ISSN 2590-9770
The Art of Discrete and Applied Mathematics 3 (2020) #P1.03
https://doi.org/10.26493/2590-9770.1263.86e
(Also available at http://adam-journal.eu)

Self-dual, self-Petrie-dual and Möbius regular maps on linear fractional groups*

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Received 13 August 2018, accepted 7 August 2019, published online 27 July 2020

Abstract

Regular maps on linear fractional groups $\mathrm{PSL}(2,q)$ and $\mathrm{PGL}(2,q)$ have been studied for many years and the theory is well-developed, including generating sets for the associated groups. This paper studies the properties of self-duality, self-Petrie-duality and Möbius regularity in this context, providing necessary and sufficient conditions for each case. We also address the special case for regular maps of type $(5,5)$. The final section includes an enumeration of the $\mathrm{PSL}(2,q)$ maps for $q \leq 81$ and a list of all the $\mathrm{PSL}(2,q)$ maps which have any of these special properties for $q \leq 49$.

Keywords: Regular map, external symmetry, self-dual, self-Petrie-dual, Möbius regular.

Math. Subj. Class. (2020): 05C25, 05C10

*The authors would like to thank Jozef Širáň for many useful discussions during the preparation of this paper.
†The author acknowledges partial support by Slovak research grants VEGA 1/0026/16, VEGA 1/0142/17, APVV-0136-12, APVV-15-0220 and APVV-17-0428.
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1 Introduction

Regular maps always display inherent symmetry by virtue of their definition. A regular map can have further symmetry properties which are called *external symmetries*. These occur when a map is isomorphic to its image under a particular operation. The best known example of this is the tetrahedron, a Platonic solid which is self-dual.

A *map* is a cellular embedding of a graph on a surface and is made up of vertices, edges and faces. A *flag* is a triple incidence of edge-end, edge-side and face centre. Informally we can visualise each flag as a triangle with its corners at the vertex, the centre of the face and the midpoint of the edge. Thus there are four flags incident to any edge and the whole surface is covered by flags. We consider the symmetries of a map by reference to its flags. An *automorphism* of a map is an arbitrary permutation of its flags such that all adjacency relationships of the flags are preserved. The map is *regular* if the group of automorphisms acts regularly on the flags, that is the group is fixed-point-free and transitive. An implication of this is that each vertex of a regular map has a given valency, say k, and the face lengths are all equal, say to l. Henceforth we will refer to maps of type (k, l) where k is the vertex degree and l is the face length of the regular map.

For further details about the theory of regular maps see [3, 9, 12, 15, 16].

Every regular map has an associated *dual map* which is also a regular map. Informally, the dual map is created by forming a vertex at the centre of each original face and considering each of the original vertices as the centre of a face. Each edge of the dual map is thereby formed by linking a pair of neighbouring vertices across one of the original edges.

A different type of dual, the *Petrie dual* of a map has the same edges and vertices as the original map but the faces are different. That is, the underlying graph is the same, but the embedding is different. The boundary walk of a face of the Petrie dual map can be described informally as follows:

1. Starting from a vertex on the original map, trace along one side of an incident edge until you get to the midpoint of that edge;
2. Cross over to the other side of the edge and continue tracing along the edge in the same direction as before. When you approach a vertex, sweep the corner and continue along the next edge until you reach its midpoint;
3. Repeat step 2 until you rejoin the face boundary walk where you started.

When the associated dual or Petrie dual map is isomorphic to the original map, we call the map *self-dual* or *self-Petrie-dual* respectively. This paper explores necessary and sufficient conditions for a regular map with automorphism group $\mathrm{PSL}(2, q)$ or $\mathrm{PGL}(2, q)$, where q is odd, to have each of these external symmetries.

Another property of interest in the theory of regular maps is *Möbius regularity*. This concept was introduced by S. Wilson in [17] who originally named them *cantankerous*. A regular map is Möbius regular if any two distinct adjacent vertices are joined by exactly two edges and any open set supporting these edges contains a Möbius strip. Clearly such a map must have even vertex degree k and we will establish the further conditions under which a regular map on $\mathrm{PSL}(2, q)$ or $\mathrm{PGL}(2, q)$ is Möbius regular.

In Section 2 we state some of the background material and results which we will need. Section 3 investigates regular maps of type (p, p), (k, p) and (p, l) when k and l are coprime to p. Type (p, p) is self-dual but not self-Petrie-dual nor Möbius regular, and type (p, l) can be self-Petrie-dual but not Möbius regular. Type (k, p) can be self-Petrie-dual or Möbius regular. In Section 4 we address the necessary and sufficient conditions for a map of type (k, l) to be self-dual, self-Petrie-dual and Möbius regular respectively. Section 5 highlights a special case, namely maps of type $(5, 5)$ whose orientation-preserving automorphism groups turn out to be isomorphic to A_5 and Section 6 comments on and lists examples of maps with some or all of these properties.

2 Background information, notes and notation

This paper is founded on work done by M. Conder, P. Potočnik and J. Širáň in [4] which provides a detailed analysis of reflexible regular hypermaps for triples (k, l, m) on projective two-dimensional linear groups including explicit generating sets for the associated groups. In particular this paper is concerned only with maps, not hypermaps, and so, without loss of generality, we let $m = 2$.

The group of automorphisms of a regular map is generated by three involutions, two of which commute, where the three involutions can be thought of as local reflections in the boundary lines of a given flag which preserve all the adjacency relationships between flags. As shown in Figure 1 the involutions act locally on the given flag as follows: X as a reflection in the edge bisector; Y as a reflection across the edge; Z as a reflection in the angle bisector at the vertex. The dots on the diagram indicate where there may be further vertices, edges and faces while the dashed lines outline each of the flags of this part of the map.

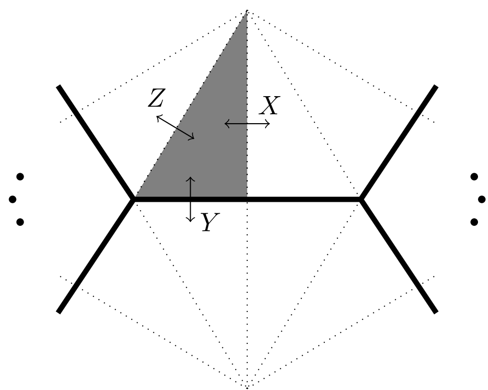

Figure 1: The action of automorphisms X, Y and Z on the shaded flag.

The study of regular maps is equivalent to the study of group presentations of the form

$$G \cong \langle X, Y, Z \mid X^2, Y^2, Z^2, (YZ)^k, (ZX)^l, (XY)^2, \ldots \rangle,$$

see [16]. The dots indicate the potential for further relations not listed, and we assume the orders shown are indeed the true orders of those elements in the group.

