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Automorphism groups of Walecki tournaments with zero and odd signatures

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Abstract

Walecki tournaments were defined by Alspach in 1966. They form a class of regular tournaments that posses a natural Hamilton directed cycle decomposition. It has been conjectured by Kelly in 1964 that every regular tournament possesses such a decomposition. Therefore Walecki tournaments speak in favor of the conjecture. A second interest in Walecki tournaments arises from the mapping between cycles of the complementing circular shift register and isomorphism classes of Walecki tournaments. The problem of enumerating non-isomorphic Walecki tournaments has not been solved to date. We characterize the arc structure of Walecki tournaments whose corresponding binary sequences have zero and odd signature. Automorphism groups are determined for zero signature Walecki tournaments and for odd signature Walecki tournaments with the zero signature Walecki subtournaments.

Walecki tournaments possess a broad range of subtournaments isomorphic to some Walecki tournament. Subtournaments of odd signature Walecki tournaments induced by the outsets of the central vertex are proven to be either regular or almost regular.

Keywords: Tournaments, Hamilton directed cycles, automorphism groups.

Math. Subj. Class.: 05C20, 05C45, 05E18, 20B25

1 Introduction

Walecki tournaments were defined by Alspach in 1966. They form a class of regular tournaments that posses a natural Hamilton directed cycle decomposition. It has been conjectured by Kelly in 1964 that every regular tournament possesses such a decomposition. An approximate version of Kelly's conjecture has been proven by Kühn et al. [9]. Kelly's conjecture has been verified for regular tournaments on n vertices, whenever n is sufficiently

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large in [8]. Counting Walecki-type Hamiltonian cycle systems up to isomorphism has been solved by Brugnoli [6]. The problem of enumerating non-isomorphic Walecki tournaments has not been solved to date. It was published as an open problem by Alspach [2]. Research of this paper continues the work of Aleš [1].

We first define cycles for the complementing circular shift register on binary sequences of length n (see Subsection 1.1). Walecki tournaments are defined in Subsection 1.2. Section 2 determines the arc structure of Walecki tournaments, while Section 3 focuses on Walecki tournaments with zero signature. A specific permutation is proven to be an automorphism for Walecki tournaments. For n odd, transitive subtournaments induced by outsets and insets of specific vertices are discussed in Subsection 3.1.1 and, for n even, almost-regular and transitive subtournaments are studied in Subsection 3.1.2. Section 3.2 contains characterization of automorphism groups of Walecki tournaments with zero signature. Section 4 contains results on odd signature Walecki tournaments and Subsection 4.2 contains characterization of automorphism groups of Walecki tournaments with odd signature with a zero subsignature.

For theoretical background on tournaments we refer the reader to Beineke and Reid [5], Moon [12], and for topics on permutation groups to Burnside [7] and Wielandt [13].

Automorphism groups of Walecki tournaments were computed with algorithm NAUTY (No AUTomorphisms, Yes?). Dr. Brendan McKay has made the graph isomorphism program NAUTY available to the academic community and it proved to be an indispensible tool in this research (see McKay [10, 11]).

1.1 Cycles of the complementing circular shift register

Let E_n denote the set of all binary sequences $e = (e_1, e_2, \ldots, e_n)$ with $e_i = 0$ or 1 for all i. When considering particular binary sequences we will use $e_1 e_2 \cdots e_n$ to denote $(e_1, e_2, \ldots, e_n)$. We use standard notation $\overline{e_i}$ to denote $(e_i + 1)$ modulo 2 and $\overline{e}$ to denote $(\overline{e_1}, \overline{e_2}, \ldots, \overline{e_n})$.

Let $R\colon E_n \to E_n$, a *complementing circular shift register* operator, be defined by $(R(e))_i = e_{i+1}$, if $1 \leq i \leq n-1$ and $(R(e))_n = \overline{e_1}$. For an integer $k \geq 2$, we have $(R^k(e))_i = (R(R^{k-1}(e)))_i$. It is clear that $R^k(e) \in E_n$ for all $k \geq 1$. If $e \in E_n$, the *period* of e is defined to be the smallest positive integer k such that $R^k(e) = e$. That is, $R^k(e) = e$, and $R^j(e) \neq e$ for $2 \leq j \leq k-1$.

A subset $\{e, R(e), \ldots, R^{m-1}(e)\}$ of E_n is called an *m-cycle* of operator R if e has period m. Define $e \sim_R f$ if and only if $f = R^k(e)$ for $e, f \in E_n$ and some integer k. It is easy to verify that $\sim_R$ is an *equivalence relation* on E_n. Let $[e]_R$, or $[e]$ if no confusion will arise, denote the equivalence class containing $e \in E_n$ under the relation $\sim_R$. If not otherwise stated, we will consider the lexicographically smallest element of $[e]_R$ to be the canonical equivalence class representative. This representation is canonical since no two distinct binary sequences have the same lexicographic order.

Let f and h be sequences in E_{n_1} and E_{n_2}, respectively, and let $e = fh$ denote the sequence of length $n_1 + n_2$ in $E_{n_1+n_2}$.

For completness reasons we state results obtained by Alspach [4] which have direct implications on the structure of Walecki tournaments with non-trivial automorphism groups.

Lemma 1.1 (Alspach, 1966). *Let $e \in E_n$. If a positive integer k divides n such that n/k is odd and if*

$$e = f\overline{f}f\overline{f}\ldots\overline{f}f, \textit{ for } f \in E_k,$$

then $R^{2k}(e) = e$. Moreover, e and f have the same period.

Lemma 1.2 (Alspach, 1966)**.** *If $e \in E_n$ has period m, where $m < 2n$, then $m = 2r$ where r divides $n, n/r$ is odd, $r < n$, and $e = f\overline{f}f\overline{f}\ldots\overline{f}f$ such that $f \in E_r$ and the period of f is $2r = m$.*

1.2 Definition of Walecki tournaments

Let $[v(0), v(1), \ldots, v(2n)]$ be a given undirected Hamilton cycle of order $2n+1$ on the vertex set $\{v(0), v(1), \ldots, v(2n)\}$. Let $\tau \in \mathbb{S}_{2n+1}$ be the permutation of $\{v(0), v(1), \ldots, v(2n)\}$ defined by $\tau = (v(0))(v(1)v(2)v(4)v(6)\cdots v(2n-2)v(2n)v(2n-1)v(2n-3)\cdots v(5)v(3))$. We define the v-*labeling* of the vertices of Walecki tournaments as follows: $2n$ vertices on the circumference of a circle are labeled as $v(1), v(2), v(4), \ldots, v(2n-2), v(2n), v(2n-1), \ldots, v(5)$, $v(3)$ with the central vertex labeled $v(0)$.

The permutation τ corresponds to the clockwise rotation of the vertices on the circumference of the circle with $v(0)$ as a fixed point in the center. Figure 1 shows the action of the permutation $\tau \in \mathbb{S}_{2n+1}$ on vertices of the given Hamilton cycle. A Walecki tourna-

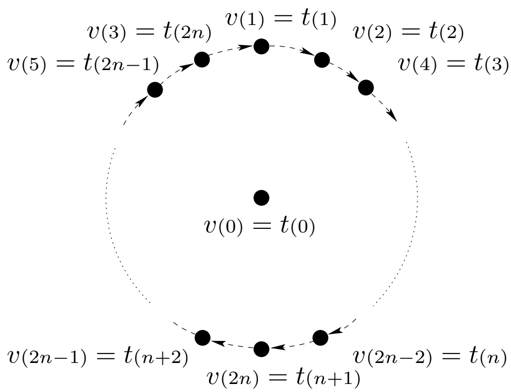

Figure 1: The diagram shows the action of the permutation $\tau \in \mathbb{S}_{2n+1}$ on vertices of the Walecki tournament $W(e)$. Vertex $v(0)$ is fixed by τ.

ment on $2n+1$ vertices is then defined by assigning to each of the n *undirected* Hamilton cycles $H_1 = [v(0), v(1), \ldots, v(2n)]$, $H_2 = [\tau(v(0)), \tau(v(1)), \ldots, \tau(v(2n))]$, $\ldots, H_n = [\tau^{n-1}(v(0)), \tau^{n-1}(v(1)), \ldots, \tau^{n-1}(v(2n))]$ one of the two possible orientations. For example, Figure 2 shows the directed Hamilton cycle $\overrightarrow{H_1} = [v(0), v(1), \ldots, v(2n)]$. Directed Hamilton cycle $\overleftarrow{H_1}$ has all the arcs of $\overrightarrow{H_1}$ reversed, that is, $\overleftarrow{H_1} = [v(0), v(2n), \ldots, v(1)]$. It is easy to see that the union of n Hamilton directed cycles indeed forms a regular tournament.

Let $e \in E_n$ be a binary sequence of length n. Define components of e as follows, $e_i = 0$, if $v(0) \to \tau^{i-1}(v(1))$, and $e_i = 1$, if $v(0) \leftarrow \tau^{i-1}(v(1))$, for $1 \le i \le n$, where arrows $\to$ and $\leftarrow$ denote arcs in a tournament. This establishes a one-to-one correspondence between all 2^n possible orientations of n Hamilton cycles and the elements of E_n. For example, Figure 3 shows Walecki tournaments $W(0)$ and $W(00)$ on 3 and 5 vertices, respectively, Figure 4 shows Walecki tournament $W(000)$ on 7 vertices, and Figure 5 shows Walecki tournament $W(0000)$ on 9 vertices, where $W(e)$ denotes a Walecki tournament with Hamilton cycles directed according to elements of the sequence e.

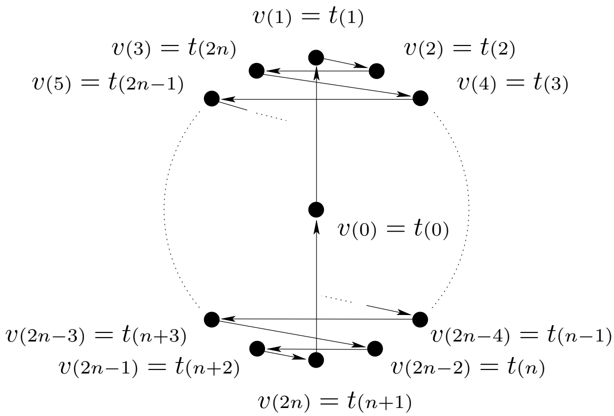

Figure 2: The directed Hamilton cycle $\overrightarrow{H_1} = [v(0), v(1), \ldots, v(2n)]$.

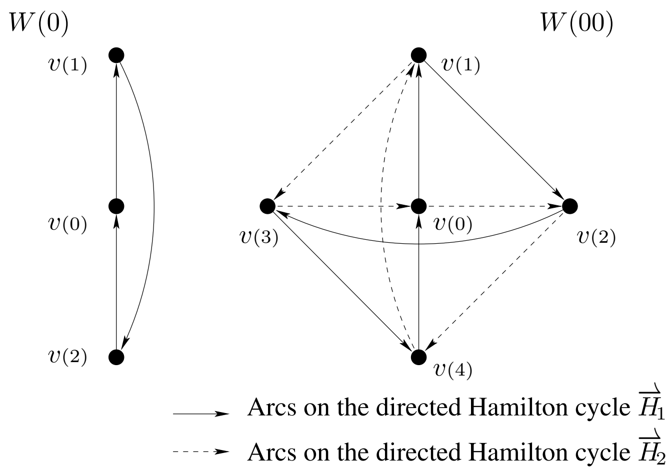

Figure 3: Walecki tournaments $W(0)$ and $W(00)$.

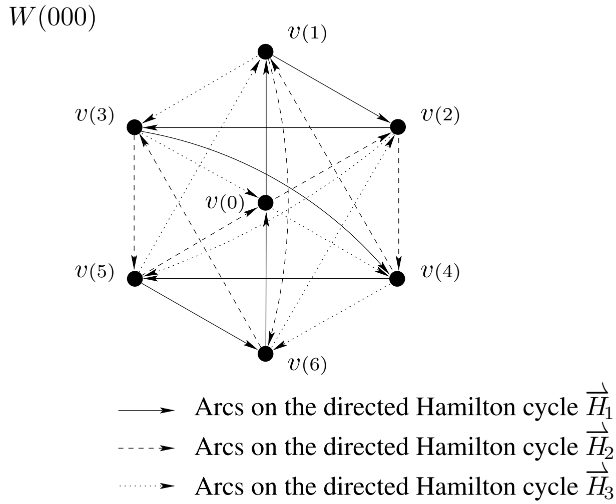

Figure 4: Walecki tournament $W(000)$.

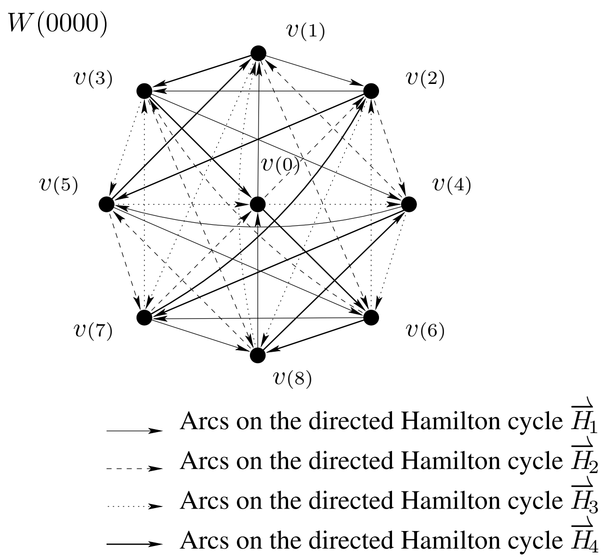

Figure 5: Walecki tournament $W(0000)$.

We state a necessary condition for isomorphism of two Walecki tournaments which can be easily proven using isomorphism τ^k for $1 \leq k \leq 2n$.

