THE ART OF DISCRETE AND APPLIED MATHEMATICS

ISSN 2590-9770
The Art of Discrete and Applied Mathematics 2 (2019) #P1.02
https://doi.org/10.26493/2590-9770.1261.9a0
(Also available at http://adam-journal.eu)

On König-Egerváry collections of maximum critical independent sets*

Vadim E. Levit
Department of Computer Science, Ariel University, Israel

Eugen Mandrescu
Department of Computer Science, Holon Institute of Technology, Israel

Received 20 April 2017, accepted 13 May 2018, published online 8 August 2018

Abstract

Let G be a simple graph with vertex set $V(G)$. A set $S \subseteq V(G)$ is *independent* if no two vertices from S are adjacent. Let $\mathrm{Ind}(G)$ denote the family of all independent sets.

The graph G is said to be *König-Egerváry* if $\alpha(G) + \mu(G) = |V(G)|$, where $\alpha(G)$ denotes the size of a maximum independent set and $\mu(G)$ is the cardinality of a maximum matching. A family $\Gamma \subseteq \mathrm{Ind}(G)$ is a *König-Egerváry collection* if $|\bigcup \Gamma| + |\bigcap \Gamma| = 2\alpha(G)$.

The number $d(X) = |X| - |N(X)|$ is the *difference* of $X \subseteq V(G)$, and a set $A \in \mathrm{Ind}(G)$ is *critical* if $d(A) = \max\{d(I) : I \in \mathrm{Ind}(G)\}$.

In this paper, we show that if the family of all maximum critical independent sets of a graph G is a König-Egerváry collection, then G is a König-Egerváry graph. This result generalizes one of our conjectures verified by Short in 2016.

Keywords: Maximum independent set, critical set, kernel, nucleus, core, corona, diadem, König-Egerváry graph.

Math. Subj. Class.: 05C69, 05C70, 05A20

1 Introduction

Throughout this paper G is a finite simple graph with vertex set $V(G)$ and edge set $E(G)$. If $X \subseteq V(G)$, then $G[X]$ is the subgraph of G induced by X. By $G - W$ we mean either the subgraph $G[V(G) - W]$, if $W \subseteq V(G)$, or the subgraph obtained by deleting the edge set W, for $W \subseteq E(G)$. In either case, we use $G - w$, whenever $W = \{w\}$.

*We would like to thank the anonymous reviewers for their constructive comments, which helped us to improve the presentation of the paper.

E-mail addresses: levitv@ariel.ac.il (Vadim E. Levit), eugen_m@hit.ac.il (Eugen Mandrescu)

The *neighborhood* $N(v)$ of a vertex $v \in V(G)$ is the set $\{w : w \in V(G)$ and $vw \in E(G)\}$. The *neighborhood* $N(A)$ of $A \subseteq V(G)$ is $\{v \in V(G) : N(v) \cap A \neq \emptyset\}$, and $N[A] = N(A) \cup A$.

A set $S \subseteq V(G)$ is *independent* if no two vertices from S are adjacent, and by $\mathrm{Ind}(G)$ we mean the family of all the independent sets of G. An independent set of maximum size is a *maximum independent set* of G, and $\alpha(G) = \max\{|S| : S \in \mathrm{Ind}(G)\}$.

Let $\Omega(G)$ denote the family of all maximum independent sets,

$$\mathrm{core}(G) = \bigcap \{S : S \in \Omega(G)\} \text{ [12], and } \mathrm{corona}(G) = \bigcup \{S : S \in \Omega(G)\} \text{ [3].}$$

If $A \in \Omega(G[N[A]])$, then A is a *local maximum independent set* of G [13].

A *matching* is a set M of pairwise non-incident edges of G. A matching of maximum cardinality, denoted $\mu(G)$, is a *maximum matching*.