The surface on which a regular map is embedded could be orientable or non-orientable. If the regular map is on an orientable surface then G has a subgroup of index two which corresponds to the orientation preserving automorphisms. Instead of the group generated by these three involutions X, Y and Z, we can consider the group of orientation-preserving automorphisms which is generated by the two rotations $R = YZ$ and $S = ZX$. On a non-orientable surface these two elements will still generate the full automorphism group and we can say that studying these maps is equivalent to studying groups which have presentaions of the form $\langle R, S \mid R^k, S^l, (RS)^2, \ldots \rangle$.

We focus on regular maps of type (k, l) where the associated group $G \cong \langle X, Y, Z \rangle$ is isomorphic to $\mathrm{PSL}(2, q)$ or $\mathrm{PGL}(2, q)$ where q is a power of a given odd prime p. By [4], both k and l are either equal to p or divide $q - 1$ or $q + 1$. Following the convention and notation of [4], in the latter two cases we let ξ_κ and/or ξ_λ be primitive $2k$th or $2l$th roots of unity respectively. Note that in the case where k or l divides $q - 1$ then the corresponding primitive root is in the field $\mathrm{GF}(q)$; otherwise it is in the unique quadratic extension $\mathrm{GF}(q^2)$. We also define $\omega_i = \xi_i + \xi_i^{-1}$ for $i \in \{\kappa, \lambda\}$. Note that ω_i is thus in the field $\mathrm{GF}(q)$. We too assume that (k, l) is a *hyperbolic* pair, that is $1/k + 1/l < 1/2$. This implies that $k \geq 3$ and $l \geq 3$. The conditions in this paragraph are what we refer to as *the usual setup*.

We can consider duality and Petrie duality as operators on a map. Since the dual of a map is obtained by swapping the vertices for faces and vice versa, in terms of the involutions X, Y, Z the dual operator would fix Z and interchange X and Y. The Petrie dual operator would replace X with XY and fix Y and Z. The automorphism associated with any type of duality is an involution. This is because it acts on our map to produce the dual map, and when this automorphism is repeated we get back to the original map. Self-duality and self-Petrie-duality are therefore equivalent to the existence of precisely such involutory automorphisms of G, the group associated with the regular map.

Our paper is devoted in large part to finding conditions for the existence of involutory automorphisms which imply self-duality and/or self-Petrie-duality. The automorphism group for G is $\mathrm{P\Gamma L}(2, q)$, the semidirect product $\mathrm{PGL}(2, q) \rtimes C_e$ where $q = p^e$, [13]. Observe that, when $e = 1$, this group is essentially $\mathrm{PGL}(2, p)$ and so, in the case where $G \cong \mathrm{PGL}(2, p)$, all automorphisms are inner automorphisms. Elements $(A, j) \in \mathrm{P\Gamma L}(2, q)$ act as follows: $(A, j)(T) = A\phi_j(T)A^{-1}$ where ϕ_j is the repeated Frobenius field automorphism of the finite field, $\phi_j \colon x \to x^r$ with $r = p^j$. The function ϕ_j acts element-wise on a matrix and we use the general rule for composition in $\mathrm{P\Gamma L}(2, q)$ which is $(B, j)(A, i) = (B\phi_j(A), i + j)$.

When $(A, j) \in \mathrm{P\Gamma L}(2, p^e)$ is an involution, it must be such that $(A, j)(A, j) = (A\phi_j(A), 2j)$ is the identity, so $2j \equiv 0 \pmod{e}$. One case is when there is no field automorphism involved, that is $j = 0$ and $A^2 = I$. Alternatively $e = 2j$ is even, and then we need $\phi_j(A) = A^{-1}$. This is summarised in Lemma 2.1.

Lemma 2.1. $(A, j) \in \mathrm{P\Gamma L}(2, p^e)$ *is an involution if and only if one of the following conditions holds:*

1. $j = 0$ *and* $A^2 = I$
2. $2j = e$ *and* $\phi_j(A) = A^{-1}$.

Explicit generating sets are known for regular maps with automorphism group G isomorphic to $\mathrm{PSL}(2, q)$ or $\mathrm{PGL}(2, q)$, and for details we refer the interested reader to [4]. We present the results for maps of each type as required.

We will need to consider performing operations on the elements X, Y and Z of G. As such we denote elements of the group $\mathrm{PSL}(2, q)$ or $\mathrm{PGL}(2, q)$, by a representative matrix with square brackets. This allows us to perform the necessary calculations. We can then determine whether or not two resulting matrices are equivalent within G, that is whether or not they correspond to the same element of the group G. A pair of matrices are in the same equivalence class, that is they represent the same *element* of G, if one is a scalar multiple of the other. We use curved brackets for matrix representatives for X, Y and Z.

Lemma 2.1 can then be used to find conditions for the elements of the matrix part A of an involutory automorphism (A, j) as follows.

Lemma 2.2. *Let $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$. The automorphism denoted $(A, j) \in \mathrm{P\Gamma L}(2, p^e)$ is an involution, if and only if a, b, c, d satisfy the following equations, with $r = p^j$ for $j = 0$ or $2j = e$.*

1. $a^{r+1} = d^{r+1}$,
2. $bc^r = cb^r$,
3. $ab^r + bd^r = 0$,
4. $ca^r + dc^r = 0$.

Proof. By Lemma 2.1, and letting $r = p^j$ we have

$$A\phi_j(A) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a^r & b^r \\ c^r & d^r \end{bmatrix} = \begin{bmatrix} a^{r+1} + bc^r & ab^r + bd^r \\ ca^r + dc^r & cb^r + d^{r+1} \end{bmatrix} = I.$$

By comparing the leading diagonal entries we see that $a^{r+1} + bc^r = cb^r + d^{r+1}$. Applying the field automorphism ϕ_j yields $a^{1+r} + b^r c = c^r b + d^{1+r}$. Subtracting these two equations, and remembering that q is odd, we get the first two equations, while looking at the off-diagonal immediately gives rise to the final two equations. □

When we are establishing the conditions under which a regular map is Möbius regular we will rely on the following group-theoretic result, proved in [11] by Li and Širáň. Note that implicit in this necessary and sufficient condition is that for a map of type (k, l) to be Möbius regular k must be even.