Proposition 1.3 (Alspach, 1966)**.** *Let n be a positive integer and let $e \in E_n$. If k is an integer such that $1 \leq k \leq 2n$, then*

$$W(e) \cong W(R^k(e)).$$

Let η denote the permutation $\tau^n \in \mathbb{S}_{2n+1}$. That is,

$$\eta = (v(0))(v(1)\ v(2n))(v(2)\ v(2n-1)) \ldots (v(n)\ v(n+1)).$$

Notice that $\tau(v(0)) = \eta(v(0)) = v(0)$. The n *directed* cycles of $W(e)$ are

$$H_k = [\tau^{k-1}(\eta^{e_k}(v(0))), \tau^{k-1}(\eta^{e_k}(v(1))), \ldots, \tau^{k-1}(\eta^{e_k}(v(2n)))],$$

for $1 \leq k \leq n$.

In order to simplify notation we introduce the t-labeling $\{t(0), t(1), \ldots, t(2n)\}$ of the vertices of Walecki tournaments, where $t(0) = v(0)$ and $t(i) = \tau^{i-1}(v(1))$, $1 \leq i \leq 2n$ (see Figure 2). The action of τ on $\{t(0), t(1), \ldots, t(2n)\}$ is given by $\tau = (t(0))(t(1)\ t(2) \cdots t(2n))$.

2 Arc structure of Walecki tournaments

In 1964 Kelly conjectured that every regular tournament admits a decomposition into Hamilton directed cycles (see Moon [12]). Walecki tournaments are regular and possess a natural Hamilton directed cycle decomposition (see Alspach [3]). The class of Walecki tournaments speaks in favor of the above conjecture. Therefore knowledge of their structure would be of importance. They posses a rich collection of induced subtournaments ranging from transitive to regular as we shall see later. In some instances outsets of $v(0)$ induce regular or almost regular subtournaments. In other cases outsets of $v(0)$ induce subtournaments whose scores differ for at most 2.

We will first state various results which give insight into the arc structure of an arbitrary Walecki tournament. The reader can find the proofs in [1].

Proposition 2.1. *Walecki tournaments are self-complementary and*

$$\mathrm{AntiAut}(W(e)) = \mathrm{Aut}(W(e))\eta,$$

where AntiAut *denotes the antiautomorphism group of a tournament.*

Proposition 2.2. *Let T be a tournament and let $V(T)$ denote the vertex set of T. For $v \in V(T)$ and $g \in \mathrm{Aut}(T)_v$,*

$$g(N^+(v)) = N^+(v) \quad \textit{and} \quad g(N^-(v)) = N^-(v),$$

where $\mathrm{Aut}(T)_v$ denotes the subgroup of $\mathrm{Aut}(T)$ which fixes v.

Proposition 2.3. *Let $W(e)$ be a Walecki tournament of order $2n+1$ and let k be an integer such that $1 \leq k \leq 2n$. If $t(0) \to t(k) \in \overrightarrow{H_k}$, then $t(n+k) \to t(0) \in \overrightarrow{H_k}$.*

Proposition 2.4. *Let $W(e)$ be a Walecki tournament of order $2n+1$ and let i and j be integers such that $0 \leq i, j \leq 2n-1$. If $t(i) \to t(j)$, then $t(n+i) \leftarrow t(n+j)$.*

The following result is used in many proofs about the structure of Walecki tournaments. It uses the binary sequence $e \in E_n$ to determine the direction of a particular arc in $W(e)$. The arcs are grouped according to the Hamilton directed cycle they belong to.

Lemma 2.5. *Let $e \in E_n$ and let $W(e)$ be the corresponding Walecki tournament. Let i and j be integers such that $1 \leq i < j \leq 2n$. In the case when $j - i$ is even, let $k = i + 1 + (j - i)/2$.*

- *If $1 \leq k \leq n$, then $t_{(j+e_k)} \to t_{(i)} \to t_{(j+\bar{e}_k)}$ and $t_{(j+n+e_k)} \leftarrow t_{(i+n)} \leftarrow t_{(j+n+\bar{e}_k)}$.*
- *If $n + 1 \leq k \leq 2n$, then $t_{(j-n+e_{k-n})} \to t_{(i-n)} \to t_{(j-n+\bar{e}_{k-n})}$ and $t_{(j+e_{k-n})} \leftarrow t_{(i)} \leftarrow t_{(j+\bar{e}_{k-n})}$.*

In the case when $j - i$ is odd, let $\ell = i + 1 + (j - i - 1)/2$.

- *If $1 \leq \ell \leq n$, then $t_{(j-e_\ell)} \leftarrow t_{(i)} \leftarrow t_{(j-\bar{e}_\ell)}$ and $t_{(j+n-e_\ell)} \to t_{(i+n)} \to t_{(j+n-\bar{e}_l)}$.*
- *If $n + 1 \leq \ell \leq 2n$, then $t_{(j-n-e_{\ell-n})} \leftarrow t_{(i-n)} \leftarrow t_{(j-n-\bar{e}_{\ell-n})}$ and $t_{(j-e_{\ell-n})} \to t_{(i)} \to t_{(j-\bar{e}_{l-n})}$.*

Proof. Let i and j be as in the conditions of the Proposition 2.4. We first consider the case when $j - i$ is even. Let $k = i + 1 + (j - i)/2$. The structure of the Hamilton directed cycle $\overrightarrow{H_k}$ implies that if $e_k = 0$, then $t_{(j)} \to t_{(i)}$ and $t_{(i)} \to t_{(j+1)}$ (see Figure 6). On the other

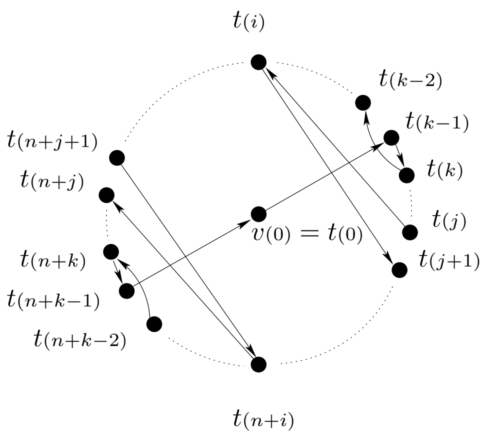

Figure 6: The diagram shows the case when $j - i$ is even and $e_k = 0$ from the proof of Lemma 2.5.

hand, if $e_k = 1$, then $t_{(j+1)} \to t_{(i)}$ and $t_{(i)} \to t_{(j)}$, implying $t_{(j+e_k)} \to t_{(i)} \to t_{(j+\bar{e}_k)}$. Proposition 2.4 implies $t_{(j+n+e_k)} \leftarrow t_{(i+n)} \leftarrow t_{(j+n+\bar{e}_k)}$. To prove the next two statements substitute above $j - n$ for j and $i - n$ for i. The remaining cases are proven similarly. $\square$

3 Zero signature Walecki tournaments

Let $[e]_R$ be an equivalence class of binary n-sequences under the relation $\sim_R$ defined in Subsection 1.1. We say that $[e]_R$ has *zero signature* if the lexicographically smallest sequence of $[e]_R$ is $(0, 0, \ldots, 0)$. Tournaments corresponding to the zero signature sequences, as we shall see, have a surprisingly simple automorphism group. In the case of odd n,

$$\sigma = (t_{(0)})(t_{(1)}t_{(2)} \cdots t_{(n)})\ (t_{(2n)}t_{(2n-1)} \cdots t_{(n+1)}) \in \mathbb{S}_{2n+1}$$

is in fact an automorphism of $W((0,0,\ldots,0))$ as proven in Theorem 3.3. We will prove in Theorem 3.4 and Theorem 3.6 that Walecki tournaments of order $2n+1$ with *zero signature* possess transitive subtournaments of order n that are induced by the outset of vertex $v_{(1)}$. When n is odd they also contain circulant subtournaments of order n induced by the outset of vertex $v_{(0)}$. This furthermore implies that these subtournaments are regular. For example see Figure 4 which shows Walecki tournament $W(000)$.

The next couple results are needed for characterizing the arc structure of Walecki tournaments for the case when n is odd, $n \geq 3$, and $e = (0,0,\ldots,0) \in E_n$. We omit straightforward proofs.

Proposition 3.1. *Let $e \in E_n$ and $n \geq 3$. Consider the Walecki tournament $W(e)$. If $e_i = e_{i+1}$ and $1 \leq i \leq n-1$, then τ is dominance-preserving on $\overrightarrow{H_i}$ and τ^{-1} is dominance-preserving on $\overrightarrow{H}_{i+1}$.*

Notice that permutation τ is not an automorphism of $W(e)$.

Lemma 3.2. *Let $e \in E_n$ and $n \geq 3$. Consider the Hamilton directed cycle $\overrightarrow{H_i}$, $1 \leq i \leq n$, in the Walecki tournament $W(e)$. Let $u \to w$ be any arc of $\overrightarrow{H_i}$ of the form $u = \tau^{i-1}(v_{(2j)})$ and $w = \tau^{i-1}(v_{(2j+1)})$ or $u = \tau^{i-1}(v_{(2j+1)})$ and $w = \tau^{i-1}(v_{(2j+2)})$, $1 \leq j \leq n-2$. If $\rho \in \mathbb{S}_{2n+1}$ is a permutation of $\{v_{(0)}, v_{(1)}, \ldots, v_{(2n)}\}$ such that $\rho = \tau$ on $\tau^{i-1}(v_{(2j)})$, $1 \leq j \leq n-2$, and $\rho = \tau^{-1}$ on $\tau^{i-1}(v_{(2j+1)})$, $1 \leq j \leq n-2$. Then ρ is dominance-preserving on the arc $u \to w$.*

Let n be odd and let the permutation $\sigma \in \mathbb{S}_{2n+1}$ be defined by

$$\sigma = (t_{(0)})(t_{(1)}\, t_{(2)} \cdots t_{(n)})(t_{(2n)}\, t_{(2n-1)} \cdots t_{(n+1)}),$$

(see Figure 7). That is,

$$\sigma = (v_{(0)})(v_{(1)}\, v_{(2)}\, v_{(4)} \cdots v_{(2n-4)}\, v_{(2n-2)})(v_{(3)}\, v_{(5)} \cdots v_{(2n-3)}\, v_{(2n-1)}\, v_{(2n)}).$$

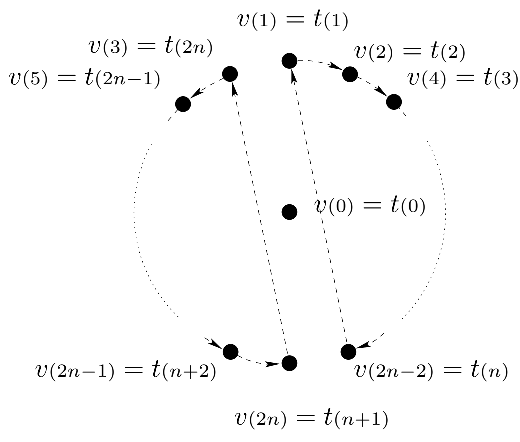

Figure 7: The action of the permutation $\sigma \in \mathbb{S}_{2n+1}$.

Theorem 3.3. *Let $e = (0,0,\ldots,0)$, be the binary n-sequence of all 0s. If n is odd, and $n \geq 3$, then σ is an automorphism of $W(e)$.*

Proof. We want to show that σ is dominance-preserving on all of $W(e)$. By definition, σ fixes $t(0)$, cyclically permutes the vertices of the outset of $t(0)$, and cyclically permutes the vertices of the inset of $t(0)$. Thus, σ is dominance-preserving on the arcs incident with $t(0)$. Figure 7 shows the action of $\sigma \in \mathbb{S}_{2n+1}$ on vertices of the Walecki tournament $W(e)$, for $e = (0, 0, \ldots, 0) \in E_n, n$ odd and $n \geq 3$.

Note that σ restricted to $V^+ = N^+(t(0)) - \{t(n)\} = \{t(1), t(2), \ldots, t(n-1)\}$ has the same action as τ. It then follows from Proposition 3.1 that σ is dominance-preserving on any arc both of whose vertices lie in V^+ because such an arc is not in $\overrightarrow{H_n}$. Similarly, σ restricted to $V^- = N^-(t(0)) - \{t(n+1)\} = \{t(n+2), t(n+3), \ldots, t(2n)\}$ has the same action as τ^{-1}. Again it follows from Proposition 3.1 that σ is dominance-preserving on any arc both of whose vertices lie in V^- because such an arc is not in $\overrightarrow{H_1}$. (Figure 1 shows the action of the permutation τ on vertices of the Walecki tournament $W(e)$.)

By Lemma 3.2, σ is dominance-preserving on any arc with one end vertex in V^+ and the other end vertex in V^- because σ acts like τ on V^+ and τ^{-1} on V^-. It remains to show that σ is dominance-preserving on any arc incident with $t(n) = v(2n-2)$ or $t(n+1) = v(2n)$. Since $e_n = 0$ we have $v(2n-2) \to v(2n) \in \overrightarrow{H_n}$. Furthermore, $\sigma(v(2n-2)) = v(1) = \tau^0(v(1)) = \tau^{n-1}(v(2n-1))$ and $\sigma(v(2n)) = v(3) = \tau^{-1}(v(1)) = \tau^{n-1}(v(2n))$ imply $\sigma(v(2n-2)) \to \sigma(v(2n)) \in \overrightarrow{H_n}$.