For $X \subseteq V(G)$, the number $|X| - |N(X)|$ is the *difference* of X, denoted $d(X)$. The *critical difference* $d(G)$ is $\max\{d(X) : X \subseteq V(G)\}$. The number $\max\{d(I) : I \in \mathrm{Ind}(G)\}$ is the *critical independence difference* of G, denoted $id(G)$. Clearly, $d(G) \geq id(G)$. It was shown in [23] that $d(G) = id(G)$ holds for every graph G. If A is an independent set in G with $d(X) = id(G)$, then A is a *critical independent set* [23].

Theorem 1.1 ([20]). *Every local maximum independent set is a subset of a maximum independent set.*

Proposition 1.2 ([15]). *Each critical independent set is a local maximum independent set.*

Combining Theorem 1.1 and Proposition 1.2 one can conclude with the following.

Corollary 1.3 ([4]). *Every critical independent set can be enlarged to a maximum independent set.*

For a graph G, let us denote

$$\begin{aligned}
\ker(G) &= \bigcap \{A : A \text{ is a critical independent set}\} \text{ [16],}\\
\mathrm{MaxCritIndep}(G) &= \{S : S \text{ is a maximum critical independent set}\},\\
\mathrm{nucleus}(G) &= \bigcap \mathrm{MaxCritIndep}(G) \text{ [8], and}\\
\mathrm{diadem}(G) &= \bigcup \mathrm{MaxCritIndep}(G) \text{ [18].}
\end{aligned}$$

Clearly, $\ker(G) \subseteq \mathrm{nucleus}(G)$ and, by Corollary 1.3, the inclusion $\mathrm{diadem}(G) \subseteq \mathrm{corona}(G)$ is true for each graph G.

Theorem 1.4 ([16]). *For a graph G, the following assertions are true:*

(i) $\ker(G) \subseteq \mathrm{core}(G)$;

(ii) if A and B are critical in G, then $A \cup B$ and $A \cap B$ are critical as well.

Let us consider the graphs G_1 and G_2 from Figure 1: $\mathrm{core}(G_1) = \{a, b, c, d\}$ and it is a critical set, while $\mathrm{core}(G_2) = \{x, y, z, w\}$ and it is not critical.

Moreover,

$$\ker(G_1) = \{a, b, c\} \subset \mathrm{core}(G_1) \subset \{a, b, c, d, g\} = \mathrm{nucleus}(G_1),$$

as $\mathrm{MaxCritIndep}(G_1) = \{\{a, b, c, d, e, g\}, \{a, b, c, d, f, g\}\}$.

In addition, notice that $\mathrm{diadem}(G_1) \subsetneq \mathrm{corona}(G_1)$.

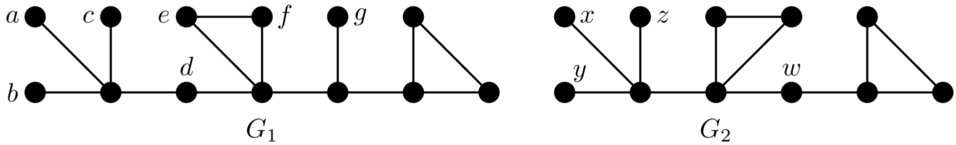

Figure 1: Both G_1 and G_2 are not König-Egerváry graphs.

Theorem 1.5 ([7]). *Let* $\emptyset \neq \Gamma \subseteq \Omega(G)$*. If* $\bigcup \Gamma$ *is critical, then* $\bigcap \Gamma$ *is critical as well.*

It is well known that $\alpha(G) + \mu(G) \leq |V(G)|$ holds for every graph G. Recall that if $\alpha(G) + \mu(G) = |V(G)|$, then G is a *König-Egerváry graph* [5, 22]. For example, each bipartite graph is a König-Egerváry graph. Various properties of König-Egerváry graphs can be found in [2, 6, 9, 14, 17].

Theorem 1.6 ([11, 15]). *For a graph G, the following assertions are equivalent:*

(i) G *is a König-Egerváry graph;*

(ii) there exists some maximum independent set which is critical;

(iii) each of its maximum independent sets is critical.

If Γ, Γ' are two collections of sets, we write $\Gamma' \lhd \Gamma$ if $\bigcup \Gamma' \subseteq \bigcup \Gamma$ and $\bigcap \Gamma \subseteq \bigcap \Gamma'$ [8]. Clearly, the relation $\lhd$ is a preorder. The following theorem extends and generalizes some findings from [19].