Lemma 2.3. *A regular map is Möbius regular if and only if $XR^{\frac{k}{2}}X = R^{\frac{k}{2}}Y$ where $R = YZ$.*

3 Regular maps on linear fractional groups of type (p, p), (k, p) and (p, l) where p is an odd prime

For odd prime p, by Proposition 3.1 in [4], maps of the type (p, p) have the following representatives for X, Y and Z, where $\alpha^2 = -1$:

$$X_1 = -\alpha \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}, \quad Y_1 = -\alpha \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad Z_1 = \alpha \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Proposition 3.1. *With the usual setup, a map of type (p, p) is self-dual.*

Proof. For self duality we need $G \cong \langle X, Y, Z \rangle$ to admit an automorphism such that X and Y are interchanged, and Z is fixed. So the question is: can we find an automorphism $(A, j) \in \mathrm{P\Gamma L}(2, q)$ such that $A\phi_j(X)A^{-1} = Y$, $A\phi_j(Y)A^{-1} = X$, and $A\phi_j(Z)A^{-1} = Z$. It is easy to verify that $(A, 0)$, where A has the form $A = \left[\begin{smallmatrix} 0 & 1 \\ 2 & 0 \end{smallmatrix}\right]$ satisfies these conditions, so this type of map is self-dual. □

Proposition 3.2. *With the usual setup, a map of type (p, p) is not self-Petrie-dual.*

Proof. In order to be self-Petrie-dual, the group G needs to admit an involutory automorphism (B, j) which fixes Z and Y, and exchanges X with XY.

First notice that $\phi_j(Z) = Z$ and $\phi_j(Y) = Y$ so if B exists, it must be of a form which commutes with both Z and Y. To commute with Z, the necessarily non-identity element B must be either $B_1 = \left[\begin{smallmatrix} 0 & b \\ c & 0 \end{smallmatrix}\right]$ or $B_2 = \left[\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right]$. Note that $0 \notin \{a, b, c, d\}$ and $a \neq d$. As shown below, neither of these commute with Y.

$$B_1 Y = -\alpha \begin{bmatrix} 0 & -b \\ c & -c \end{bmatrix} \neq Y B_1 = -\alpha \begin{bmatrix} -c & b \\ -c & 0 \end{bmatrix}$$

$$B_2 Y = -\alpha \begin{bmatrix} a & -a \\ 0 & -d \end{bmatrix} \neq Y B_2 = -\alpha \begin{bmatrix} a & -d \\ 0 & -d \end{bmatrix}$$

Hence this type of map is not self-Petrie-dual. □

Remark 3.3. Maps of type (k, p) and (p, l) where k and l are coprime to p clearly cannot be self-dual since the vertex degree and face lengths differ.

Proposition 3.4. *With the usual setup, and for k coprime to p, a map of type (k, p) is self-Petrie-dual if and only if $k \mid 2(r \pm 1)$ and $\pm\omega_\kappa^{(r+1)} = 4\xi_\kappa^{(r\pm1)}$ when the corresponding signs in each $(r \pm 1)$ are read simultaneously, and where $r = p^j$ and $j = 0$ or $2j = e$.*

Proof. When k is coprime to p, [8] tells us that a map of type (k, p) has the following triple of generating matrices corresponding to X, Y, and Z:

$$X_2 = \eta\alpha \begin{pmatrix} -\omega_\kappa & -2\xi_\kappa \\ 2\xi_\kappa^{-1} & \omega_\kappa \end{pmatrix}, \quad Y_2 = -\alpha \begin{pmatrix} 0 & \xi_\kappa \\ \xi_\kappa^{-1} & 0 \end{pmatrix}, \quad Z_2 = \alpha \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where $\alpha^2 = -1$ and $\eta = (\xi_\kappa - \xi_\kappa^{-1})^{-1}$.

Suppose the map is self-Petrie-dual and $(B, j) = (\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right], j)$ is the associated involutory automorphism. In order to fix Z we must have $a = d$ and $b = c$ or $a = -d$ and $b = -c$. In order to fix Y we find that either $a = 0$ or $c = 0$ in which case we need $k \mid 2(p^j + 1)$ or $k \mid 2(p^j - 1)$ respectively. When $a = 0$, the involution then interchanges X with XY if and only if $\pm\omega_\kappa^{(r+1)} = 4\xi_\kappa^{(r+1)}$. When $c = 0$, the involution then interchanges X with XY if and only if $\pm\omega_\kappa^{(r+1)} = 4\xi_\kappa^{(r-1)}$. □

Proposition 3.5. *Under the usual setup, and with l coprime to p, a regular map of type (p, l) is self-Petrie-dual if and only if $\omega_\lambda^2 = -\omega_\lambda^{2r}$ where $r = p^j$ and $2j = e$.*

Proof. Using a similar argument to the above applied to the appropriate matrix triple from [8], namely

$$X_3 = \alpha \begin{pmatrix} 0 & \omega_\lambda^{-1} \\ \omega_\lambda & 0 \end{pmatrix}, \quad Y_3 = -\alpha \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Z_3 = \alpha \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix},$$

we find that the only allowable form for B is the identity. The necessary non-trivial field automorphism applied to the X and XY interchange then yields the stated condition. □

Remark 3.6. For odd p, a regular map of type (p, p) or (p, l) is not Möbius regular. This is immediate from the fact that each pair of adjacent vertices in a Möbius regular map is joined by exactly two edges, hence the vertex degree must be even.

Proposition 3.7. *Under the usual setup for even k, a map of type (k,p) is Möbius regular if and only if $\omega_\kappa^2+4=0$*

Proof. A regular map is Möbius regular if and only if the equation $XR^{\frac{k}{2}}X = R^{\frac{k}{2}}Y$ is satisfied. We assume k is even, since if k is odd then the map is certainly not Möbius regular. In this case

$$R=[Y_2Z_2]=\begin{bmatrix}\xi_\kappa & 0\\ 0 & \xi_\kappa^{-1}\end{bmatrix}$$

so we have $R^{\frac{k}{2}}=\left[\begin{smallmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{smallmatrix}\right]$, where $\alpha^2=-1$. Hence the map is Möbius regular if and only if these matrices are equivalent:

$$XR^{\frac{k}{2}}X=-\eta^2\alpha\begin{bmatrix}\omega_\kappa^2+4 & 4\omega_\kappa\xi_\kappa\\ -4\omega_\kappa\xi_\kappa^{-1} & -(\omega_\kappa^2+4)\end{bmatrix}\quad\text{and}\quad R^{\frac{k}{2}}Y=\begin{bmatrix}0 & \xi_\kappa\\ -\xi_\kappa^{-1} & 0\end{bmatrix}.$$

These matrices are equivalent if and only if $\omega_\kappa^2+4=0$. □

4 Regular maps on linear fractional groups of type (k,l) where both k and l are coprime to p

In this case we have different generating triples for the group G. As per Proposition 3.2 in [4], the triple (X,Y,Z) has representatives as defined below where $D=\omega_\kappa^2+\omega_\lambda^2-4$, $\beta=-1/\sqrt{-D}$ and $\eta=(\xi_\kappa-\xi_\kappa^{-1})^{-1}$.