We divide the proof for the arcs incident with either $v(2n-2)$ or $v(2n)$ into two cases. Let $u \in V^- \cup V^+$. First we consider arcs $u \to v(2n-2)$ and $u \to v(2n)$. We use Lemma 2.5 extensively. We remind the reader that arcs $\tau^{k-1}(v(i)) \to \tau^{k-1}(v(i+1)), 1 \leq i \leq 2n-1$, belong to the Hamilton directed cycle H_k. In the figures accompanying this proof we use arrows - - - - ➤ to denote the action of σ and arrows ———➤ to denote arcs of the tournament.

Case 1.1. Let $u \in V^-$. If k is an integer such that $1 \leq k \leq (n-1)/2$, then vertices $t(n+2k+1)$ and $t(n+2k)$ belong to V^-, and arcs $t(n+2k+1) = \tau^{k-1}(v(2(n-k)-1)) \to t(n) = \tau^{k-1}(v(2(n-k)))$ and $t(n+2k) = \tau^{k-1}(v(2(n-k)+1)) \to t(n+1) = \tau^{k-1}(v(2(n-k+1)))$ belong to the Hamilton directed cycle $\overrightarrow{H_k}$ (see Figure 8). Now, σ is dominance-preserving on these

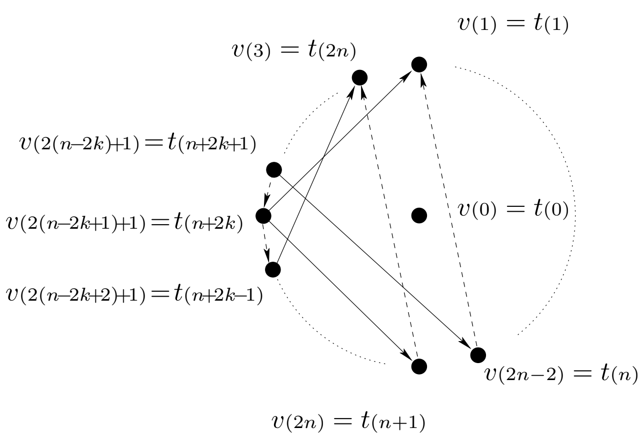

Figure 8: The diagram shows the action of the permutation $\sigma \in \mathbb{S}_{2n+1}$ on arcs from Case 1.1 for the proof of Theorem 3.3.

two arcs since $\sigma(t_{(n+2k+1)}) = t_{(n+2k)} = \tau^{k+(n-1)/2}(v_{(n+2k-1)})$ and $\sigma(t_{(n)}) = t_{(1)} = \tau^{k+(n-1)/2}(v_{(n+2k)})$ imply

$$\sigma(v_{(2(n-2k)+1)}) \to \sigma(v_{(2n-2)}) \in \overrightarrow{H}_{k+(n+1)/2}.$$

Also $\sigma(t_{(n+2k)}) = t_{(n+2k-1)} = \tau^{k+(n-3)/2}(v_{(n+2k-1)})$ and $\sigma(t_{(n+1)}) = t_{(2n)} = \tau^{k+(n-3)/2}(v_{(n+2k)})$ imply

$$\sigma(v_{(2(n-2k+1)+1)}) \to \sigma(v_{(2n)}) \in \overrightarrow{H}_{k+(n-1)/2}.$$

Case 1.2. Let $u \in V^+$. If k is an integer such that $(n+1)/2 \le k \le n-1$, then vertices $t_{(2k-n+1)}$ and $t_{(2k-n+2)}$ belong to V^+, and arcs $t_{(2k-n+1)} \to t_{(n)}$ and $t_{(2k-n+2)} \to t_{(n+1)}$ belong to the Hamilton directed cycle $\overrightarrow{H}_k$ (see Figure 9). Similarly as before we have

$$\sigma(v_{(2(2k-n))}) \to \sigma(v_{(2n-2)}) \in \overrightarrow{H}_{k-(n-3)/2}$$

and

$$\sigma(v_{(2(2k-n-1))}) \to \sigma(v_{(2n)}) \in \overrightarrow{H}_{k-(n-1)/2}.$$

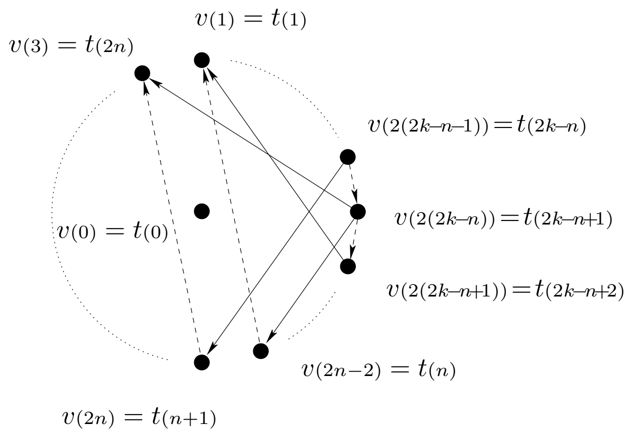

Figure 9: The action of the permutation $\sigma \in \mathbb{S}_{2n+1}$ on arcs from Case 1.2 for the proof of Theorem 3.3.

Next we consider arcs $v_{(2n-2)} \to u$ and $v_{(2n)} \to u$ for $u \in V^- \cup V^+$. We state all cases but leave the proofs to the reader.

Case 2.1. Let $u \in V^-$. If k is an integer such that $2 \le k \le (n-1)/2$, then vertices $v_{(2(n-2k+1)+1)}$ and $v_{(2(n-2k+2)+1)}$ belong to V^-. We have to consider vertices $v_{(3)}$ and $v_{(2n-1)} \in V^-$ as a special case.

Case 2.2. Let $u \in V^+$. If k is an integer such that $(n+3)/2 \le k \le n-1$, then vertices $v_{(2(2k-n-1))}$ and $v_{(2(2k-n+2))}$ belong to V^+. We consider vertex $v_{(1)} \in V^+$ as a special case.

Therefore $u \to w$ implies $\sigma(u) \to \sigma(w)$ for every arc $u \to w$ in $W(e)$ and so $\sigma \in \mathrm{Aut}(W(e))$. □

The importance of σ in the theory of automorphism groups of Walecki tournaments was previously unknown. However, once zero signature sequences were determined as a potential source of Walecki tournaments with non-trivial automorphism groups, permutation σ became a natural candidate for a generator.

3.1 Subtournaments of zero signature Walecki tournaments

Next we characterize specific subtournaments of zero signature Walecki tournaments, which prove to be transitive for n odd and almost regular or transitive for n even.

3.1.1 Transitive subtournaments for n odd

In the following result we prove transitivity of subtournaments of Walecki tournament with zero signature for n odd. The linear orderings of subsets of vertices that induce transitive subtournaments are given in the proof.

Theorem 3.4. *Let $T = W(e)$ for $e = (0, 0, \ldots, 0) \in E_n$, n odd, and $n \geq 3$. For $t_{(i)} \in N^+(t_{(0)})$ and $t_{(j)} \in N^-(t_{(0)})$ the tournaments*

$$\langle N^+(t_{(i)})\rangle, \quad T\langle N^-(t_{(j)})\rangle, \quad T\langle N^-(t_{(i)}) - \{t_{(0)}\}\rangle, \quad \textit{and} \quad T\langle N^+(t_{(j)}) - \{t_{(0)}\}\rangle$$

are transitive subtournaments of T.

Proof. Proposition 2.1 tells us that $W(e) \cong \overline{W(e)}$. Since $\sigma \in \mathrm{Aut}(T)$, it suffices to prove the theorem for the vertex $t_{(1)} \in N^+(t_{(0)})$. Let us consider the outset of vertex $t_{(1)}$. The arcs $v_{(2i+1)} \to v_{(2i+2)}$ lie in $\overrightarrow{H_1}$ for $0 \leq i \leq n-1$ so that $t_{(1)} = \tau^i(v_{(2i+1)}) \to \tau^i(v_{(2i+2)}) = \tau^{2i+1}(v_{(1)}) \in \overrightarrow{H}_{i+1}$. Hence

$$N^+(t_{(1)}) = \{t_{(2i+2)} \mid 0 \leq i \leq n-1\}. \tag{3.1}$$

We prove that the vertices of $N^+(t_{(1)})$ in the order $t_{(2n)}, t_{(2)}, t_{(2n-2)}, t_{(4)}, \ldots, t_{(2n-2i)}, t_{(2i+2)}, \ldots, t_{(n+3)}, t_{(n-1)}, t_{(n+1)}$ determine the score sequence $(s_j)_{j=0}^{n-1}$, where $s_j = j$ for $0 \leq j \leq n-1$. That is, $s_{2i} = s(t_{(2n-2i)}) = 2i$ for $0 \leq i \leq (n-3)/2$, $s_{2i+1} = s(t_{(2i+2)}) = 2i+1$ for $0 \leq i \leq (n-3)/2$, and $s_{n-1} = s(t_{(n+1)}) = n-1$. We prove this by showing that all arcs in the subtournament $T\langle N^+(t_{(1)})\rangle$ point from right to left in the ordering of the vertices given above. Figure 10 shows seven different types of arcs considered. We divide the proof into several cases and show details for some of them. In all of them the index i is an integer such that $0 \leq i \leq (n-3)/2$.

Case 1.1. Since $t_{(n+1)} = \tau^{(n+1)/2+i}(v_{(n-2i-1)})$ and $t_{(2i+2)} = \tau^{(n+1)/2+i}(v_{(n-2i)})$, the arcs of type $t_{(n+1)} \to t_{(2i+2)}$ belong to cycles $\overrightarrow{H}_{(n+3)/2+i}$.

We omit proofs of $t_{(n+1)} \to t_{(2n-2i)} \in H_{(n+1)/2-i}$ and $t_{(2i)} \to t_{(2n-2i)} \in \overrightarrow{H_1}$. In the remaining cases the index j is in the range $i \leq j \leq (n-3)/2$.

Case 1.2. Since $t_{(2j+2)} = \tau^{i+j+1}(v_{(2(j-i))})$ and $t_{(2i+2)} = \tau^{i+j+1}(v_{(2(j-i)+1)})$, it is $t_{(2j+2)} \to t_{(2i+2)} \in \overrightarrow{H}_{i+j+2}$.

We omit proofs of $t_{(2j+2)} \to t_{(2n-2i)} \in \overrightarrow{H}_{j-i+1}$, $t_{(2n-2j)} \to t_{(2i+2)} \in \overrightarrow{H}_{n+i-j+1}$, and $t_{(2n-2j)} \to t_{(2n-2i)} \in \overrightarrow{H}_{n-i-j}$.

It follows that the scores of vertices in the subtournament $T\langle N^+(t_{(1)})\rangle$ are $s_j = j$ for $0 \leq j \leq n-1$. Thus, the subtournament $T\langle N^+(t_{(1)})\rangle$ is transitive.

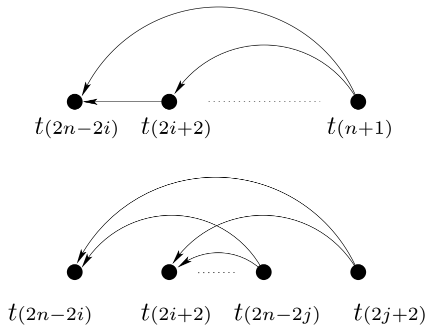

Figure 10: Seven different types of arcs from Case 1.1 and Case 1.2 of the proof of transitivity of the tournament $T\langle N^+(v_{(1)})\rangle$ from Theorem 3.4.

Next we consider the set of vertices $N^-(t_{(1)}) - \{t_{(0)}\}$. Since $N^+(t_{(1)}) = \{t_{(2i+2)} \mid 1 \leq i \leq n-1\}$, it follows that $N^-(t_{(1)}) - \{t_{(0)}\} = \{t_{(2i+1)} \mid 1 \leq i \leq n-1\}$. We will prove that the labeling of the vertices of $N^-(t_{(1)}) - \{t_{(0)}\}$ in the order $t_{(2n-1)}, t_{(3)}, t_{(2n-3)}, t_{(5)}, \dots, t_{(2n-2i+1)}, t_{(2i+1)}, \dots, t_{(n+2)}, t_{(n)}$ determines the score sequence $(s_j)_{j=0}^{n-1}$, where $s_j = j$ for $0 \leq j \leq n-2$. That is, $s_{2i-2} = s(t_{(2n-2i+1)}) = 2i-2,$ for $1 \leq i \leq \frac{n-1}{2}$, and $s_{2i-1} = s(t_{(2i+1)}) = 2i-1,$ for $1 \leq i \leq \frac{n-1}{2}$. Similarly as in the previous case one can prove that all arcs in the subtournament $T\langle N^-(t_{(1)}) - \{t_{(0)}\}\rangle$ point from right to left in the ordering of the vertices given above. Thus, the subtournament $T\langle N^-(t_{(1)}) - \{t_{(0)}\}\rangle$ is transitive. □

3.1.2 Almost-regular and transitive subtournaments for n even

When n is even σ is not an automorphism because an automorphism group of a tournament has to have an odd order.

Theorem 3.5. *Let $T = W(e)$ for $e = (0, 0, \dots, 0) \in E_n$, n even, and $n \geq 4$. The subtournaments $T\langle N^+(t_{(0)})\rangle$ and $T\langle N^-(t_{(0)})\rangle$ are almost regular.*

Proof. It follows from the definition of Walecki tournaments that $N^+(t_{(0)}) = \{t_{(i)} \mid 1 \leq i \leq n\}$. Equation (3.1) also holds for n even. Hence, $N^+(t_{(1)}) = \{t_{(2i)} \mid 1 \leq i \leq n\}$, implying $N^+(t_{(0)}) \cap N^+(t_{(1)}) = \{t_{(2i)} \mid 1 \leq i \leq n/2\}$. Therefore, $|N^+(t_{(0)}) \cap N^+(t_{(1)})| = n/2$. It is easy to verify that $N^+(t_{(2)}) = \{t_{(2i+1)} \mid 1 \leq i \leq n-1\} \cup \{t_{(2n)}\}$, which implies $N^+(t_{(0)}) \cap N^+(t_{(2)}) = \{t_{(2i+1)} \mid 1 \leq i \leq n/2-1\}$. Furthermore, $|N^+(t_{(0)}) \cap N^+(t_{(2)})| = n/2 - 1$.