Theorem 1.7 ([8]). *Let* $\emptyset \neq \Gamma \subseteq \Omega(G)$.

(i) If $\Gamma' \subseteq \mathrm{Ind}(G)$ *is such that* $\Gamma' \lhd \Gamma$*, then* $\left|\bigcap \Gamma'\right| + \left|\bigcup \Gamma'\right| \leq \left|\bigcap \Gamma\right| + \left|\bigcup \Gamma\right|$.

(ii) $2\alpha(G) \leq \left|\bigcap \Gamma\right| + \left|\bigcup \Gamma\right|$.

(iii) If, in addition, G is a König-Egerváry graph, then $\left|\bigcap \Gamma\right| + \left|\bigcup \Gamma\right| = 2\alpha(G)$*, and, in particular,* $|\mathrm{corona}(G)| + |\mathrm{core}(G)| = 2\alpha(G)$.

Let us notice that if $S \in \mathrm{Ind}(G)$, then $G[N[S]]$ is not necessarily a König-Egerváry graph. For instance, consider the graph G_1 from Figure 1, and $S_1 = \{d, g\}$, $S_2 = \{d, e, g\}$. Then, $G_1[N[S_1]]$ is a König-Egerváry graph, while $G_1[N[S_2]]$ is not a König-Egerváry graph.

Theorem 1.8 ([11]). *For every graph G, there is some* $X \subseteq V(G)$*, such that:*

(i) $X = N[S]$ *for every* $S \in \mathrm{MaxCritIndep}(G)$*;*

(ii) $G[X]$ *is a König-Egerváry graph.*

In other words, Theorem 1.8*(i)* claims that $X = N[S]$ does not depend on the choice of $S \in \mathrm{MaxCritIndep}(G)$. The *critical independence number* of G is defined as $\alpha'(G) = \max\{|S| : S \in \mathrm{MaxCritIndep}(G)\}$ [11].

Recently, the following conjectures were validated in [21].

Conjecture 1.9 ([8]). *If* $|\mathrm{nucleus}(G)| + |\mathrm{diadem}(G)| = 2\alpha(G)$*, then* G *is a König-Egerváry graph.*

Conjecture 1.10 ([7]). *If* $|\mathrm{diadem}(G)| = |\mathrm{corona}(G)|$*, then* G *is a König-Egerváry graph.*

Conjecture 1.11 ([7]). $|\ker(G)| + |\mathrm{diadem}(G)| \leq 2\alpha(G)$ *for every graph* G*.*

An alternative proof of the inequality $|\ker(G)| + |\mathrm{diadem}(G)| \leq 2\alpha(G)$ may be found in [1].

In this paper, we involve these findings in a more general framework, where they appear as corollaries.

2 Results

Lemma 2.1. *If* $S \in \mathrm{MaxCritIndep}(G)$ *and* $X = N[S]$*, then*

$$\mathrm{MaxCritIndep}(G) \lhd \Omega\left(G[X]\right).$$

Proof. By Proposition 1.2, we get that $\alpha\left(G[X]\right) = |S|$. Since, in accordance with Theorem 1.8*(i)*, $X = N[A]$ for each $A \in \mathrm{MaxCritIndep}(G)$, we may conclude that $\mathrm{MaxCritIndep}(G) \subseteq \Omega\left(G[X]\right)$. Hence, $\mathrm{MaxCritIndep}(G) \lhd \Omega\left(G[X]\right)$. □

There is a graph G with $\mathrm{MaxCritIndep}(G) \subsetneq \Omega\left(G[X]\right)$, $S \in \mathrm{MaxCritIndep}(G)$, and $X = N[S]$. For instance, the graph G from Figure 2 has

$$\mathrm{MaxCritIndep}(G) = \left\{\{a,b,c,d,e,g\}, \{a,b,c,d,f,g\}\right\}, \; X = N[\{a,b,c,d,e,g\}],$$

while $\{a,b,c,d,e,k\} \in \Omega\left(G[X]\right) - \mathrm{MaxCritIndep}(G)$.