$$X_4=\eta\beta\begin{pmatrix}D & D\omega_\lambda\xi_\kappa\\ -\omega_\lambda\xi_\kappa^{-1} & -D\end{pmatrix},\ Y_4=\beta\begin{pmatrix}0 & \xi_\kappa D\\ \xi_\kappa{}^{-1} & 0\end{pmatrix},\ \text{and } Z_4=\beta\begin{pmatrix}0 & D\\ 1 & 0\end{pmatrix}$$

We will also consider the pair of matrices which represent R and S, the rotations around a vertex and a face respectively, which by Proposition 2.2 in [4] are:

$$R_4=\begin{pmatrix}\xi_\kappa & 0\\ 0 & \xi_\kappa^{-1}\end{pmatrix}\text{ and } S_4=\eta\begin{pmatrix}-\omega_\lambda\xi_\kappa^{-1} & -D\\ 1 & \omega_\lambda\xi_\kappa\end{pmatrix}.$$

At this point we note that there is an exception for maps of type $(5,5)$, which is addressed in Section 5.

Theorem 4.1. *Under the usual setup, a regular map of type (k,k) is self-dual if and only if $\omega_\lambda=\pm\omega_\kappa^r$ where $r=p^j$, and $j=0$ or $2j=e$.*

Proof. Suppose the map is self-dual.

There is an involutory automorphism of G which fixes Z and interchanges X and Y. This is equivalent to interchanging the rotations $R^{-1}=(YZ)^{-1}=ZY$ and $S=ZX$ around a vertex and a face respectively. That is, there is an automorphism (A,j) which interchanges $\pm R^{-1}$ with S. Here the $\pm$ takes into account both representative elements for R. So $A(\pm\phi_j(R^{-1}))A^{-1}=S$. Remembering that conjugation preserves traces this implies $\pm\phi_j\mathrm{tr}(R^{-1})=\mathrm{tr}(S)$ which immediately yields the condition $\pm\omega_\kappa^r=\omega_\lambda$.

Conversely suppose $\pm\omega_\kappa^r=\omega_\lambda$.

We note that $\omega_\kappa^{2r}=\omega_\lambda^2\iff\omega_\kappa^2=\omega_\lambda^{2r}$ and so $D^r=(\omega_\kappa^2+\omega_\lambda^2-4)^r=\omega_\kappa^{2r}+\omega_\lambda^{2r}-4=\omega_\lambda^2+\omega_\kappa^2-4=D$.

We aim to find an involutory automorphism (A,j) which demonstrates this map is self-dual. Consider $A=\begin{bmatrix}a & D\\ -1 & -a\end{bmatrix}$ which, by Lemma 2.2, so long as $a^r=a$, satisfies all the equations necessary for the element (A,j) to be involutory. Notice that (A,j) also fixes Z. We also need X and Y to be interchanged by the automorphism in which case the following matrices are equivalent.

$$A\phi_j(X) = \eta^r\beta^r \begin{bmatrix} D(a-\omega_\lambda^r\xi_\kappa^{-r}) & D(a\omega_\lambda^r\xi_\kappa^r - D) \\ \omega_\lambda^r\xi_\kappa^{-r}a - D & D(a-\omega_\lambda^r\xi_\kappa^r) \end{bmatrix} \text{ and } YA = \beta \begin{bmatrix} -\xi_\kappa D & -a\xi_\kappa D \\ a\xi_\kappa^{-1} & D\xi_\kappa^{-1} \end{bmatrix}$$

Ratio of elements in the leading diagonal: $-\xi_\kappa^2 = (a-\omega_\lambda^r\xi_\kappa^{-r})/(a-\omega_\lambda^r\xi_\kappa^r)$

Ratio of elements in the left column: $-D\xi_\kappa^2/a = D(a-\omega_\lambda^r\xi_\kappa^{-r})/(\omega_\lambda^r\xi_\kappa^{-r}a - D)$

Ratio of elements in the top row: $1/a = (a-\omega_\lambda^r\xi_\kappa^{-r})/(a\omega_\lambda^r\xi_\kappa^r - D)$

The last ratio listed yields the following quadratic in a: $0 = a^2 - a\omega_\lambda^r(\xi_\kappa^{-r}+\xi_\kappa^r) + D$, that is $0 = a^2 - a\omega_\lambda^r\omega_\kappa^r + D$. This is consistent with all the necessary ratios. All that remains is to check that a value of a satisfying this quadratic is invariant under the repeated Frobenius field automorphism. The discriminant $\Delta = \omega_\kappa^2\omega_\kappa^{2r} - 4(\omega_\kappa^2 + (\omega_\kappa^r)^2 - 4) = ((\omega_\kappa^r)^2-4)(\omega_\kappa^2-4)$. Furthermore the expression for $a = (\omega_\lambda^r\omega_\kappa^r \pm \sqrt{\Delta})/2$ is invariant under the transformation $x \to x^r$ as required. Hence the map is self-dual. □

Theorem 4.2. *With the usual setup, where k, l are coprime to p, a map of type (k, l) is self-Petrie-dual if and only if one of the following conditions is fulfilled:*

1. $\omega_\lambda^2 = -D$

2. $q = r^2 = p^{2j}$, $\omega_\lambda^{2r} = -D$ and $k|(r \pm 1)$.

Proof. First suppose the map is self-Petrie-dual. So there exists $(B, j) \in P\Gamma L(2, q)$ such that $B\phi_j(X)B^{-1} = XY$, $B\phi_j(Y)B^{-1} = Y$, and $B\phi_j(Z)B^{-1} = Z$. By comparing the traces of $\phi_j(ZX)$ and ZXY we get the necessary conditon: $\omega_\lambda^{2r} = -D$.

For the rest of the proof we split the situation into two cases: the first when $j = 0$ and we do not consider any field automorphism, and the second case where a field automorphism is included.

Case 1: $j = 0$.

Suppose $\omega_\lambda^2 = -D$. Notice that $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ fixes both Y and Z. The map is self-Petrie-dual if $BX = \eta\beta \begin{bmatrix} D & D\omega_\lambda\xi_\kappa \\ \omega_\lambda\xi_\kappa^{-1} & D \end{bmatrix}$ and $XYB = \eta\beta^2 \begin{bmatrix} D\omega_\lambda & -D^2\xi_\kappa \\ -D\xi_\kappa^{-1} & D\omega_\lambda \end{bmatrix}$ are also equivalent. Comparing these and applying our assumption that $\omega_\lambda^2 = -D$ we conclude this map is self-Petrie-dual.

Case 2: $2j = e$. By Lemma 2.1 we include the repeated Frobenius automorphism.

Suppose $\omega_\lambda^{2r} = -D$ and $k|(r \pm 1)$.