The scores of the remaining vertices in $N^+(t_{(0)})$ can be obtained similarly, then we omit the proofs. They alternate between $n/2$ and $n/2-1$ which proves that $T\langle N^+(t_{(0)})\rangle$ is almost regular. Since T is self-complementary, this completes the proof. □

Theorem 3.6. *Let $T = W(e)$ for $e = (0, 0, \dots, 0) \in E_n$, n even, and $n \geq 4$. For $t^+ \in N^+(t_{(0)})$ and $t^- \in N^-(t_{(0)})$ the tournaments*

$$T\langle N^+(t^+)\rangle, \quad T\langle N^-(t^+) - \{t_{(0)}\}\rangle, \quad T\langle N^+(t^-) - \{t_{(0)}\}\rangle, \quad \textit{and} \quad T\langle N^-(t^-)\rangle$$

are transitive subtournaments of T.

Proof. Proposition 2.1 tells us that $T \cong \overline{T}$. Hence, it suffices to prove the theorem for vertices in $N^+(t(0))$. The proof of transitivity of $T\langle N^+(t(1))\rangle$ and $T\langle N^-(t(1)) - \{t(0)\}\rangle$ is similar to the proof of Theorem 3.4, the difference being that n is even. This changes the proof in two ways. First, the vertices of $N^+(t(1))$ in the order $t(2n), t(2), t(2n-2), t(4), \ldots, t(2n-2i), t(2i+2), \ldots, t(n+2), t(n)$ determine the score sequence $(0, 1, 2, \ldots, n-1)$.

Furthermore, since n is even we have $\sigma \notin \mathrm{Aut}(T)$. Therefore, one has to prove that the subtournaments $T\langle N^+(t^+)\rangle$ and $T\langle N^-(t^+) - \{t(0)\}\rangle$ are transitive for all $t^+ \in N^+(t(0))$. The proofs are similar to the initial case and we omit them. $\square$

3.2 Automorphism groups of Walecki tournaments with zero signature

The arc structure of Walecki tournaments with zero signature plays a major role in determining their automorphism groups.

Theorem 3.7. *Automorphism groups of Walecki tournaments with zero signature are cyclic:*

$$\mathrm{Aut}(W(0)) = \mathbb{Z}_3, \quad \mathrm{Aut}(W(00)) = \mathbb{Z}_5, \quad \mathrm{Aut}(W(e)) = \mathbb{Z}_n, \quad \textit{for } n \textit{ odd, } n \geq 3,$$

and

$$\mathrm{Aut}(W(e)) = \mathbb{Z}_1, \quad \textit{for } n \textit{ even, } n \geq 4,$$

where $\mathbb{Z}_n$ denotes the cyclic group of order n and $e = (0, 0, \ldots, 0) \in E_n$.

Proof. We leave the proof of initial cases as an exercise for the reader. Let T denote the Walecki tournament $W(e)$ and let G denote its automorphism group $\mathrm{Aut}(T)$. We use Orbit Stabilizer Theorem two times to get

$$|G| = |\mathcal{O}(v(0))||G_{v(0)}| = |\mathcal{O}(v(0))||\mathcal{O}(v(1))||G_{v(0),v(1)}|, \tag{3.2}$$

where $\mathcal{O}(v(1))$ denotes the orbit of vertex $v(1)$ for the subgroup $G_{v(0)}$ of G.

Case 1. Let us assume that n is odd, $n \geq 3$, and $e = (0, 0, \ldots, 0) \in E_n$. We first consider the cardinality of $\mathcal{O}(v(0))$. $T\langle N^+(v(0))\rangle$ is an almost regular tournament. Therefore, it is not transitive. On the other hand, $T\langle N^+(v(i))\rangle$ is transitive for $v(i) \in N^+(v(0))$ (see Theorem 3.4). Thus, $v(0)$ cannot be mapped to a vertex from $N^+(v(0))$ by elements of G. Proposition 2.1 implies $T \cong \overline{T}$ with the graph anti-automorphism τ^n. Therefore, $v(0)$ cannot be mapped to a vertex from $N^-(v(0))$ by elements of G. We have proven that $v(0)$ must be fixed under the action of G, and thus

$$|\mathcal{O}(v(0))| = 1. \tag{3.3}$$

The fact that $v(0)$ cannot be mapped to any vertex in $N^-(v(0))$ can also be proven directly for $n \geq 5$. Let us consider $T\langle N^+(v(0)) - v(i)\rangle$ for $v(i) \in N^+(v(0))$. $T\langle N^+(v(0))\rangle$ is regular of degree $(n-1)/2$. Thus, $T\langle N^+(v(0)) - v(i)\rangle$ has $(n-1)/2$ vertices of degree $(n-1)/2$ and $(n-1)/2$ vertices of degree $(n-3)/2$. Therefore, if $n \geq 5$, the subtournament $T\langle N^+(v(0)) - \{v(i)\}\rangle$ is not transitive. However, $T\langle N^+(v(j)) - \{v(0)\}\rangle$ is transitive for $v(j) \in N^-(v(0))$. Therefore, $v(0)$ cannot be mapped to $v(j) \in N^-(v(0))$ by elements of G if $n \geq 5$. We cannot use the same argument for $n = 3$ since the subtournament $T\langle N^+(v(0)) - v(i)\rangle$, for $v(i) \in N^+(v(0))$, is a tournament on two vertices and is therefore transitive.

Next we determine $|\mathcal{O}(v(1))|$. Since $v(0)$ is a fixed point for any element ρ in G, $\rho(N^+(v(0))) = N^+(v(0))$. Hence, $\rho(v(1)) \in N^+(v(0))$ and $|\mathcal{O}(v(1))| \leq |N^+(v(0))| = n$. We proved that the permutation $\sigma \in \mathbb{S}_{2n+1}$ of $V(T)$ defined by

$$\sigma = \big(v(1)\ v(2)\ v(4) \cdots v(2n{-}4)\ v(2n{-}2)\big)\big(v(3)\ v(5) \cdots v(2n{-}3)\ v(2n{-}1)\ v(2n)\big),$$

is an element in G. Since $\sigma(v(0)) = v(0)$, $\sigma \in G_{v(0)}$. Hence, $\langle \sigma \rangle \subseteq G_{v(0)}$. The orbit of $v(1)$ for σ is $N^+(v(0))$ implying

$$|\mathcal{O}(v(1))| = n. \tag{3.4}$$

Last we prove that $G_{v(0),v(1)} = id$. The subtournaments $T\langle N^+(v(1))\rangle$ and $T\langle N^-(v(1)) - \{v(0)\}\rangle$ are transitive, implying that any automorphism $\rho \in G_{v(0),v(1)}$ fixes all other vertices. Figure 11 shows the partition of the vertices of T with respect to the outsets and insets of vertices $v(0)$ and $v(1)$. Therefore, $G_{v(0),v(1)} = id$, that is,

$$|G_{v(0),v(1)}| = 1. \tag{3.5}$$

Equations (3.2), (3.3), (3.4), and (3.5) imply that $|G| = n$. Now, $\langle \sigma \rangle \subseteq G_{v(0)} \subseteq G$ and since $\langle \sigma \rangle \cong \mathbb{Z}_n$ we have $G \cong \mathbb{Z}_n$.

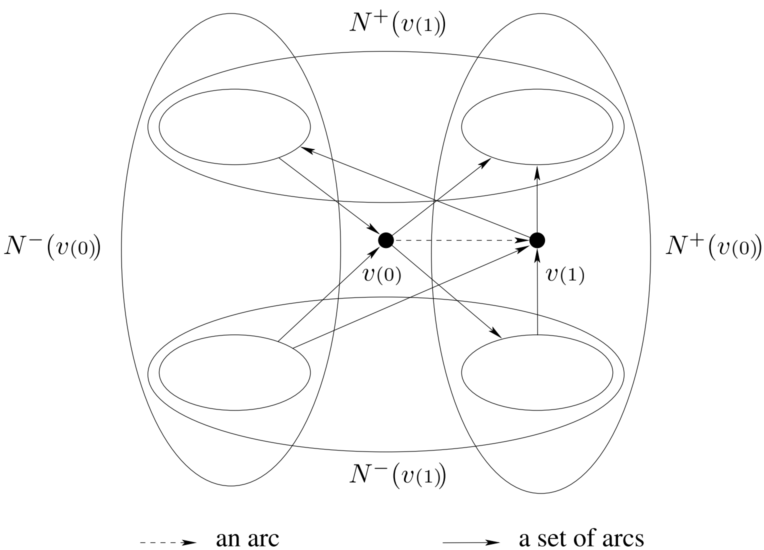

Figure 11: The partition of the vertices of the Walecki tournament $W(e)$, for $e = (0, 0, \dots, 0) \in E_n$, n odd, and $n \geq 3$, with respect to the outsets and insets of vertices $v(0)$ and $v(1)$. Only the arcs essential for the proof of Theorem 3.7 are drawn.

Case 2. Let us assume n is even, $n \geq 4$, and $e = (0, 0, \dots, 0) \in E_n$. We first consider the cardinality of $\mathcal{O}(v(0))$. The subtournament $T\langle N^+(v(0))\rangle$ is almost regular (see Theorem 3.5). Therefore, it is not transitive. On the other hand, $T\langle N^+(v(i))\rangle$ is transitive for $v(i) \in N^+(v(0))$ (see Theorem 3.6). Thus, $v(0)$ cannot be mapped to a vertex from $N^+(v(0))$ by elements of G.

Proposition 2.1 implies implies $T \cong \overline{T}$ via the graph anti-automorphism τ^n. Therefore, $v(0)$ cannot be mapped to a vertex from $N^-(v(0))$ by elements of G. We have proven that

$v_{(0)}$ must be fixed under the action of G, and thus

$$|\mathcal{O}(v_{(0)})| = 1. \tag{3.6}$$

Next we determine $|\mathcal{O}(v_{(1)})|$. Since $v_{(0)}$ is a fixed point for any element ρ in G, $\rho(N^+(v_{(0)})) = N^+(v_{(0)})$. Hence, $\rho(v_{(1)}) \in N^+(v_{(0)})$. As seen in the proof of Theorem 3.5, $|N^+(v_{(0)}) \cap N^+(v_{(1)})| = n/2$ and $|N^+(v_{(0)}) \cap N^-(v_{(1)})| = n/2 - 1$. Furthermore, $T\big\langle N^+(v_{(1)})\big\rangle$ and $T\big\langle N^-(v_{(1)})\big\rangle$ are both transitive which implies that $T\big\langle N^+(v_{(0)}) \cap N^+(v_{(1)})\big\rangle$ and $T\big\langle N^+(v_{(0)}) \cap N^-(v_{(1)})\big\rangle$ are transitive. Moreover, $v_{(1)}$ is dominated by $N^+(v_{(0)}) \cap N^-(v_{(1)})$ implying that $T\big\langle (N^+(v_{(0)}) \cap N^-(v_{(1)})) \cup \{v_{(1)}\}\big\rangle$ is transitive. Let $X = N^+(v_{(0)}) \cap N^+(v_{(1)})$ and $Y = (N^+(v_{(0)}) \cap N^-(v_{(1)})) \cup \{v_{(1)}\}$. Now, vertices of X have score $n/2 - 1$ in $T\big\langle N^+(v_{(0)})\big\rangle$. Similarly, vertices of Y have score $n/2$ in $T\big\langle N^+(v_{(0)})\big\rangle$. Therefore, X and Y have to be fixed setwise. Hence,

$$|\mathcal{O}(v_{(1)})| = 1. \tag{3.7}$$

Last we prove that $G_{v(0),v(1)} = id$. Subtournaments $T\big\langle N^+(v_{(1)})\big\rangle$ and $T\big\langle N^-(v_{(1)}) - \{v_{(0)}\}\big\rangle$ are transitive and thus any automorphism fixing both $v_{(0)}$ and $v_{(1)}$ fixes all other vertices. Therefore, $G_{v(0),v(1)} = id$ which implies

$$|G_{v(0),v(1)}| = 1. \tag{3.8}$$

Equations (3.2), (3.6), (3.7), and (3.8) imply that $|G| = 1$ and $G \cong \mathrm{Z}_1$. This completes the proof. □

4 Odd signature Walecki tournaments

A sequence $f\overline{f}\ldots\overline{f}f \in E_n$, for $f \in E_r$ and $n/r > 1$ has an *odd signature*. Notice, that n/r is odd. If there exists an element in the equivalence class $[e]_R$ of odd signature, then all elements in $[e]_R$ have odd signature. We say that such an equivalence class has odd signature.

Furthermore, a sequence $f\overline{f}\ldots f\overline{f}$, for $f \in E_r$ and $n/2r > 1$ has an *even signature*. Let e be such a sequence. Not all sequences of the equivalence class $[e]_R$ have an even signature. For example, sequences $(0,0,1,0)$ and $(0,1,0,1)$ both belong to the same equivalence class. However, only the latter has an even signature. We say that an equivalence class $[e]_R$ has even signature if there exists a sequence in $[e]_R$ with even signature.

To simplify terminology we will refer to an equivalence class with a given signature as a "sequence" with that signature. We call a sequence *periodic* if it has either zero, odd, or even signature. All other sequences are called *aperiodic*. We will furthermore simplify terminology by referring to a Walecki tournament whose corresponding binary sequence has odd signature, for example, as a Walecki tournament with odd signature.