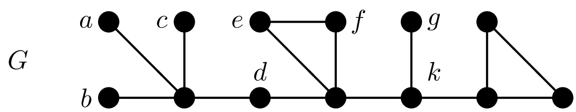

Figure 2: $d(\{a,b,c,d,e,k\}) = 1 < 2 = d(G)$.

Corollary 2.2 ([21]). *If* $S \in \mathrm{MaxCritIndep}(G)$ *and* $X = N[S]$*, then*

$$\mathrm{diadem}(G) \subseteq \mathrm{diadem}(G[X]) \quad \textit{and} \quad \mathrm{nucleus}(G[X]) \subseteq \mathrm{nucleus}(G).$$

Proof. In accordance with Theorem 1.8*(ii)*, $G[X]$ is a König-Egerváry graph. Hence, Theorem 1.6*(iii)* implies that $\mathrm{MaxCritIndep}(G[X]) = \Omega\left(G[X]\right)$. Therefore, Lemma 2.1 ensures that $\mathrm{MaxCritIndep}(G) \lhd \mathrm{MaxCritIndep}(G[X])$, which, by definition, means $\mathrm{diadem}(G) \subseteq \mathrm{diadem}(G[X])$ and $\mathrm{nucleus}(G[X]) \subseteq \mathrm{nucleus}(G)$. □

Lemma 2.3. *If* $\emptyset \neq \Gamma' \subseteq \mathrm{MaxCritIndep}(G)$ *and* $\emptyset \neq \Gamma \subseteq \Omega(G)$*, then*

$$\left|\bigcap \Gamma'\right| + \left|\bigcup \Gamma'\right| \leq 2\alpha'(G) \leq 2\alpha(G) \leq \left|\bigcap \Gamma\right| + \left|\bigcup \Gamma\right|.$$

Proof. Let $S \in \mathrm{MaxCritIndep}(G)$ and $X = N[S]$. Since $\Gamma' \subseteq \mathrm{MaxCritIndep}(G)$, and, by Lemma 2.1, $\mathrm{MaxCritIndep}(G) \lhd \Omega(G[X])$, we get $\Gamma' \lhd \Omega(G[X])$. According to Theorem 1.8*(ii)*, $G[X]$ is a König-Egerváry graph. Now, using Theorem 1.7*(ii)–(iii)*, we obtain

$$\left|\bigcap \Gamma'\right| + \left|\bigcup \Gamma'\right| \leq |\mathrm{core}(G[X])| + |\mathrm{corona}(G[X])| \\ = 2\alpha(G[X]) = 2\alpha'(G) \leq 2\alpha(G) \leq \left|\bigcap \Gamma\right| + \left|\bigcup \Gamma\right|,$$

as claimed. □

If $\Gamma' = \mathrm{MaxCritIndep}(G)$ and $\Gamma = \Omega(G)$, Lemma 2.3 immediately implies the following.

Corollary 2.4 ([21]). $|\mathrm{nucleus}(G)| + |\mathrm{diadem}(G)| \leq 2\alpha(G)$ *for every graph* G.

Since $\ker(G) \subseteq \mathrm{nucleus}(G)$, Corollary 2.4 validates Conjecture 1.11.

Let us recall that a family of independent sets Γ is a *König-Egerváry collection* if

$$\left|\bigcap \Gamma\right| + \left|\bigcup \Gamma\right| = 2\alpha(G) \text{ [8]}.$$

If there exists a König-Egerváry collection $\Gamma \subseteq \Omega(G)$, this does not oblige G to be a König-Egerváry graph. For instance, the graph G from Figure 3 satisfies $|\mathrm{corona}(G)| + |\mathrm{core}(G)| = 2\alpha(G)$, i.e., $\Omega(G)$ is a König-Egerváry collection, while G is not a König-Egerváry graph.