The map is self-Petrie-dual if there is an involutory automorphism which not only fixes Z but also fixes Y and interchanges XY with X. We hope to find $(B, j) = (\begin{bmatrix} a & b \\ c & d \end{bmatrix}, j)$, the associated element of $P\Gamma L$. In addition to the conditions for a, b, c, d established in Lemma 2.2, we require $B\phi_j(Z)B^{-1} = Z$ and $B\phi_j(Y)B^{-1} = Y$. In order to fix Z we must have $bd = acD^r$ and $D(d^2 - c^2D^r) = a^2D^r - b^2$. Fixing Y yields two further equations: $bd = ac\xi_\kappa^{2r}D^r$ and $\xi_\kappa^2 D(d^2\xi_\kappa^{-r} - c^2\xi_\kappa^r D^r) = a^2\xi_\kappa^r D^r - b^2\xi_\kappa^{-r}$. Since $\xi_\kappa^{2r} \neq 1$ and $D \neq 0$, notice that $bd = ac\xi_\kappa^{2r}D^r = acD^r$ tells us that either $a = 0$ or $c = 0$.

If $a = 0$, we immediately see $d = 0$ too, and so we can assume $b = 1$ without loss of generality. The equations for a, b, c, d tell us that to fix Z we have $c^2 = \frac{1}{D^{r+1}}$ and to fix Y we have $c^2\xi^{2r+2} = \frac{1}{D^{r+1}}$. So this automorphism exists only if $\xi_\kappa^{2r+2} = 1$. By definition ξ_κ is a primitive $2k$th root of unity and $\xi_\kappa^{2r+2} = 1 \iff 2k|(2r+2) \iff k|(r+1)$, which is the case by our assumption.

$$B\phi_j(XY) = \eta^r\beta^{2r}\begin{bmatrix}-D^r\xi^{-r} & \mp D^r\sqrt{-D}\\ \pm cD^r\sqrt{-D} & cD^{2r}\xi_\kappa^r\end{bmatrix} \text{ and } XB = \eta\beta\begin{bmatrix}cD\omega_\lambda\xi_\kappa & D\\ -cD & -\omega_\lambda\xi_\kappa^{-1}\end{bmatrix}$$

are also equivalent if the map is self-Petrie-dual so we compare the ratios of the elements in turn. This yields $\pm c = \frac{1}{\omega_\lambda\xi_\kappa^{r+1}\sqrt{-D}} = \frac{\omega_\lambda\sqrt{-D}}{D^{r+1}\xi_\kappa^{r+1}}$ which is true only if $D^r = -\omega_\lambda^2$, which is again the case by our assumption. These conditions are consistent with our other requirements for the value of c, (namely that $c^r = c$) so we have an automorphism demonstrating that this map is self-Petrie-dual.

If on the other hand $c = 0$ then we have $b = 0$ and we assume $a = 1$ without loss of generality. Fixing Z yields $d^2D = D^r$. Fixing Y yields $d^2D = \xi_\kappa^{2r-2}D^r$. So the map is self-Petrie-dual only if $\xi_\kappa^{2r-2} = 1$, which is the case by our assumption.

Now $B\phi_j(XY) = \eta^r\beta^{2r}D^r\begin{bmatrix}\omega_\lambda^r & D^r\xi_\kappa^r\\ -d\xi_\kappa^{-r} & -d\omega_\lambda^r\end{bmatrix}$ and $XB = \eta\beta\begin{bmatrix}D & dD\omega_\lambda\xi_\kappa\\ -\omega_\lambda\xi_\kappa^{-1} & -dD\end{bmatrix}$.

Again, the map is self-Petrie-dual if these two elements are equivalent, that is if both $D^r\xi_\kappa^{r-1} = d\omega_\lambda^{r+1}$ and $\omega_\lambda^{r+1}\xi_\kappa^{r-1} = dD$. Applying our assumption $\omega_\kappa^{2r} = -D$, we conclude the map is self-Petrie-dual. □

Using the fact that when $q = p$ the conditions are often much simpler to state, the preceding two results, Theorem 4.1 and Theorem 4.2, indicate a *sufficient* condition for a regular map of type (k, k) to be both self-dual and self-Petrie-dual, namely $\omega_\kappa^2 = \omega_\lambda^2 = -D$. Corollary 4.3 shows this becomes a tractable sufficient condition for both self-duality and self-Petrie-duality.

Corollary 4.3. *If $\omega = \omega_\kappa = \omega_\lambda$ and $3\omega^2 = 4$ then the associated map is both self-dual and self-Petrie-dual.*

We now turn our attention to the conditions for Möbius regularity.

Proposition 4.4. *With the usual setup, a regular map of type (k, l) is Möbius regular if and only if k is even and $\omega_\kappa^2 + 2\omega_\lambda^2 = 4$.*

Proof. By Lemma 2.3, a regular map is Möbius regular if and only if the equation $XR^{\frac{k}{2}}X = R^{\frac{k}{2}}Y$ is satisfied. In this case $R = \begin{bmatrix}\xi_\kappa & 0\\ 0 & \xi_\kappa^{-1}\end{bmatrix}$. So $R^{\frac{k}{2}} = \begin{bmatrix}\alpha & 0\\ 0 & -\alpha\end{bmatrix}$, where $\alpha^2 = -1$. Then $XR^{\frac{k}{2}}X = R^{\frac{k}{2}}Y$ is satisfied if and only if

$$\eta^2\beta^2\begin{bmatrix}\alpha D^2 + \alpha D\omega_\lambda^2 & 2\alpha D^2\omega_\lambda\xi_\kappa\\ -2\alpha D\omega_\lambda\xi_\kappa^{-1} & -\alpha D\omega_\lambda^2 - \alpha D^2\end{bmatrix} = \beta\begin{bmatrix}0 & \alpha\xi_\kappa D\\ -\alpha\xi_\kappa^{-1} & 0\end{bmatrix}.$$

The elements on the leading diagonal must be zero, which yields just one equation: $\eta^2\beta^2\alpha D(D + \omega_\lambda^2) = 0$. The ratio between the non-zero entries is the same for both matrices and so no further conditions arise.

We conclude that for even k, the map is Möbius regular if and only if $D = -\omega_\lambda^2$, which is equivalent to $\omega_\kappa^2 + 2\omega_\lambda^2 = 4$. □

It is not surprising to see some similarity between conditions for self-Petrie-duality and Möbius regularity since we know all Möbius regular maps (with any automorphism group) are also self-Petrie-dual [17]. However, since there are alternative conditions which imply self-Petrie-duality, the converse is not true – not all self-Petrie-dual regular maps are Möbius regular.