Let $e \in E_n$, let n be divisible by r, and $m = 2r$. We introduce a partition of $V(W(e)) - \{t_{(0)}\}$ into m-sets $M_1, M_2, \ldots, M_{n/r}$, where $M_i = \{t_{((i-1)m+j)} \mid 1 \le j \le m\}$, for $1 \le i \le n/r$, and $|M_i| = m$. First we prove the following result that relates the structure of Walecki tournament $W(e)$ and Walecki tournament $W(f)$, where e has either odd signature $e = f\overline{f}\ldots\overline{f}f \in E_n$ or even signature $e = f\overline{f}\ldots f\overline{f} \in E_n$, and $f \in E_r$ has either a zero signature or is aperiodic.

Theorem 4.1. *Let $n \ge 1, f \in E_r$, and let r divide n. If $e = f\overline{f}\ldots\overline{f}f \in E_n$ or $e = f\overline{f}\ldots f\overline{f} \in E_n$, then $W(e)\langle\{t_{(0)}\} \cup M_1\rangle \cong W(f)$.*

Proof. Let $1 \leq k \leq r$. Let $t_{(i)}$ denote a vertex of $W(e)$ and let $\overline{t}_{(i)}$ denote a vertex of $W(f)$, likewise for $v_{(i)}$ and $\overline{v}_{(i)}$. We define a function $\psi \colon \{t_{(0)}\} \cup M_1 \to V(W(f))$ by $\psi(t_{(0)}) = \overline{t}_{(0)}$ and $\psi(t_{(i)}) = \overline{t}(i)$, for $1 \leq i \leq 2r$. Clearly, ψ is a bijection. We will show that the Hamilton directed cycle $\overrightarrow{H_k}$ in $W(f)$ is a union of ψ-images of directed paths belonging to Hamilton directed cycles $\overrightarrow{H_k}$ and $\overrightarrow{H_{r+k}}$ in $W(e)$.

Let $\overrightarrow{P_k}$ denote the directed path $[t_{(0)}, t_{(k)}, \ldots, t_{(2k)}]$ on $\overrightarrow{H_k}$ and let $\overrightarrow{P_{r+k}}$ denote the directed path $[t_{(0)}, t_{(r+k)}, \ldots, t_{(2k)}]$ on $\overrightarrow{H_{r+k}}$ (see Figure 12). A ψ-image of a directed path $\overrightarrow{P}$ is a directed path comprised of the ψ-images of vertices of $\overrightarrow{P}$. The ψ-image of $\overrightarrow{P_k}$ is $\overrightarrow{P_k'}$. Similarly, the ψ-image of $\overrightarrow{P_{r+k}}$ is $\overrightarrow{P_{r+k}'}$.

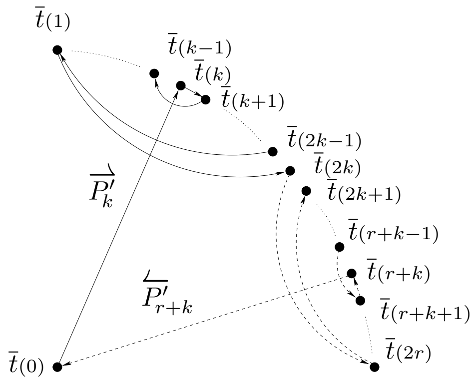

Figure 12: Hamilton directed cycle from the proof of Theorem 4.1.

The definition of Walecki tournaments implies that ψ is dominance-preserving on paths $\overrightarrow{P_k}$ and $\overrightarrow{P_{r+k}}$. The signature of $e = f\overline{f} \ldots \overline{f}f$ implies that if $e_k = 0$, then $e_{r+k} = 1$ and $\overrightarrow{H_k'} = \overrightarrow{P_k'} \cup \overleftarrow{P_{r+k}'}$, where $\overleftarrow{P_{r+k}'}$ denotes the path $\overrightarrow{P_{r+k}'}$ with all of its arcs reversed (see Figure 12). If $e_k = 1$, then $e_{r+k} = 0$ and $\overleftarrow{H_k'} = \overleftarrow{P_{r+k}'} \cup \overrightarrow{P_k'}$. Therefore, $W(e)\langle\{t_{(0)}\} \cup M_1\rangle \cong W(f)$. □

Figure 13 shows Walecki tournament $W(000111000)\langle\{t_{(0)} \cup M_1\}\rangle \cong W(000)$, an example of a Walecki tournament from Theorem 4.1.

If we consider the case when $e \in E_n$ has period $m < 2n$, then Lemma 1.2 implies that $m = 2r$, n/r is odd, $e = f\overline{f} \ldots \overline{f}f \in E_n$, and $f \in E_r$. The special form of e implies various symmetries in the corresponding Walecki tournament.

Lemma 4.2. *Let $n \geq 5$ and let $e \in E_n$ with period $m = 2r < 2n$. If k is an integer such that $1 \leq k \leq r$, then $t_{(2ri+rf_k+k)} \in N^+(t_{(0)})$ and $t_{(2ri+r\overline{f}_k+k)} \in N^-(t_{(0)})$, for $0 \leq i \leq n/r - 1$.*

Proof. Let k be an integer such that $1 \leq k \leq r$. Since $e = f\overline{f} \ldots \overline{f}f$ it follows that $e_{2ri+k} = f_k$ for $0 \leq i \leq (n/r-1)/2$. Therefore, if $f_k = 0$, then $t_{(2ri+k)} \in N^+(t_{(0)})$ and if $f_k = 1$, then $t_{(2ri+k)} \in N^-(t_{(0)})$ for $0 \leq i \leq (n/r-1)/2$.

On the other hand, $e_{2ri+r+k} = \overline{f}_k$ for $0 \leq i \leq (n/r-1)/2 - 1$. Now, if $f_k = 0$, then $t_{(2ri+r+k)} \in N^-(t_{(0)})$ and if $f_k = 1$, then $t_{(2ri+r+k)} \in N^+(t_{(0)})$. If $t_{(0)} \to t_{(k)} \in \overrightarrow{H_k}$, then $t_{(n+k)} \to t_{(0)} \in \overrightarrow{H_k}$, which proves the remaining cases. □

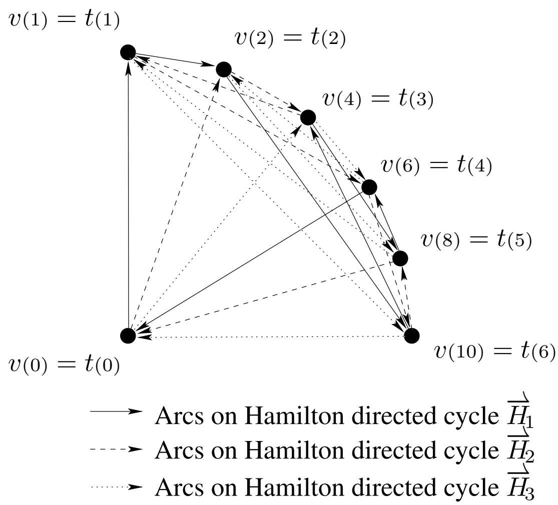

Figure 13: Walecki tournament $W(000)$ as an induced subtournament of $W(000111000)$.

Lemma 4.3. *Let $n \geq 5$ and let $e \in E_n$ with period $m = 2r < 2n$. If k is an integer such that $1 \leq k \leq r$, then $t(\overline{f}_k + 2(2ri+k)-1) \in N^+(t(1))$ and $t(f_k + 2(2ri+k)-1) \in N^-(t(1))$, for $0 \leq i \leq (n/r - 1)/2$.*

Moreover, $t(f_k + 2(2ri+r+k)-1) \in N^+(t(1))$ and $t(\overline{f}_k + 2(2ri+r+k)-1) \in N^-(t(1))$, for $0 \leq i \leq (n/r - 1)/2 - 1$.

Proof. Let $e \in E_n$ have period $m < 2n$. Lemma 1.2 implies that $m = 2r$, n/r is odd, $e = f\overline{f}\ldots\overline{f}f \in E_n$, and $f \in E_r$. The special form of e tells us that $e_{2ri+k} = f_k$ for $0 \leq i \leq (n/r - 1)/2$. The structure of the Hamilton directed cycle $\overrightarrow{H_k}$ implies that if $f_k = 0$, then $t(2(2ri+k)-1) \in N^-(t(1))$ and $t(2(2ri+k)) \in N^+(t(1))$. Furthermore, if $f_k = 1$, then $t(2(2ri+k)-1) \in N^+(t(1))$ and $t(2(2ri+k)) \in N^-(t(1))$. Therefore, $t(2(2ri+k)+\overline{f}_k-1) \in N^+(t(1))$ and $t(2(2ri+k)+f_k-1) \in N^-(t(1))$.

The special form of e also implies $e_{2ri+r+k} = \overline{f}_k$ for $0 \leq i \leq (n/r - 1)/2 - 1$, then the remaining cases can be proven similarly as above. □

Now, let $e \in E_n$ have odd signature. Theorem 4.1 implies the existence of n/r Walecki subtournaments $T\langle\{t(0)\} \cup M_i\rangle$, for $1 < i < n/r$, of tournament $T = W(e)$. We determine subtournaments of $W(e)$ which are isomorphic to some Walecki tournament with odd signature. This is a generalization of Theorem 4.1 for odd signature Walecki tournaments. The vertices that induce the subtournament are chosen on the circumference in clockwise order starting at the vertex $t(1)$.

Theorem 4.4. *Let $n \geq 5$ and let $e = f\overline{f}\ldots\overline{f}f \in E_n$ with period $m = 2r < 2n$, and $f \in E_r$. If ℓ is an odd integer such that $1 \leq \ell \leq n/r - 2$ then, $W(e)\big\langle\{t(0)\} \cup \{M_1 \cup M_2 \cup \cdots \cup M_\ell\}\big\rangle \cong W(e')$, where $e' = f\overline{f}\ldots\overline{f}f \in E_{\ell r}$.*

Proof. If $\ell = 1$, the conclusion is just Theorem 4.1. Let ℓ be an odd integer such that $3 \leq \ell \leq n/r - 2$ and let $e' = f\overline{f}\ldots\overline{f}f \in E_{\ell r}$. Let $t(i)$ denote a vertex of $W(e)$ and $\overline{t}(i)$ denote a vertex of $W(e')$, likewise for $v(i)$ and $\overline{v}(i)$. We define a function $\psi\colon \{t(0)\} \cup M_1 \cup M_2 \cup \cdots \cup M_\ell \to V(W(e'))$ by $\psi(t(0)) = \overline{t}(0)$ and $\psi(t(i)) = \overline{t}(i)$, for $0 \leq i \leq 2\ell r - 1$.

Clearly, ψ is a bijection. We will show that the Hamilton directed cycle $\overrightarrow{H_k}$ in $W(e')$ is a union of ψ-images of directed paths belonging to Hamilton directed cycles $\overrightarrow{H_k}$ and $\overrightarrow{H_{\ell r+k}}$ in $W(e)$ when $\ell r + k \leq n$, otherwise, $\overrightarrow{H'_k}$ is a union of ψ-images of directed paths belonging to Hamilton cycles $\overrightarrow{H_k}$ and $\overrightarrow{H}_{n-\ell r+k}$.

Case 1. Let us first consider the case when $\ell r + k \leq n$. The ψ-image of $\overrightarrow{P_k}$ is the path $\overrightarrow{P'_k}$ as in the proof of the Theorem 4.1, $\psi(\overrightarrow{P_k}) = \overrightarrow{P'_k}$. The ψ-image of $\overrightarrow{P}_{\ell r+k}$ is $\overrightarrow{P'_{\ell r+k}}$. The definition of Walecki tournaments implies that ψ is dominance-preserving on paths $\overrightarrow{P_k}$ and $\overrightarrow{P}_{\ell r+k}$. By assumption, the difference between $\ell r + k$ and k ia an *odd* multiple of r. Furthermore, odd signature of e implies that if $e_k = 0$ then $e_{\ell r+k} = 1$ and $\overrightarrow{H'_k} = \overrightarrow{P'_k} \cup \overleftarrow{P'_{\ell r+k}}$ (see Figure 14). If $e_k = 1$ then $e_{\ell r+k} = 0$ and $\overrightarrow{H'_k} = \overleftarrow{P'_{\ell r+k}} \cup \overleftarrow{P'_k}$.

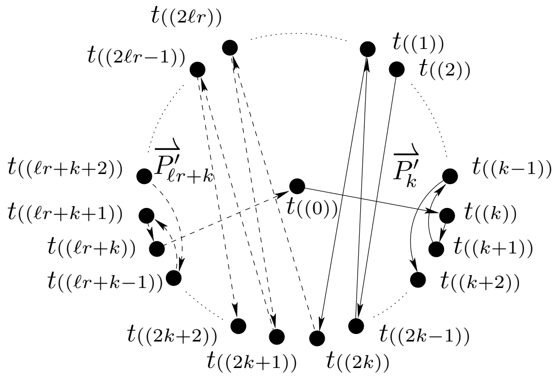

Figure 14: Hamilton directed cycle constructed from directed paths from Case 1 of the proof of Theorem 4.4.