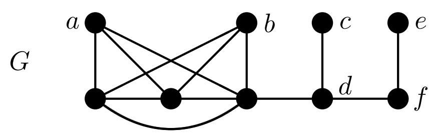

Figure 3: $\mathrm{core}(G) = \{a, b\}$ and $\mathrm{corona}(G) = \{a, b, c, d, e, f\}$.

Theorem 2.5. *For a graph G, the following assertions are equivalent:*

(i) G is a König-Egerváry graph;

(ii) every non-empty family of maximum critical independent sets of G is a König-Egerváry collection;

(iii) there is a König-Egerváry collection of maximum critical independent sets of G.

Proof. *(i)* $\Longrightarrow$ *(ii)*: By Theorem 1.6, we obtain $\mathrm{MaxCritIndep}(G) = \Omega(G)$. Further, in accordance with Theorem 1.7*(iii)*, each $\emptyset \neq \Gamma' \subseteq \mathrm{MaxCritIndep}(G)$ is a König-Egerváry collection.

(ii) $\Longrightarrow$ *(iii)*: Clear.

(iii) $\Longrightarrow$ *(i)*: Let $\Gamma' \subseteq \mathrm{MaxCritIndep}(G)$ be a König-Egerváry collection, $S \in \Gamma'$ and $X = N[S]$. Since, by Lemma 2.1, $\mathrm{MaxCritIndep}(G) \lhd \Omega(G[X])$, we arrive at the conclusion that $\Gamma' \lhd \Omega(G[X])$, and hence,

$$\left|\bigcap \Gamma'\right| + \left|\bigcup \Gamma'\right| \leq |\mathrm{nucleus}(G[X])| + |\mathrm{diadem}(G[X])|.$$

According to Theorem 1.8*(ii)*, $G[X]$ is a König-Egerváry graph. Using Theorem 1.7*(iii)*, we infer that

$$2\alpha(G) = \left|\bigcap \Gamma'\right| + \left|\bigcup \Gamma'\right|$$
$$\leq |\mathrm{nucleus}(G[X])| + |\mathrm{diadem}(G[X])| = 2\alpha(G[X]) \leq 2\alpha(G).$$

Consequently, we obtain $\alpha(G[X]) = \alpha(G)$, which ensures that G is a König-Egerváry graph. □

Since $|\mathrm{nucleus}(G)| + |\mathrm{diadem}(G)| = 2\alpha(G)$ means that $\mathrm{MaxCritIndep}(G)$ is a König-Egerváry collection, Theorem 2.5 immediately implies the following.

Corollary 2.6 ([21]). *If* $|\mathrm{nucleus}(G)| + |\mathrm{diadem}(G)| = 2\alpha(G)$*, then* G *is a König-Egerváry graph.*

It is worth mentioning that Corollary 2.6 validates Conjecture 1.9.

If $\emptyset \neq \Gamma \subseteq \Omega(G)$, then none of $\bigcap \Gamma$ and $\bigcup \Gamma$ is necessarily critical. For instance, consider the graph G from Figure 3, and $\Gamma = \{\{a, b, c, e\}, \{a, b, c, f\}\} \subseteq \Omega(G)$.

Lemma 2.7. *Let* $\Gamma \subseteq \Omega(G)$ *and* $\emptyset \neq \Gamma' \subseteq \mathrm{MaxCritIndep}(G)$*, be such that for every* $A \in \Gamma'$ *there exists* $S \in \Gamma$ *enjoying* $A \subseteq S$*. If* $\bigcap \Gamma$ *is a critical set, then the following assertions are true:*

(i) $\bigcap \Gamma \subseteq \bigcap \Gamma'$;

(ii) $\Gamma' \lhd \Gamma$;

(iii) $\left|\bigcap \Gamma'\right| + \left|\bigcup \Gamma'\right| \leq \left|\bigcap \Gamma\right| + \left|\bigcup \Gamma\right|$;

(iv) $\bigcap \Gamma' = \bigcap \Gamma$*, if, in addition,* $\bigcup \Gamma' = \bigcup \Gamma$.

Proof.