5 Regular maps of type $(5, 5)$ whose orientation-preserving automorphism group $\langle R, S\rangle$ is isomorphic to A_5

Adrianov's [1] enumeration of regular hypermaps on $\mathrm{PSL}(2, q)$ includes a constant which deals with the special case which occurs for maps of type $(5, 5)$.

For us to be considering a map of type (k, l) we must have $2k|(q \pm 1)$ and $2l|(q \pm 1)$, and it is known, see [10], that $\mathrm{PSL}(2, q)$ has subgroup A_5 when $q \equiv \pm 1 \pmod{10}$. The constant in Adrianov's enumeration, which is 2 for maps of type $(5, 5)$ and zero otherwise, is subtracted to account for the cases when the group $\langle R, S\rangle$ collapses into the subgroup $A_5 \leq \mathrm{PSL}(2, q)$.

The following result, with the usual definitions for ω_κ and ω_λ, indicates when the orientation-preserving automorphism group of a type $(5, 5)$ map is not the linear fractional group that we might expect, and as such addresses an omission in [4].

Proposition 5.1. *The group $\langle R, S\rangle$ of a regular map of type $(5, 5)$, generated by the representative matrices R_4 and S_4, is isomorphic to A_5 if and only if $\omega_\lambda \neq \omega_\kappa$.*

Proof. From [14] we know a presentation of the group A_5 is: $\langle a, b | a^5, b^5, (ab)^2, (a^4 b)^3\rangle$.

Considering the group $\langle R, S | R^5, S^5, (RS)^2, \dots\rangle$, it is clear that this will be isomorphic to A_5 if and only if the condition $(R^4 S)^3 = I$ is also satified. This is the case if and only if $R^{-1}S$ has order 3.

$$R^{-1}S = \eta \begin{bmatrix} \xi_\kappa^{-1} & 0 \\ 0 & \xi_\kappa \end{bmatrix} \begin{bmatrix} -\omega_\lambda \xi_\kappa^{-1} & -D \\ 1 & \omega_\lambda \xi_\kappa \end{bmatrix} = \eta \begin{bmatrix} -\omega_\lambda \xi_\kappa^{-2} & -\xi_\kappa^{-1} D \\ \xi_\kappa & \omega_\lambda \xi_\kappa^2 \end{bmatrix}$$

$$(R^{-1}S)^3 = \eta^3 \begin{bmatrix} \omega_\lambda(2D\xi_\kappa^{-2} - \omega_\lambda^2 \xi_\kappa^{-6} - D\xi_\kappa^2) & D\xi_\kappa^{-2}(D\xi_\kappa + \omega_\lambda^2(\xi_\kappa - \xi_\kappa^5 - \xi_\kappa^{-3})) \\ -(D\xi_\kappa + \omega_\lambda^2(\xi_\kappa - \xi_\kappa^5 - \xi_\kappa^{-3})) & \omega_\lambda(D\xi_\kappa^{-2} + \omega_\lambda^2 \xi_\kappa^6 - 2D\xi_\kappa^2) \end{bmatrix}$$

The off diagonal elements are both zero if and only if $D\xi_\kappa + \omega_\lambda^2(\xi_\kappa - \xi_\kappa^5 - \xi_\kappa^{-3}) = 0$. This condition is equivalent to $(\omega_\kappa + 2)(\omega_\kappa - 2)(1 + \omega_\kappa\omega_\lambda)(1 - \omega_\kappa\omega_\lambda) = 0$ and we know that $\omega_\kappa \neq \pm 2$ so long as $k \neq p$.

The leading diagonal entries are equal if and only if $D(\xi_\kappa^{-2} + \xi_\kappa^2) = \omega_\lambda^2(\xi_\kappa^6 + \xi_\kappa^{-6})$. Applying $\xi_\kappa^{10} = 1$ and eliminating D shows this is equivalent to

$$(\omega_\kappa^2 - 4)(\omega_\kappa^2 - \omega_\kappa^2\omega_\lambda^2 + \omega_\lambda^2 - 2) = 0.$$

Assume $\omega_\kappa^2\omega_\lambda^2 = 1$. The off-diagonals are clearly zero, and the leading diagonal entries are equal since $(\omega_\kappa^2 - \omega_\kappa^2\omega_\lambda^2 + \omega_\lambda^2 - 2) = \omega_\kappa^{-2}(\omega_\kappa^4 - 3\omega_\kappa^2 + 1) = 0$ is always the case since the expression inside the bracket is the sum of powers of ξ_κ^2, a 5th root of unity. Then $R^{-1}S$ has order 3.

Conversely, assume $(R^{-1}S)^3 = I$. Then we instantly have $\omega_\kappa^2\omega_\lambda^2 = 1$ since $p \neq 5$.

We conclude that $(R^{-1}S)^3 = I$ if and only if $\omega_\kappa\omega_\lambda = \pm 1$. By considering the two possible values for ω_κ and ω_λ we see that this will happen if and only if $\omega_\kappa \neq \omega_\lambda$. □

6 Tables of results and comments

In the following tables, produced using the computer package GAP [7], we list for given $q \leq 49$, all the $\mathrm{PSL}(2, q)$ maps which have one or more of the properties we have addressed in the paper, the ticks indicating when the map has each property. The tables are ordered by the characteristic of the field, and the elements ξ_κ, ξ_λ, ω_κ and ω_λ are expressed as powers of a primitive element ξ in the field $\mathrm{GF}(q^2)$. For a given k, l, only one map is shown in each equivalence class under the action of the automorphism group. Extended tables of results detailing the $\mathrm{PSL}(2,q)$ regular maps for $q \leq 81$ are available in the ancillary file to [5].

For interest we also include an enumeration in Table 2 which shows how many $\mathrm{PSL}(2,q)$ maps there are with each of these combinations of properties for $q \leq 81$.

Table 1: All the $\mathrm{PSL}(2,q)$ maps which have one or more of the properties we have addressed in the paper.