Case 2. Let $\ell r + k > n$. Note that $n - \ell r < k \leq \ell r$. Define $\overrightarrow{P_k}$ as in the previous case and let $\overrightarrow{P}_{\ell r+k-n}$ denote the directed path $[t(2k), t(2\ell r), t(2k{+}1), t(2\ell r{-}1), t(2k{+}1), \ldots, t(\ell r{+}k{+}2), t(\ell r{+}k{-}1), t(\ell r{+}k{+}1), t(\ell r{+}k), t(0)]$ on $\overrightarrow{H}_{\ell r+k-n}$. The ψ-image of $\overrightarrow{P}_{\ell r+k-n}$ is $\overrightarrow{P'_{\ell r+k-n}}$. By the definition of Walecki tournaments ψ is dominance-preserving on the path $\overrightarrow{P}_{\ell r+k}$. Now, n/r is odd and, by assumption, ℓ is also odd which implies that the difference $\ell r - n = (\ell - n/r)r$ is an even multiple of r. Furthermore, the odd signature of e implies that if $e_k = 0$ then $e_{\ell r+k-n} = 0$ and $\overrightarrow{H'_k} = \overrightarrow{P'_k} \cup \overrightarrow{P'_{\ell r+k-n}}$ (see Figure 15). If $e_k = 1$ then $e_{\ell r+k-n} = 1$ and $\overrightarrow{H'_k} = \overleftarrow{P'_{\ell r+k-n}} \cup \overleftarrow{P'_k}$. This completes the proof. □

The importance of τ^m for the automorphism groups of Walecki tournaments with odd signature was previously unknown. The subtournaments $T\langle\{t(0)\} \cup M_i\rangle$, for $1 \leq i \leq n/r$, are isomorphic to $W(f)$. This suggests that the permutation τ^m, which is a product of $m = 2r$ disjoint cycles of length n/r, might be an automorphism of $W(e)$.

Proposition 4.5. *Let* $n \geq 5$. *If* $e \in E_n$ *has period* $m < 2n$, *then* $\tau^m \in \mathrm{Aut}(W(e))$.

Proof. We have $m = 2r$, n/r is odd, $e = f\overline{f} \ldots \overline{f}f \in E_n$, $f \in E_r$, and e and f have the same period $2r$, and $R^m(e) = R^{2r}(e) = e$. The statement follows by Theorem 1.3. □

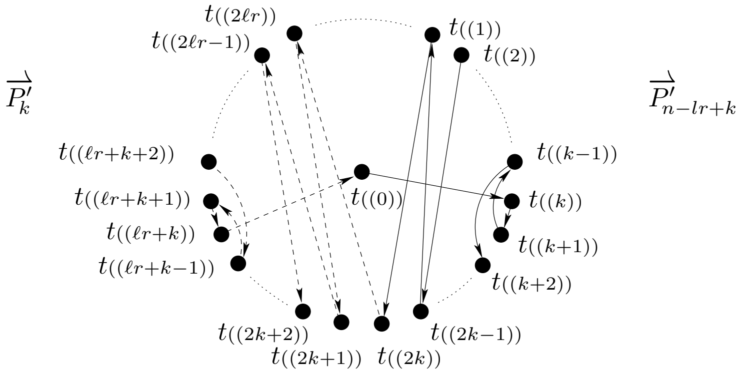

Figure 15: Hamilton directed cycle constructed from directed paths from Case 2 of the proof of Theorem 4.4.

4.1 Subtournaments of odd signature Walecki tournaments

A second partition of $V(W(e)) - \{t_{(0)}\}$ is now introduced. The partition is defined by the orbits $O_1, O_2, \ldots, O_m$, for the permutation τ^m acting on $V(W(e)) - \{t_{(0)}\}$. The orbits have length n/r and can be expressed as follows:

$$O_\ell = \{t_{(im+\ell)} \mid 0 \le i \le n/r - 1\}, \tag{4.1}$$

for $1 \le \ell \le m$ and $m = 2r$.

Let $1 \le k \le m$. One can easily prove that if $f_k = 0$, then $O_k \subseteq N^+(t_{(0)})$ and $O_{r+k} \subseteq N^-(t_{(0)})$. Moreover, if $f_k = 1$, then $O_k \subseteq N^-(t_{(0)})$ and $O_{r+k} \subseteq N^+(t_{(0)})$. In order to simplify the notation we introduce the permutation $\zeta \in \mathbb{S}_{2r}$ acting on the set $\{1, 2, \ldots, 2r\}$ defined by $\zeta = (1\ r+1)(2\ r+2)\cdots(r\ 2r)$. That is, $\zeta(i) = i + r$ for $1 \le i \le r$, and $\zeta(i) = i - r$ for $r + 1 \le i \le 2r$. Note that for $1 \le k \le r$,

$$O_{\zeta^{f_k}(k)} = \{t_{(im+\zeta^{f_k}(k))} \mid 0 \le i \le n/r - 1\}. \tag{4.2}$$

It follows from the observations above that $O_{\zeta^{f_k}(k)} \subseteq N^+(t_{(0)})$ and $O_{\zeta^{\bar{f}_k}(k)} \subseteq N^-(t_{(0)})$. Therefore,

$$N^+(t_{(0)}) = \bigcup_{k=1}^{r} O_{\zeta^{f_k}(k)} \tag{4.3}$$

and

$$N^-(t_{(0)}) = \bigcup_{k=1}^{r} O_{\zeta^{\bar{f}_k}(k)}. \tag{4.4}$$

Orbits and m-sets are *orthogonal* in the sense that each orbit contains exactly one vertex of each m-set, and vice-versa.

The arc structure of subtournaments induced by the orbits O_ℓ, for the permutation τ^m, is determined by the value of e_ℓ, for $1 \le \ell \le m$.

Theorem 4.6. *Let T denote the Walecki tournament $W(e)$ for $e \in E_n$ and $n \ge 5$. If e has period $m < 2n$, then the orbits $O_1, O_2, \ldots, O_m$ for the permutation τ^m acting on*

$V(W(e)) - \{t(0)\}$ induce regular tournaments $T\langle O_1\rangle, T\langle O_2\rangle, \ldots, T\langle O_m\rangle$. If ℓ is an integer such that $1 \le \ell \le m$, the subtournaments $T\langle O_\ell \cap N^+(t(1))\rangle$ and $T\langle O_\ell \cap N^-(t(1))\rangle$ are transitive and the directions of all their arcs are determined by e_ℓ.

Proof. Under the conditions for the sequence $e \in E_n$, Proposition 4.5 implies that $O_1, O_2, \ldots, O_m$ are orbits for the permutation $\tau^m \in \mathrm{Aut}(T)$ which proves that the subtournaments $T\langle O_1\rangle, T\langle O_2\rangle, \ldots, T\langle O_m\rangle$ are regular. To prove the rest of the theorem we first consider the subtournaments $T\langle O_1 \cap N^+(t(1))\rangle$ and $T\langle O_1 \cap N^-(t(1))\rangle$. We make use of Lemma 4.2 and Lemma 4.3.

Let $m = 2r$. Lemma 1.1 implies that $e = f\overline{f}\ldots\overline{f}f$ where $f \in E_r$. Vertices of an arbitrary orbit were determined in (4.1) and (4.2).

We may assume $f_1 = 0$ for if not we may work with $\overline{W(e)}$ instead. We first consider the orbit containing vertex $t(1)$. Since $f_1 = 0$ we have $O_{\varsigma^{f_1}(1)} = O_1 \subseteq N^+(t(0))$ and $O_1 = \{t(2ri+1) \mid 0 \le i \le n/r - 1\}$. Let $Y^+ = O_1 \cap N^+(t(1))$ and $Y^- = O_1 \cap N^-(t(1))$. Since $e_{r+1} = \overline{f}_1 = 1$, we have $t(2r+1) \in Y^+$. Similarly, $e_{2r+1} = f_1 = 0$ implies $t(4r+1) \in Y^-$. Furthermore, $\tau^m \in \mathrm{Aut}(T)$ implies $Y^+ = \{t(4ri+2r+1) \mid 0 \le i \le (n/r-3)/2\}$ and $Y^- = \{t(4ri+1) \mid 1 \le i \le (n/r-1)/2\}$.

Let us consider Y^+. We will prove that considering the vertices of Y^+ in the order $t(2r+1), t(6r+1), \ldots, t(2n-4r+1)$, all arcs point from right to left in the subtournament $T\langle Y^+\rangle$. Let $0 \le i \le (n/r-3)/2$. The equalities

$$\tau^{4rj+2r}(1) = \tau^{2r(j+i+1)}(\tau^{2r(j-i)}(1)) = \tau^{2r(j+i+1)}(4r(j-i))$$

and

$$\tau^{4ri+2r}(1) = \tau^{2r(j+i+1)}(\tau^{2r(i-j)}(1)) = \tau^{2r(j+i+1)}(4r(j-i)+1)$$

imply that the arc $t(4rj+2r+1) \to t(4ri+2r+1)$ belongs to the Hamilton directed cycle $\overrightarrow{H}_{2r(j+i+1)+1}$. This proves the subtournament $T\langle Y^+\rangle$ transitive. Moreover, $\tau^m(Y^+) = Y^-$ implies that the subtournament $T\langle Y^-\rangle$ is also transitive.

Next we consider orbits in $N^+(t(0))$ that do not contain vertex $t(1)$. Equation (4.3) implies that $O_{\varsigma^{f_k}(k)} \subseteq N^+(t(0))$, for $2 \le k \le r$. Let us first assume that $f_k = 0$. Using Equation (4.2) we have $O_{\varsigma^{f_k}(k)} = O_k = \{t(2ri+k) \mid 0 \le i \le n/r - 1\}$. We further divide the proof depending on the parity of k.

Case 1. Let k be odd. Clearly, $k+1$ is even so $e_{(k+1)/2} = f_{(k+1)/2}$ and $e_{r+(k+1)/2} = \overline{f}_{(k+1)/2}$ determine the membership of $t(k)$ and $t(2r+k)$, respectively, in $N^+(t(1))$ and $N^-(t(1))$.

Case 1.1. If $f_{(k+1)/2} = 0$, we have $t(k) \in N^-(t(1))$, and $\overline{f}_{(k+1)/2} = 1$ implies $t(2r+k) \in N^+(t(1))$. It follows that $Y' = \{t(4ri+k) \mid 0 \le i \le (n/r-1)/2\} \subseteq N^-(t(1))$ and $Y'' = \{t(4ri+2r+k) \mid 0 \le i \le (n/r-3)/2\} \subseteq N^+(t(1))$. Similarly as above we can prove that the subtournaments $T\langle Y'\rangle$ and $T\langle Y''\rangle$ are transitive.

Case 1.2. If $f_{(k+1)/2} = 1$, then $t(k) \in N^+(t(1))$ and $\overline{f}_{(k+1)/2} = 0$ which implies that $t(2r+k) \in N^-(t(1))$. Since the sequence e has the form $f\overline{f}\ldots\overline{f}f$ it follows that $Y' \subseteq N^+(t(1))$ and $Y'' \subseteq N^-(t(1))$. As in the previous case we can show that the subtournaments $T\langle Y'\rangle$ and $T\langle Y''\rangle$ are transitive. A change in the value of $f_{(k+1)/2}$ results in the reversal of arcs associated with $t(1)$. However, the arcs between vertices of Y' depend on f_k only. The same reasoning applies to Y''.

Case 2. Let k be even. This implies that $e_{k/2} = f_{k/2}$ and $e_{r+k/2} = \overline{f}_{k/2}$ determine the membership of $t_{(k)}$ and $t_{(2r+k)}$, respectively, in $N^+(t_{(1)})$ and $N^-(t_{(1)})$.

Case 2.1. If $f_{k/2} = 0$, then $t_{(k)} \in N^+(t_{(1)})$, and $\overline{f}_{k/2} = 1$ implies $t_{(2r+k)} \in N^-(t_{(1)})$. Similarly as in the previous case we have $Y^{'} \subseteq N^+(t_{(1)})$, $Y^{''} \subseteq N^-(t_{(1)})$, and the subtournaments $T\langle Y^{'}\rangle$ and $T\langle Y^{''}\rangle$ are transitive.

Case 2.2. Using similar arguments as above, we can prove that if $f_{k/2} = 1$, then the sets $Y^{'}$ and $Y^{''}$ determine transitive subtournaments $T\langle Y^{'}\rangle$ and $T\langle Y^{''}\rangle$, respectively. This completes the proof for $f_k = 0$.

Assume now $f_k = 1$. This implies $O_{\zeta^{f_k}(k)} = O_{r+k} = \{t_{(2ri+r+k)} \mid 0 \le i \le n/r-1\}$. In a way similar to the previous case we define $Y^{'} = \{t_{(4ri+r+k)} \mid 0 \le i \le (n/r-1)/2\}$ and $Y^{''} = \{t_{(4ri+3r+k)} \mid 0 \le i \le (n/r-3)/2\}$, and prove that $T\langle Y^{'}\rangle$ and $T\langle Y^{''}\rangle$ are transitive subtournaments. The proof is similar to the case $f_k = 0$, however, the direction of all arcs considered is reversed since $e_k = f_k = 1$. This completes the proof for orbits $O_{\zeta^{f_k}(k)} \subseteq N^+(t_{(0)})$. The result for orbits $O_{\zeta^{\overline{f}_k}(k)} \subseteq N^-(t_{(0)})$ follows since $T \cong \overline{T}$. $\square$

Next we consider the subtournaments induced by outsets and insets of vertices in a Walecki tournament with odd signature $e = f\overline{f}\ldots\overline{f}f \in E_n$, whose defining sequence is $f = (0,0,\ldots,0) \in E_r$. We show that if the vertex is distinct from $t_{(0)}$ then its outset induces a subtournament that is not regular for n odd. However, the outset of $t_{(0)}$ induces a regular subtournament. Similarly, for n even the outset of $t_{(0)}$ induces an almost regular subtournament but the outset of any other vertex induces a subtournament that is not almost regular. This implies that $t_{(0)}$ must be fixed for any automorphism of a Walecki tournament with an odd signature $e = f\overline{f}\ldots\overline{f}f$ where $f = (0,0,\ldots,0)$.