(i) Let $A \in \Gamma'$ and $S \in \Gamma$, such that $A \subseteq S$. Since $\bigcap \Gamma \subseteq S$, it follows that $A \cup \bigcap \Gamma \subseteq S$, and hence, $A \cup \bigcap \Gamma$ is independent. By Theorem 1.4, we get that $A \cup \bigcap \Gamma$ is a critical independent set. Since $A \subseteq A \cup \bigcap \Gamma$ and A is a maximum critical independent set, we infer that $\bigcap \Gamma \subseteq A$. Thus, $\bigcap \Gamma \subseteq A$ for every $A \in \Gamma'$. Therefore, $\bigcap \Gamma \subseteq \bigcap \Gamma'$.

(ii) By Part *(i)*, we know that $\bigcap \Gamma \subseteq \bigcap \Gamma'$. According to the hypothesis, every element of Γ' is included in some element of Γ. Hence, we deduce that $\bigcup \Gamma' \subseteq \bigcup \Gamma$.

(iii) The inequality follows from Part *(ii)* and Theorem 1.7*(i)*.

(iv) Part *(iii)* implies $\left|\bigcap \Gamma'\right| \leq \left|\bigcap \Gamma\right|$, and using Part *(i)*, we obtain $\bigcap \Gamma = \bigcap \Gamma'$. □

Proposition 2.8. *Let* $\Gamma \subseteq \Omega(G)$ *and* $\emptyset \neq \Gamma' \subseteq \mathrm{MaxCritIndep}(G)$ *be such that for every* $A \in \Gamma'$ *there exists* $S \in \Gamma$ *such that* $A \subseteq S$*. If* $\bigcup \Gamma' = \bigcup \Gamma$*, then* G *is a König-Egerváry graph.*

Proof. Since, by Theorem 1.4*(ii)*, $\bigcup \Gamma'$ is critical, we get that $\bigcup \Gamma$ is critical. Hence, according to Theorem 1.5, we infer that $\bigcap \Gamma$ is critical. Applying Lemma 2.7, we obtain $\bigcap \Gamma = \bigcap \Gamma'$. Further, we have

$$\begin{aligned} 2\alpha(G) &\leq \left|\bigcap \Gamma\right| + \left|\bigcup \Gamma\right| = \left|\bigcap \Gamma'\right| + \left|\bigcup \Gamma'\right| \\ &\leq |\mathrm{core}(G[X])| + |\mathrm{corona}(G[X])| = 2\alpha(G[X]) \leq 2\alpha(G). \end{aligned}$$

Consequently, $\left|\bigcap \Gamma'\right| + \left|\bigcup \Gamma'\right| = 2\alpha(G)$, which ensures, by Theorem 2.5, that G is a König-Egerváry graph. □

If $\Gamma' = \mathrm{MaxCritIndep}(G)$ and $\Gamma = \Omega(G)$, Proposition 2.8 immediately implies the following.

Corollary 2.9 ([21]). *If* $\mathrm{diadem}(G) = \mathrm{corona}(G)$*, then* G *is a König-Egerváry graph.*

It is worth mentioning that Corollary 2.9 validates Conjecture 1.10.

3 Conclusions

In this paper we focus on interconnections between unions and intersections of maximum critical independents sets of a graph. In [21], the question arises about polynomial-time complexity of computing the following lower bound for the independence number

$$|\mathrm{nucleus}(G)| + |\mathrm{diadem}(G)| \leq 2\alpha\,(G)\,.$$

Actually, Lemma 2.3 tells us that $\alpha'(G)$ is a better lower bound, since

$$|\mathrm{nucleus}(G)| + |\mathrm{diadem}(G)| \leq 2\alpha'(G) \leq 2\alpha\,(G)\,,$$

while $\alpha'(G)$ is polynomially computable [10]. It seems promising to pursue upper bounds for $\alpha\,(G)$ in terms of $\alpha'(G)$. Let us call G a *k-bounded graph* if $\alpha\,(G) \leq k \cdot \alpha'(G)$. For instance, König-Egerváry graphs are 1-bounded, in accordance with Theorem 1.6. It is worth emphasizing that the independence number can be computed in polynomial time for König-Egerváry graphs, since in this case $\alpha(G) = \alpha'(G)$.