q	k	l	$\log_\xi \xi_\kappa$	$\log_\xi \xi_\lambda$	$\log_\xi \omega_\kappa$	$\log_\xi \omega_\lambda$	SD	SPD	MR
3^2	5	5	8	8	30	30	✓		
3^3	7	7	52	52	420	420	✓		
3^3	13	7	140	156	28	532		✓	
3^3	13	13	28	28	560	560	✓		
3^3	13	13	28	252	560	672		✓	
3^3	13	13	140	140	28	28	✓		
3^3	14	14	26	26	476	476	✓		
5	5	5					✓		
5^2	3	13	104	72	0	494		✓	
5^2	4	13	78	168	390	26		✓	
5^2	6	13	52	24	546	260		✓	
5^2	12	12	26	26	416	416	✓		
5^2	12	13	26	216	416	130		✓	✓
5^2	13	13	24	24	260	260	✓		
5^2	13	13	24	120	260	52	✓		
5^2	13	13	72	72	494	494	✓		
5^2	13	13	72	264	494	598	✓		
5^2	13	13	168	168	26	26	✓		
5^2	13	13	168	216	26	130	✓		
7	7	7					✓		
7^2	4	25	300	144	400	1250		✓	
7^2	5	5	240	240	1350	1350	✓		
7^2	6	24	200	250	1400	2050		✓	
7^2	7	24		50		500		✓	
7^2	7	25		432		1500		✓	
7^2	7	25		816		100		✓	
7^2	8	25	150	48	2200	1450		✓	
7^2	8	25	450	528	600	950		✓	

q	k	l	$\log_\xi \xi_\kappa$	$\log_\xi \xi_\lambda$	$\log_\xi \omega_\kappa$	$\log_\xi \omega_\lambda$	SD	SPD	MR
7^2	12	12	100	100	750	750	✓		
7^2	12	25	100	144	750	1250		✓	✓
7^2	24	12	50	100	500	750		✓	✓
7^2	24	24	50	50	500	500	✓		
7^2	24	24	50	350	500	1100	✓		
7^2	24	24	250	250	2050	2050	✓		
7^2	24	24	250	550	2050	1150	✓		
7^2	24	25	250	912	2050	700		✓	✓
7^2	25	25	48	48	1450	1450	✓		
7^2	25	25	48	336	1450	550	✓		
7^2	25	25	144	144	1250	1250	✓		
7^2	25	25	144	1008	1250	1550	✓		
7^2	25	25	432	432	1500	1500	✓		
7^2	25	25	432	624	1500	900	✓		
7^2	25	25	528	528	950	950	✓		
7^2	25	25	528	1104	950	1850	✓		
7^2	25	25	816	816	100	100	✓		
7^2	25	25	816	912	100	700	✓		
11	5	5	12	12	36	36	✓		
11	5	5	36	36	24	24	✓	✓	
11	5	6	12	10	36	108		✓	
11	6	6	10	10	108	108	✓		
11	11	11					✓		
13	6	6	14	14	112	112	✓		
13	7	7	12	12	126	126	✓		
13	7	7	12	60	126	56		✓	
13	7	7	36	36	70	70	✓	✓	
13	7	7	60	60	56	56	✓		
13	7	13	60		56			✓	
13	13	13					✓		
17	8	8	18	18	36	36	✓		
17	8	8	54	54	90	90	✓		
17	8	9	54	80	90	54		✓	✓
17	8	17	18		36			✓	✓
17	9	9	16	16	216	216	✓		
17	9	9	80	80	54	54	✓		

q	k	l	$\log_\xi \xi_\kappa$	$\log_\xi \xi_\lambda$	$\log_\xi \omega_\kappa$	$\log_\xi \omega_\lambda$	SD	SPD	MR
17	9	9	112	112	18	18	✓		
17	17	17					✓		
19	3	9	60	20	0	300		✓	
19	5	5	36	36	320	320	✓		
19	5	5	108	108	220	220	✓		
19	9	5	20	36	300	320		✓	
19	9	9	20	20	300	300	✓		
19	9	9	100	100	80	80	✓		
19	9	9	140	100	340	80		✓	
19	9	9	140	140	340	340	✓		
19	9	10	100	18	80	60		✓	
19	10	10	18	18	60	60	✓		
19	10	10	54	54	100	100	✓		
19	19	19					✓		
23	3	11	88	72	0	168		✓	
23	6	6	44	44	192	192	✓		
23	6	11	44	216	192	240		✓	✓
23	11	11	24	24	360	360	✓		
23	11	11	72	72	168	168	✓		
23	11	11	120	120	480	480	✓		
23	11	11	168	168	336	336	✓		
23	11	11	216	216	240	240	✓		
23	12	11	22	24	144	360		✓	✓
23	12	12	22	22	144	144	✓		
23	12	12	110	110	384	384	✓	✓	✓
23	23	23					✓		
29	3	15	140	28	0	480		✓	
29	5	5	84	84	180	180	✓		
29	5	5	252	252	240	240	✓		
29	5	14	84	90	180	270		✓	
29	5	29	252		240			✓	
29	7	7	60	60	570	570	✓		
29	7	7	180	180	750	750	✓		
29	7	7	300	300	780	780	✓		
29	14	14	30	30	630	630	✓		
29	14	14	90	90	270	270	✓		
29	14	14	150	150	540	540	✓		

q	k	l	$\log_\xi \xi_\kappa$	$\log_\xi \xi_\lambda$	$\log_\xi \omega_\kappa$	$\log_\xi \omega_\lambda$	SD	SPD	MR
29	15	7	308	180	300	750		✓	
29	15	7	364	300	810	780		✓	
29	15	14	196	30	90	630		✓	
29	15	15	28	28	480	480	✓		
29	15	15	28	308	480	300		✓	
29	15	15	196	196	90	90	✓		
29	15	15	308	308	300	300	✓		
29	15	15	364	364	810	810	✓		
29	29	29					✓		
31	5	5	96	96	128	128	✓		
31	5	5	288	288	352	352	✓		
31	8	8	60	60	160	160	✓		
31	8	8	180	180	704	704	✓		
31	8	15	60	352	160	320		✓	✓
31	8	16	180	30	704	256		✓	✓
31	15	15	32	32	416	416	✓		
31	15	15	224	224	512	512	✓		
31	15	15	352	352	320	320	✓		
31	15	15	416	416	672	672	✓		
31	16	5	210	288	448	352		✓	✓
31	16	8	90	60	576	160		✓	✓
31	16	15	150	416	544	672		✓	✓
31	16	16	30	30	256	256	✓		
31	16	16	30	150	256	544		✓	✓
31	16	16	90	90	576	576	✓		
31	16	16	150	150	544	544	✓		
31	16	16	210	210	448	448	✓		
31	31	31					✓		
37	6	6	114	114	1178	1178	✓		
37	9	9	76	76	190	190	✓		
37	9	9	380	380	1292	1292	✓		
37	9	9	532	532	1254	1254	✓		
37	18	18	38	38	836	836	✓		
37	18	18	190	190	266	266	✓		
37	18	18	266	266	76	76	✓		
37	19	9	180	380	798	1292		✓	
37	19	9	468	76	646	190		✓	