Theorem 4.7. *Let $T = W(e)$ for $e = f\overline{f}\ldots\overline{f}f \in E_n$, $n \ge 5$, n/r odd, and $f = (0,0,\ldots,0) \in E_r$. For $v \in V(W(e)) - \{t_{(0)}\}$, the tournaments $T\big\langle N^+(v)\big\rangle$ are not regular and not almost regular subtournaments of T for n odd and n even, respectively.*

Proof. Since $W(e) \cong \overline{W(e)}$, it suffices to prove the theorem for vertices in $N^+(t_{(0)})$. Furthermore, since $\tau^m \in \mathrm{Aut}(T)$, it is sufficient to prove the theorem for the vertices in $N^+(t_{(0)}) \cap M_1$. Let $M' = M_1 \cup M_3 \cup \cdots \cup M_{n/r}$ and $M'' = M_2 \cup M_4 \cup \cdots \cup M_{n/r-1}$.

We first assume $r > 1$ and consider $t_{(1)} \in N^+(t_{(0)}) \cap M_1$. We will count the vertices in $N^+(t_{(1)}) \cap N^+(t_{(2n)})$. First we determine the vertices in $N^+(t_{(1)})$. Since $f = (0,0,\ldots,0)$, Theorem 4.1 implies

$$N^+(t_{(1)}) \cap M_1 = \{t_{(2i+2)} \mid 0 \le i \le r-1\}. \tag{4.5}$$

Using Lemma 2.5 we have

$$N^+(t_{(1)}) \cap M_2 = \{t_{(2r+2i+1)} \mid 0 \le i \le r-1\}. \tag{4.6}$$

Let $X' = N^+(t_{(1)}) \cap M'$ and $X'' = N^+(t_{(1)}) \cap M''$. The odd signature of the sequence e and equalities (4.5) and (4.6) imply

$$X' = \{t_{(4rj+2i+2)} \mid 0 \le i \le r-1, 0 \le j \le (n/r-1)/2\} \tag{4.7}$$

and

$$X'' = \{t_{(4rj+2r+2i+1)} \mid 0 \le i \le r-1, 0 \le j \le (n/r-3)/2\}. \tag{4.8}$$

Clearly, $N^+(t_{(1)}) = X' \cup X''$. Next we determine the vertices in $N^+(t_{(2n)})$. We have

$$N^+(t_{(2n)}) \cap M_1 = \{t_{(2i+1)} \mid 1 \leq i \leq r-1\} \tag{4.9}$$

and

$$N^+(t_{(2n)}) \cap M_2 = \{t_{(2r+1)}\} \cup \{t_{(2r+2i+2)} \mid 0 \leq i \leq r-1\}. \tag{4.10}$$

Let Y' denote $N^+(t_{(2n)}) \cap M'$ and let Y'' denote $N^+(t_{(2n)}) \cap M''$. The odd signature of the sequence e and equalities (4.9) and (4.10) imply

$$Y' = \{t_{(4rj+2i+1)} \mid 1 \leq i \leq r-1, 0 \leq j \leq (n/r-3)/2\}. \tag{4.11}$$

Let $Y'' = \overline{Y}'' \cup \overline{\overline{Y}}'' =$ where

$$\overline{Y}'' = \{t_{(4rj+2r+1)} \mid 0 \leq j \leq (n/r-3)/2\} \tag{4.12}$$

and

$$\overline{\overline{Y}}'' = \{t_{(4rj+2r+2i+2)} \mid 0 \leq i \leq r-1, 0 \leq j \leq (n/r-3)/2\}. \tag{4.13}$$

Clearly, $N^+(t_{(2n)}) = \{t_{(0)}\} \cup Y' \cup Y''$.

Comparing the indices of the vertices in equalities (4.7), (4.8), (4.11), (4.12), and (4.13) for vertices in $N^+(t_{(1)})$ and $N^+(t_{(2n)})$, we deduce $(X' \cup X'') \cap (Y' \cup Y'') = \overline{\overline{Y}}$, which implies $N^+(t_{(1)}) \cap N^+(t_{(2n)}) = \overline{Y}''$. Hence, the score of vertex $t_{(2n)}$ in $T\langle N^+(t_{(1)})\rangle$ equals $|\overline{Y}''| = (n/r-1)/2$. If $r > 1$, then $(n/r-1)/2 < n/2 - 1$ which implies that $T\langle N^+(t_{(1)})\rangle$ is not regular or almost regular when n is odd or even, respectively. The proofs for the remaining vertices of $N^+(t_{(0)}) \cap M_1$ are similar and we omit them.

The arc structure of $T\langle N^+(t_{(1)})\rangle$ is different in the case when $r = 1$, that is, when $e = (0,1,0,1,\ldots,0,1,0) \in E_n$. Notice that n/r odd implies that n has to be odd. In order to verify non-regularity of $T\langle N^+(t_{(1)})\rangle$, we consider $N^+(t_{(1)}) \cap N^+(t_{(3)})$. The signature of e implies

$$N^+(t_{(1)}) = \{t_{(2n)}\} \cup \{t_{(4i+2)},\ t_{(4i+3)} \mid 0 \leq i \leq (n-3)/2\}. \tag{4.14}$$

Since $\tau^2 \in \mathrm{Aut}(W(e))$ we have

$$N^+(t_{(3)}) = \{t_{(2)}\} \cup \{t_{(4i+4)},\ t_{(4i+5)} \mid 0 \leq i \leq (n-3)/2\}. \tag{4.15}$$

Comparing the indices of the vertices in equalities (4.7), (4.8), (4.14), (4.15), for vertices in $N^+(t_{(1)})$ and $N^+(t_{(3)})$ we deduce that $N^+(t_{(1)}) \cap N^+(t_{(3)}) = \{t_{(2)}\}$. Hence, the score of vertex $t_{(3)}$ in $T\langle N^+(t_{(1)})\rangle$ equals 1 implying that $T\langle N^+(t_{(1)})\rangle$ is not regular. This completes the proof. □

4.1.1 Regular subtournaments for n odd

Next we determine the structure of arcs in the subtournaments induced by $N^+(t_{(0)})$ and $N^-(t_{(0)})$. There are two cases to be considered depending on the parity of n.

Let $e = f\overline{f}\ldots\overline{f}f \in E_n$, n odd, be an odd signature sequence with a zero subsignature $f = (0,0,\ldots,0) \in E_r$, and let T denote $W(e)$. We will consider the subtournaments $T\langle N^+(t_{(0)})\rangle$ and $T\langle N^-(t_{(0)})\rangle$. We know that the out-neighbours and in-neighbours of $t_{(0)}$ are determined by f and $\overline{f}$. Therefore, r vertices of each m-set belong to $N^+(t_{(0)})$ and the other r of them to $N^-(t_{(0)})$. On the other hand, the construction of Walecki tournaments implies that out of two consecutive vertices $t_{(j)}$ and $t_{(j+1)}$ exactly one is an out-neighbour of the vertex $t_{(i)}$, whenever $j-i$ is even. Therefore one would hope that the score of each vertex in $T\langle N^+(t_{(0)})\rangle$ is no more than $2n/4 = n/2$. This would imply the regularity of the subtournaments $T\langle N^+(t_{(0)})\rangle$ and $T\langle N^-(t_{(0)})\rangle$.

Theorem 4.8. *Let n be odd and let T denote the Walecki tournament $W(e)$ for $e = f\overline{f}\ldots\overline{f}f \in E_n$, n/r odd, and $f = (0,0,\ldots,0) \in E_r$. If e has period $m < 2n$, then the subtournaments $T\langle N^+(t_{(0)})\rangle$ and $T\langle N^-(t_{(0)})\rangle$ are regular.*

Proof. The result is clearly true for $n \le 3$. Let T be a tournament as stated in the conditions of the theorem and $n \ge 5$. In order to prove that $T\langle N^+(t_{(0)})\rangle$ is regular, we first determine the score of the vertex $t_{(1)}$ in $T\langle N^+(t_{(0)})\rangle$. Let Y denote the set $N^+(t_{(0)}) \cap N^+(t_{(1)})$. We are interested in the cardinality of Y. Since $m < 2n$, Lemma 1.2 implies that $m = 2r$ and n/r is odd. Now, n odd implies that r is also odd. We proceed by proving that $|Y \cap (M_i \cup M_{i+1})| = r$, for $1 \le i \le n/r - 1$. We use Lemma 4.2 and Lemma 4.3 extensively.

Let us consider the cardinality of $Y \cap (M_1 \cap M_2)$. We first determine the vertices in $N^+(t_{(1)}) \cap M_1$. If i is an integer such that $1 \le i \le r-1$, then $e_{i+1} = f_{i+1}$. Because $f_{i+1} = 0$, we have

$$t_{(2i+1)} \in N^-(t_{(1)}) \quad \text{and} \quad t_{(2i+2)} \in N^+(t_{(1)}). \tag{4.16}$$

Since $T \cong \overline{T}$, we may assume that $e_1 = f_1 = 0$. Thus, $t_{(2)} \in N^+(t_{(1)})$. Next we consider $N^+(t_{(0)}) \cap M_1$. If j is an integer such that $1 \le j \le r-1$, then $e_{j+1} = f_{j+1}$. Since $f_{j+1} = 0$, we have

$$t_{(j+1)} \in N^+(t_{(0)}) \quad \text{and} \quad t_{(j+1+r)} \in N^-(t_{(0)}). \tag{4.17}$$

Being $0 \le i \le r-1$, we have $2r \le 2(r+i) \le 4r-2$. The neighbours of the vertex $t_{(1)}$ in the set M_2 are given by

$$t_{(2(r+i)+2)} \in N^-(t_{(1)}) \quad \text{and} \quad t_{(2(r+i)+1)} \in N^+(t_{(1)}). \tag{4.18}$$

If k is an integer such that $0 \le k \le r-1$, then $2r \le 2r+k \le 3r-1$. The neighbours of the vertex $t_{(0)}$ in the set M_2 are given by

$$t_{(2r+k+1)} \in N^+(t_{(0)}) \quad \text{and} \quad t_{(2r+k+1+r)} \in N^-(t_{(0)}). \tag{4.19}$$

We use (4.16), (4.17), (4.18), and (4.19) in the following case study. Let i be an integer such that $1 \le i \le (r-3)/2$. Since $e_{i+1} = f_{i+1} = 0$, it follows that $t_{(2i+2)}, t_{(2r+2i+1)} \in N^+(t_{(1)})$. Similarly, $e_{i+(r+1)/2} = f_{i+(r+1)/2} = 0$ implies $t_{(r+2i+1)} \in N^+(t_{(1)})$, and $e_{i+(r+3)/2} = f_{i+(r+3)/2} = 0$ implies $t_{(3r+2i+2)} \in N^+(t_{(1)})$. Also $e_{2i+1} = f_{2i+1} = 0$ implies $t_{(2i+1)}, t_{(2r+2i+1)} \in N^+(t_{(0)})$, and $e_{2i+2} = f_{2i+2} = 0$ implies $t_{(2i+2)}, t_{(2r+2i+2)} \in N^+(t_{(0)})$. Let $1 \le i \le (r-3)/2$. Since $f_{i+1}, f_{2i+1}, f_{2i+2}, f_{i+(r+1)/2}$, and $f_{i+(r+3)/2}$ are all zero, it follows that exactly two of the vertices $t_{(2i+1)}, t_{(2i+2)}, t_{(r+2i+1)}, t_{(r+2i+2)}, t_{(2r+2i+1)}, t_{(2r+2i+2)}, t_{(3r+2i+1)}$, and $t_{(3r+2i+2)}$ belong to $Y \cap (M_1 \cup M_2)$.

We have yet to consider vertices $t_{(1)}, t_{(r)}, t_{(r+1)}, t_{(2r)}, t_{(2r+1)}, t_{(3r)}, t_{(3r+1)}$, and $t_{(4r)}$. Their membership in $N^+(t_{(1)}) \cap N^+(t_{(0)})$ is determined by the values of f_1, f_r and $f_{(r+1)/2}$, which are all zero. Notice that $e_r = f_r$, which implies $t_{(r)}, t_{(3r)} \in N^+(t_{(0)})$, and $t_{(2r)}, t_{(4r)} \in N^+(t_{(1)})$. Similarly, $e_{(r+1)/2} = f_{(r+1)/2}$ implies that if $f_{(r+1)/2} = 0$, then $t_{(r+1)}, t_{(3r)} \in N^+(t_{(1)})$. Since $f_1 = 0$, we have $e_{r+1} = \overline{f}_1 = 1$, which implies $t_{(2r+1)} \in N^+(t_{(1)})$ and $t_{(1)}, t_{(2r+1)} \in N^+(t_{(0)})$. Clearly, $t_{(1)} \notin N^+(t_{(1)})$. It follows that exactly two of the vertices $t_{(r)}, t_{(r+1)}, t_{(2r)}, t_{(2r+1)}, t_{(3r)}, t_{(3r+1)}$, and $t_{(4r)}$ belong to $Y \cap (M_1 \cup M_2)$.