Proposition 3.1. *If* $S \in \mathrm{MaxCritIndep}(G)$*, then* $2\alpha'(G) = d\,(\ker(G)) + |N[S]|$.

Proof. Since $\ker(G)$ and S are critical sets of the graph G, we obtain

$$\begin{aligned} d\,(\ker(G)) + |N[S]| &= |\ker(G)| - |N\,(\ker(G))| + |S| + |N\,(S)| \\ &= |S| - |N\,(S)| + |S| + |N\,(S)| = 2\,|S| = 2\alpha'(G), \end{aligned}$$

which completes the proof. □

By Theorem 1.6 and Proposition 3.1, if G is a König-Egerváry graph and $S \in \mathrm{MaxCritIndep}(G)$, then we get

$$2\alpha(G) = d\,(\ker(G)) + |N[S]|\,,$$

because $\alpha'(G) = \alpha(G)$, and consequently,

$$2\alpha\,(G) \leq |\ker(G)| + |N\,[S]|\,.$$

Proposition 3.2 ([10]). *There is a matching from $N(S)$ into S for every critical independent set S.*

Proposition 3.3. *If $2\alpha(G) \leq |\ker(G)| + |N[S]|$, where $S \in \mathrm{MaxCritIndep}(G)$, then G is $\frac{3}{2}$-bounded. More precisely,*

$$\alpha'(G) \leq \alpha(G) \leq \alpha'(G) + \frac{|N(\ker(G))|}{2} \leq \frac{3}{2}\alpha'(G).$$

Proof. According to Proposition 3.2, there is a matching from $N(S)$ into S, since S is critical. Hence, $|N[S]| \leq 2\alpha'(G)$. Therefore, taking account that $\ker(G)$ is a critical independent set, we obtain

$$2\alpha(G) \leq |\ker(G)| + |N[S]| \leq |\ker(G)| + 2\alpha'(G) \leq 3\alpha'(G).$$

In accordance with Proposition 3.1, we get

$$|\ker(G)| + |N[S]| - |N(\ker(G))| = 2\alpha'(G) \leq 2\alpha(G).$$

Thus

$$\begin{aligned}\frac{|\ker(G)| + |N[S]|}{2} - \frac{|N(\ker(G))|}{2} &= \alpha'(G) \leq \alpha(G) \\ &\leq \frac{|\ker(G)| + |N[S]|}{2} \\ &= \alpha'(G) + \frac{|N(\ker(G))|}{2} \leq \frac{3}{2}\alpha'(G),\end{aligned}$$

since, $\ker(G)$ is a critical set and, by Proposition 3.2, there exists a matching from $N(\ker(G))$ into $\ker(G)$. □

Let us emphasize that the bound $\alpha(G) \leq \alpha'(G) + \frac{|N(\ker(G))|}{2}$ is of polynomial-time complexity, since $\ker(G)$ [16] and $\alpha'(G)$ [10] can be computed polynomially.

References

[1] A. Bhattacharya, A. Mondal and T. S. Murthy, Problems on matchings and independent sets of a graph, *Discrete Math.* **341** (2018), 1561–1572, doi:10.1016/j.disc.2018.02.021.

[2] F. Bonomo, M. C. Dourado, G. Durán, L. Faria, L. N. Grippo and M. D. Safe, Forbidden subgraphs and the König-Egerváry property, *Discrete Appl. Math.* **161** (2013), 2380–2388, doi:10.1016/j.dam.2013.04.020.

[3] E. Boros, M. C. Golumbic and V. E. Levit, On the number of vertices belonging to all maximum stable sets of a graph, *Discrete Appl. Math.* **124** (2002), 17–25, doi:10.1016/s0166-218x(01) 00327-4.

[4] S. Butenko and S. Trukhanov, Using critical sets to solve the maximum independent set problem, *Oper. Res. Lett.* **35** (2007), 519–524, doi:10.1016/j.orl.2006.07.004.