q	k	l	$\log_\xi \xi_\kappa$	$\log_\xi \xi_\lambda$	$\log_\xi \omega_\kappa$	$\log_\xi \omega_\lambda$	SD	SPD	MR
37	19	18	252	38	1140	836		✓	
37	19	18	612	266	988	76		✓	
37	19	19	36	36	1026	1026	✓		
37	19	19	36	108	1026	418		✓	
37	19	19	108	108	418	418	✓		
37	19	19	108	252	418	1140		✓	
37	19	19	180	180	798	798	✓		
37	19	19	252	252	1140	1140	✓		
37	19	19	324	324	1064	1064	✓		
37	19	19	396	396	228	228	✓	✓	
37	19	19	468	468	646	646	✓		
37	19	19	540	324	532	1064		✓	
37	19	19	540	540	532	532	✓		
37	19	19	612	612	988	988	✓		
37	19	37	324		1064			✓	
37	37	37					✓		
41	5	5	168	168	1638	1638	✓		
41	5	5	504	504	882	882	✓		
41	5	7	168	600	1638	126		✓	
41	5	20	504	294	882	924		✓	
41	7	7	120	120	210	210	✓		
41	7	7	360	360	504	504	✓		
41	7	7	600	600	126	126	✓		
41	10	10	84	84	630	630	✓		
41	10	10	252	252	672	672	✓		
41	10	21	84	520	630	336		✓	✓
41	10	41	252		672			✓	✓
41	20	20	42	42	1302	1302	✓		
41	20	20	42	126	1302	714		✓	✓
41	20	20	126	126	714	714	✓		
41	20	20	294	294	924	924	✓		
41	20	20	378	378	420	420	✓		
41	20	21	126	680	714	1218		✓	✓
41	20	21	294	200	924	168		✓	✓
41	20	21	378	440	420	756		✓	✓
41	21	21	40	40	1428	1428	✓		
41	21	21	200	200	168	168	✓		

q	k	l	$\log_\xi \xi_\kappa$	$\log_\xi \xi_\lambda$	$\log_\xi \omega_\kappa$	$\log_\xi \omega_\lambda$	SD	SPD	MR
41	21	21	440	440	756	756	✓		
41	21	21	520	520	336	336	✓		
41	21	21	680	680	1218	1218	✓		
41	21	21	760	760	1134	1134	✓		
41	41	41					✓		
43	3	22	308	294	0	1276		✓	
43	7	7	132	132	792	792	✓		
43	7	7	396	396	220	220	✓		
43	7	7	660	660	1760	1760	✓		
43	7	21	132	484	792	1628		✓	
43	7	21	396	572	220	484		✓	
43	7	21	660	748	1760	1496		✓	
43	11	11	84	84	1452	1452	✓		
43	11	11	252	252	1012	1012	✓		
43	11	11	420	420	1540	1540	✓		
43	11	11	588	588	1584	1584	✓		
43	11	11	756	756	1804	1804	✓		
43	21	7	484	660	1628	1760		✓	
43	21	11	44	420	396	1540		✓	
43	21	11	220	588	1100	1584		✓	
43	21	11	748	84	1496	1452		✓	
43	21	11	836	252	440	1012		✓	
43	21	21	44	44	396	396	✓		
43	21	21	220	220	1100	1100	✓		
43	21	21	484	484	1628	1628	✓		
43	21	21	572	220	484	1100		✓	
43	21	21	572	572	484	484	✓		
43	21	21	748	748	1496	1496	✓		
43	21	21	836	836	440	440	✓		
43	22	22	42	42	132	132	✓		
43	22	22	126	126	308	308	✓		
43	22	22	210	210	968	968	✓		
43	22	22	294	294	1276	1276	✓		
43	22	22	378	378	1672	1672	✓		
43	43	43					✓		
47	3	23	368	528	0	1152		✓	
47	6	6	184	184	480	480	✓		

q	k	l	$\log_\xi \xi_\kappa$	$\log_\xi \xi_\lambda$	$\log_\xi \omega_\kappa$	$\log_\xi \omega_\lambda$	SD	SPD	MR
47	6	24	184	46	480	672		✓	✓
47	8	8	138	138	960	960	✓		
47	8	8	414	414	576	576	✓		
47	8	23	138	624	960	144		✓	✓
47	8	23	414	432	576	528		✓	✓
47	12	8	92	414	1200	576		✓	✓
47	12	12	92	92	1200	1200	✓		
47	12	12	460	460	1008	1008	✓		
47	12	23	460	816	1008	768		✓	✓
47	23	23	48	48	1344	1344	✓		
47	23	23	144	144	1056	1056	✓		
47	23	23	240	240	288	288	✓		
47	23	23	336	336	720	720	✓		
47	23	23	432	432	528	528	✓		
47	23	23	528	528	1152	1152	✓		
47	23	23	624	624	144	144	✓		
47	23	23	720	720	1296	1296	✓		
47	23	23	816	816	768	768	✓		
47	23	23	912	912	624	624	✓		
47	23	23	1008	1008	2016	2016	✓		
47	24	12	46	460	672	1008		✓	✓
47	24	23	322	144	1920	1056		✓	✓
47	24	23	506	48	336	1344		✓	✓
47	24	24	46	46	672	672	✓		
47	24	24	230	230	1488	1488	✓	✓	✓
47	24	24	322	322	1920	1920	✓		
47	24	24	506	506	336	336	✓		
47	47	47					✓		

Table 2: External symmetries of regular maps on $\mathrm{PSL}(2, q)$.

q	Maps	None	SD only	SP only	SD+SP	SP+MR	SD+SP+MR
3^2	3	2	1	0	0	0	0
3^3	54	48	4	2	0	0	0
3^4	381	356	15	7	0	3	0
5	1	0	1	0	0	0	0
5^2	63	52	7	3	0	1	0
7	5	4	1	0	0	0	0
7^2	264	238	16	7	0	3	0
11	16	11	3	1	1	0	0
13	33	26	4	2	1	0	0
17	58	50	6	0	0	2	0
19	70	58	8	4	0	0	0
23	113	101	8	1	0	2	1
29	183	163	13	7	0	0	0
31	209	190	13	0	0	6	0
37	315	290	16	8	1	0	0
41	382	356	18	2	0	6	0
43	430	400	20	10	0	0	0
47	515	485	20	1	0	8	1
53	663	625	25	13	0	0	0
59	820	779	27	13	1	0	0
61	879	836	28	14	1	0	0
67	1072	1024	32	16	0	0	0
71	1199	1151	32	4	0	11	1
73	1276	1227	33	3	1	12	0
79	1493	1438	37	2	0	16	0

The work in this paper has only addressed regular maps on linear fractional groups where the associated finite field has odd characteristic. Many of the calculations would look quite different if we were to consider the case when $p = 2$.

It has been an open problem for some time as to whether there exists a self-dual and self-Petrie-dual regular map for any given vertex degree k on some surface. In [2], Archdeacon, Conder and Širáň proved the existence of such a map for any even valency. The results in this paper allow Fraser, Jeans and Širáň [6] to prove the existence of a self-dual, self-Petrie-dual regular map for any given odd valency $k \geq 5$.

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