutes with each of $E_0^*, E_1^*, \ldots, E_5^*$. Recall that $A = R + L$. It follows from Lemma 6.2, Lemma 6.3 and Corollary 7.3 that σ commutes with A. The result follows. □

We would like to emphasize that together with the results in [10, 12, 15], Theorems 9.1 and 9.2 imply the following characterization.

Theorem 9.3. *Let $\Gamma = (X, \mathcal{R})$ denote a bipartite distance-regular graph with diameter $D \geq 4$ and valency $k \geq 3$. Assume Γ is not almost 2-homogeneous. We fix $x \in X$ and let $E_i^* = E_i^*(x)$ $(0 \leq i \leq D)$ and $T = T(x)$ denote the dual idempotents and the Terwilliger algebra of Γ with respect to x, respectively. Then the following* (i), (ii) *are equivalent.*

(i) *Γ has, up to isomorphism, exactly one irreducible T-module W with endpoint 2, and W is non-thin with $\dim(E_2^* W) = 1$, $\dim(E_{D-1}^* W) \leq 1$ and $\dim(E_i^* W) \leq 2$ for $3 \leq i \leq D$.*

(ii) *$\Delta_2 = 0$, and there exist complex scalars α_i, β_i $(2 \leq i \leq D-1)$ such that*

$$|\Gamma_{i-1}(x) \cap \Gamma_{i-1}(y) \cap \Gamma_1(z)| = \alpha_i + \beta_i |\Gamma_1(x) \cap \Gamma_1(y) \cap \Gamma_{i-1}(z)| \qquad (9.4)$$

for all $y \in \Gamma_2(x)$ and $z \in \Gamma_i(x) \cap \Gamma_i(y)$.

With reference to Definition 6.1, assume that $\Delta_2 = 0$ and $\Delta_3 \neq 0$. It is known that this implies $c_2 \in \{1, 2\}$, or $D \leq 5$, see [12, Theorem 4.4]. If $c_2 \in \{1, 2\}$, then the structure of irreducible T-modules with endpoint 2 was studied in detail in [12, 15]. Therefore, we are mainly interested in the case $c_2 \geq 3$. We have to mention however that we are not aware of any of such a graph. Using a computer program we found intersection arrays

$\{b_0, b_1, b_2, b_3, b_4; c_1, c_2, c_3, c_4, c_5\}$ up to valency $k = 20000$, which satisfy the following conditions: $c_2 \geq 3$, $\Delta_2 = 0$, $\Delta_3 > 0$, $\Delta_4 > 0$, $\gamma_2 \in \mathbb{N}$, $p^2_{22} \in \mathbb{N}$. None of them passed the feasibility condition $p^1_{ij} \in \mathbb{N} \cup \{0\}$, see the table below.

intersection arrays	feasibility condition
(58, 57, 49, 21, 1; 1, 9, 37, 57, 58)	$p^1_{23} = 1102/3 \notin \mathbb{N}$
(112, 111, 100, 45, 4; 1, 12, 67, 108, 112)	$p^1_{34} = 103600/67 \notin \mathbb{N}$
(186, 185, 161, 35, 1; 1, 25, 151, 185, 186)	$p^1_{23} = 6882/5 \notin \mathbb{N}$
(274, 273, 256, 120, 10; 1, 18, 154, 264, 274)	$p^1_{23} = 12467/3 \notin \mathbb{N}$
(274, 273, 256, 120, 1; 1, 18, 154, 273, 274)	$p^1_{23} = 12467/3 \notin \mathbb{N}$
(1192, 1191, 1156, 561, 28; 1, 36, 631, 1164, 1192)	$p^1_{23} = 118306/3 \notin \mathbb{N}$
(3236, 3235, 3136, 760, 1; 1, 100, 2476, 3235, 3236)	$p^1_{23} = 523423/5 \notin \mathbb{N}$

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