[5] R. W. Deming, Independence numbers of graphs – an extension of the Koenig-Egervary theorem, *Discrete Math.* **27** (1979), 23–33, doi:10.1016/0012-365x(79)90066-9.

[6] A. Jarden, V. E. Levit and E. Mandrescu, Two more characterizations of König-Egerváry graphs, *Discrete Appl. Math.* **231** (2017), 175–180, doi:10.1016/j.dam.2016.05.012.

[7] A. Jarden, V. E. Levit and E. Mandrescu, Critical and maximum independent sets of a graph, *Discrete Appl. Math.* (2018), doi:10.1016/j.dam.2018.03.058, in press.

[8] A. Jarden, V. E. Levit and E. Mandrescu, Monotonic properties of collections of maximum independent sets of a graph, *Order* (2018), doi:10.1007/s11083-018-9461-8, in press.

[9] E. Korach, T. Nguyen and B. Peis, Subgraph characterization of red/blue-split graph and Kőnig Egerváry graphs, in: *Proceedings of the Seventeenth Annual ACM-SIAM Symposium on Discrete Algorithms*, ACM, New York, 2006 pp. 842–850, doi:10.1145/1109557.1109650, held in Miami, FL, January 22 – 24, 2006.

[10] C. E. Larson, A note on critical independence reductions, *Bull. Inst. Combin. Appl.* **51** (2007), 34–46.

[11] C. E. Larson, The critical independence number and an independence decomposition, *European J. Combin.* **32** (2011), 294–300, doi:10.1016/j.ejc.2010.10.004.

[12] V. E. Levit and E. Mandrescu, Combinatorial properties of the family of maximum stable sets of a graph, *Discrete Appl. Math.* **117** (2002), 149–161, doi:10.1016/s0166-218x(01)00183-4.

[13] V. E. Levit and E. Mandrescu, A new Greedoid: the family of local maximum stable sets of a forest, *Discrete Appl. Math.* **124** (2002), 91–101, doi:10.1016/s0166-218x(01)00332-8.

[14] V. E. Levit and E. Mandrescu, On α^+-stable König-Egerváry graphs, *Discrete Math.* **263** (2003), 179–190, doi:10.1016/s0012-365x(02)00528-9.

[15] V. E. Levit and E. Mandrescu, Critical independent sets and König-Egerváry graphs, *Graphs Combin.* **28** (2012), 243–250, doi:10.1007/s00373-011-1037-y.

[16] V. E. Levit and E. Mandrescu, Vertices belonging to all critical sets of a graph, *SIAM J. Discrete Math.* **26** (2012), 399–403, doi:10.1137/110823560.

[17] V. E. Levit and E. Mandrescu, On maximum matchings in König-Egerváry graphs, *Discrete Appl. Math.* **161** (2013), 1635–1638, doi:10.1016/j.dam.2013.01.005.

[18] V. E. Levit and E. Mandrescu, Critical independent sets of a graph, 2014, arXiv:1407.7368 [cs.DM].

[19] V. E. Levit and E. Mandrescu, A set and collection lemma, *Electron. J. Combin.* **21** (2014), #P1.40, http://www.combinatorics.org/ojs/index.php/eljc/article/view/v21i1p40.

[20] G. L. Nemhauser and L. E. Trotter, Jr., Vertex packings: structural properties and algorithms, *Math. Programming* **8** (1975), 232–248, doi:10.1007/bf01580444.

[21] T. Short, On some conjectures concerning critical independent sets of a graph, *Electron. J. Combin.* **23** (2016), #P2.43, http://www.combinatorics.org/ojs/index.php/eljc/article/view/v23i2p43.

[22] F. Sterboul, A characterization of the graphs in which the transversal number equals the matching number, *J. Comb. Theory Ser. B* **27** (1979), 228–229, doi:10.1016/0095-8956(79)90085-6.

[23] C. Q. Zhang, Finding critical independent sets and critical vertex subsets are polynomial problems, *SIAM J. Discrete Math.* **3** (1990), 431–438, doi:10.1137/0403037.