We have considered all vertices in $M_1 \cup M_2$ except $t_{(2)}, t_{(r+2)}, t_{(2r+2)}$, and $t_{(3r+2)}$. Their membership in $N^+(t_{(1)}) \cap N^+(t_{(0)})$ is determined by the values of f_1, f_2 and $f_{(r+3)/2}$, which are all zero. Now, $e_1 = f_1 = 0$ implies $t_{(2)} \in N^+(t_{(1)})$, and $e_{r+1} = \overline{f}_1 = 1$ implies $t_{(2r+2)} \in N^-(t_{(1)})$. Notice that $e_2 = f_2 = 0$ implies $t_{(2)}, t_{(2r+2)} \in N^+(t_{(0)})$. Also, $e_{(r+3)/2} = f_{(r+3)/2} = 0$ implies $t_{(3r+2)} \in N^+(t_{(1)})$. Hence, exactly one of the vertices $t_{(2)}$, $t_{(r+2)}, t_{(2r+2)}$, and $t_{(3r+2)}$ belongs to $Y \cap (M_1 \cup M_2)$. It follows by above observations that

$$|Y \cap (M_1 \cup M_2)| = 2(r-3)/2 + 2 + 1 = r. \tag{4.20}$$

Similar to the previous case we can show that

$$|Y \cap (M_i \cup M_{i+1})| = 2(r-1)/2 + 1 = r,$$

for $2 \leq i \leq n/r - 1$, and

$$|Y \cap (M_1 \cup M_{n/r})| \leq 2(r-3)/2 + 1 + 1 = r - 1. \tag{4.21}$$

Let α denote $|Y \cap M_1|$. Since $|Y \cap (M_1 \cup M_2)| = r$, it follows that $|Y \cap M_2| = r - \alpha$. Since n/r is odd, we have $|Y \cap M_{n/r-1}| = r - \alpha$ and $|Y \cap M_{n/r}| = \alpha$ which implies $|Y \cap (M_1 \cup M_{n/r})| = 2\alpha \leq r - 1$. Now, r is odd implies $|Y \cap (M_1 \cup M_{n/r})| \leq r - 1$. Therefore, $2|Y| \leq (r-1) + (n-r) = n - 1$ which implies

$$s(t_{(1)}) = |N^+(t_{(0)}) \cap N^+(t_{(1)})| \leq \frac{n-1}{2} < \frac{n}{2}. \tag{4.22}$$

Similarly, we can prove that

$$s(t_{(i+rf_i)}) = |N^+(t_{(0)}) \cap N^+(t_{(i+rf_i)})| \leq \frac{n-1}{2},$$

for $2 \leq i \leq r$. That is, every vertex in $N^+(t_{(0)}) \cap M_1$ has score at most $(n-1)/2$. Since τ^m is an automorphism of T, the score of every vertex in the tournament $T\langle N^+(t_{(0)})\rangle$ is at most $(n-1)/2$. Now,

$$\binom{n}{2} = \sum_{v \in N^+(t(0)} s(v) \leq \frac{n(n-1)}{2}$$

implies $s(v) = (n-1)/2$ for every vertex $v \in N^+(t_{(0)})$. Therefore, the subtournament $T\langle N^+(t_{(0)})\rangle$ is regular. Regularity of the tournament $T\langle N^-(t_{(0)})\rangle$ follows since $T \cong \overline{T}$. □

4.1.2 Almost regular subtournaments for n even

When n is even, $T\langle N^+(t_{(0)})\rangle$ can not be regular. However, one can prove that in the case of $f = (0, 0, \ldots, 0) \in E_r$ it is almost regular. We follow an analog of the proof of Theorem 4.8. Oddly enough, the fact that n is even simplifies the proof.

Theorem 4.9. *Let n be even and let T denote the Walecki tournament $W(e)$ for $e = f\overline{f}\ldots\overline{f}f \in E_n$, n/r odd, and $f = (0,0,\ldots,0) \in E_r$. If e has period $m < 2n$, then the subtournaments $T\big\langle N^+(t_{(0)})\big\rangle$ and $T\big\langle N^-(t_{(0)})\big\rangle$ are almost regular.*

Proof. The result is clearly true for $n \leq 4$. Let T be a tournament as stated in the conditions of the theorem. We first prove that $T\big\langle N^+(t_{(0)})\big\rangle$ is almost regular. In order to do so we determine the score of the vertex $t_{(1)}$ in $T\big\langle N^+(t_{(0)})\big\rangle$. Let Y denote the set $N^+(t_{(0)}) \cup N^+(t_{(1)})$. We are interested in the cardinality of Y. Since $m < 2n$, we have $m = 2r$ and n/r is odd. Now, n even implies that r is also even. We proceed by considering sets $Y \cap (M_i \cup M_{i+1})$ for $1 \leq i \leq n/r - 1$. Similar to the proof of Theorem 4.8 we can prove $|Y \cap (M_i \cup M_{i+1})| = r$, for $1 \leq i \leq n/r - 1$, and $|Y \cap (M_{n/r} \cup M_1)| \leq r$. Let α denote $|Y \cap M_1|$. Since n/r is odd, $|Y \cap M_{n/r-1}| = r - \alpha$ and $|Y \cap M_{n/r}| = \alpha$ which implies $|Y \cap (M_1 \cup M_{n/r})| = 2\alpha$. Hence, $2\alpha \leq r$. Since r is even, we have $\alpha \leq r/2$ and $|Y \cap (M_1 \cup M_{n/r})| \leq r$. Therefore, $2|Y| \leq n$ which implies $s(t_{(1)}) = |N^+(t_{(0)}) \cap N^+(t_{(1)})| \leq n/2$. Similarly, we can prove that $s(t_{(i+rf_i)}) = |N^+(t_{(0)}) \cap N^+(t_{(i+rf_i)})| \leq n/2$, for $2 \leq i \leq r$. That is, every vertex in $N^+(t_{(0)}) \cap M_1$ has score at most $n/2$. Since $\tau^m \in \mathrm{Aut}(T)$, the score of every vertex in the subtournament $T\big\langle N^+(t_{(0)})\big\rangle$ is at most $n/2$.

Since $T \cong \overline{T}$ with anti-automorphism $\eta = \tau^n \in \mathbb{S}_{2n+1}$, where

$$\begin{aligned}\eta = \big(v_{(1)}v_{(2n)}\big)\big(v(2)v_{(2n-1)}\big)\ldots\big(v_{(n)}v_{(n+1)}\big)\\ = \big(t_{(1)}t_{(n+1)}\big)\big(t(2)t_{(n+2)}\big)\ldots\big(t_{(n)}t_{(2n-1)}\big),\end{aligned}$$

(see Figure 16), we have $\overline{T\big\langle N^-(t_{(0)})\big\rangle} \cong T\big\langle N^+(t_{(0)})\big\rangle$. Now, $\eta(t_{(1)}) = t_{(3)}$ implies that $|N^-(t_{(0)}) \cap N^+(t_{(3)})| = s(t_{(3)}) \leq n/2$ in $T\big\langle N^-(t_{(0)})\big\rangle$. Therefore, $|N^+(t_{(0)}) \cap N^-(t_{(1)})| \leq n/2$ in $T\big\langle N^+(t_{(0)})\big\rangle$. Since

$$N^+(t_{(0)}) = \big(N^+(t_{(0)}) \cap N^+(t_{(1)})\big) \cup \big(N^+(t_{(0)}) \cap N^-(t_{(1)})\big) \cup \{t_{(1)}\},$$

we have $s(t_{(1)}) = |N^+(t_{(0)}) \cap N^+(t_{(1)})| \geq n/2 - 1$ in $T\big\langle N^+(t_{(0)})\big\rangle$. Using a similar argument we can prove that the score of every vertex in $T\big\langle N^+(t_{(0)})\big\rangle$ is either $n/2$ or $n/2 - 1$.

If k denotes the number of vertices with score $n/2$ in $T\big\langle N^+(t_{(0)})\big\rangle$, then the number of vertices of degree $n/2 - 1$ equals $n - k$. The equation

$$\binom{n}{2} = \sum_{(v \in N^+ t(1))} s(v) = k\frac{n}{2} + (n-k)\frac{n-2}{2}$$

implies that $k = n/2$. In other words, the subtournament $T\langle N^+(t_{(0)})\rangle$ is almost regular. Furthermore, $T\langle N^-(t_{(0)})\rangle$ is also almost regular since $T \cong \overline{T}$. □

4.2 Automorphism groups of odd signature Walecki tournaments with a zero sub-signature

Theorem 4.10. *Let n be odd, $n \geq 5$, and let T denote the Walecki tournament $W(e)$ for $e = f\overline{f}\ldots\overline{f}f \in E_n$, n/r odd, and $f = (0,0,\ldots,0) \in E_r$, then $\mathrm{Aut}(W(e)) = \mathbb{Z}_{n/r}$.*

Proof. Let us assume that n is odd, $n \geq 5$, $e = f\overline{f}\ldots\overline{f}f \in E_n$, n/r odd, and $f = (0,0,\ldots,0) \in E_r$. Let T denote the Walecki tournament $W(e)$ and let G denote its automorphism group $\mathrm{Aut}(T)$. We use Orbit Stabilizer Theorem two times to get

$$|G| = |\mathcal{O}(t_{(0)})|\,|G_{t(0)}| = |\mathcal{O}(t_{(0)})|\,|\mathcal{O}(t_{(1)})|\,|G_{t(0),t(1)}|, \tag{4.23}$$

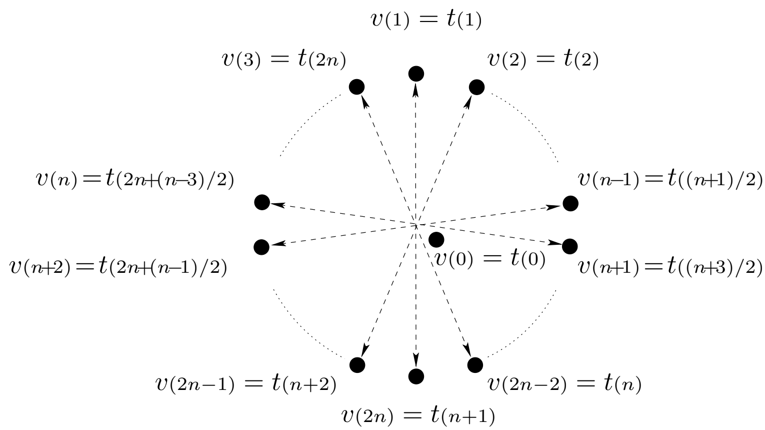

Figure 16: The diagram shows the action of the permutation $\eta = \tau^n \in \mathbb{S}_{2n+1}$ on vertices of the Walecki tournament $W(e)$, for $e \in E_n$, n odd, and $n \geq 1$. Vertex $v(0)$ is fixed by the permutation η which is an *involution* represented by two-way arrows.

where $\mathcal{O}(t(1))$ denotes the orbit of vertex $t(1)$ for the subgroup $G_{t(0)}$ of G.

We first consider the cardinality of $\mathcal{O}(t(0))$. $T\langle N^+(t(0))\rangle$ is a regular tournament by Theorem 4.8. On the other hand $T\langle N^+(t(i))\rangle$ is *not* regular for $t(i) \in N^+(t(0))$ (see Theorem 4.7). Thus, $t(0)$ cannot be mapped to a vertex from $N^+(t(0))$ by elements of G. Since $T \cong \overline{T}$ with the graph anti-automorphism τ^n, $t(0)$ cannot be mapped to a vertex from $N^-(t(0))$ by elements of G. We have proven that $t(0)$ must be fixed under the action of G, and thus

$$|\mathcal{O}(t(0))| = 1. \tag{4.24}$$

Next we determine $|\mathcal{O}(t(1))|$. Since $t(0)$ is a fixed point for any element ρ in G, $\rho(N^+(t(0))) = N^+(t(0))$. Hence, $\rho(t(1)) \in N^+(t(0))$ and $\mathcal{O}(t(1)) \subseteq N^+(t(0))$, implying

$$|\mathcal{O}(t(1))| \leq |N^+(t(0))| = n. \tag{4.25}$$

We proved that the permutation $\tau^{2r} \in \mathbb{S}_{2n+1}$ is an element of G. Since $\tau^{2r}(t(0)) = t(0)$, $\tau^{2r} \in G_{t(0)}$. Hence, $\langle\tau^{2r}\rangle \subseteq G_{t(0)}$. The orbit of $t(1)$ for $\langle\tau^{2r}\rangle$ is O_1, which implies

$$|\mathcal{O}(t(1))| \geq |O_1| = n/r. \tag{4.26}$$

If $r = 1$ then Equations (4.25) and (4.26) imply $|\mathcal{O}(t(1))| = n$. We now assume $r > 1$. We will prove that $|\mathcal{O}(t(1))| = n/r$. Suppose the contrary, $|\mathcal{O}(t(1))| > n/r$. Hence, there exists a vertex $t(i) \in \mathcal{O}(t(1)) - O_1$. Let us assume that $t(i) \in O_i$ where $1 < i \leq 2r$. Since τ^{2r} is an automorphism, we may assume $t(i) \in O_i \cap M_1$. Moreover, $f = (0, 0, \ldots, 0)$ implies $t(1), t(i) \in N^+(t(0))$, therefore, $1 < i \leq r$.

It follows that there exist $k > 1$ such that $\tau^{k(i-1)} t(i) = t(i+k(i-1)+1)$ where $r < i + k(i-1) + 1 \leq 2r$. This is a contradiction, for $t(i+k(i-1)+1)$ belongs to $N^-(t(0))$ because $e_{i+k(i-1)+1} = \overline{f}_{i+k(i-1)+1-r} = \overline{0} = 1$, however, it should belong to $N^+(t(0))$ because $t(i) \in N^+(t(0))$. Therefore,

$$|\mathcal{O}(t(1))| = n/r. \tag{4.27}$$

Last we prove that $G_{v(0),v(1)} = id$. The subtournaments $T\langle N^+(t_{(0)})\rangle$ are regular for n odd and almost regular for n even. However, the subtournaments $T\langle N^+(t_{(1)})\rangle$ are not regular and not almost regular for n odd and n even, respectively, implying that any automorphism $\rho \in G_{t(0),t(1)}$ fixes all other vertices. Therefore, $G_{v(0),v(1)} = id$, that is,

$$|G_{t(0),t(1)}| = 1. \tag{4.28}$$

Equations (4.23), (4.24), (4.27), and (4.28) imply that $|G| = n/r$. Now, $\langle \tau^{2r} \rangle \subseteq G_{t(0)} \subseteq G$ and since $\langle \tau^{2r} \rangle \cong \mathrm{Z}_{n/r}$ we have

$$G \cong \mathrm{Z}_{n/r}$$

as required. □

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