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Axiomatic characterization of transit functions of weak hierarchies*

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Abstract

Transit functions provide a unified approach to study notions of intervals, convexities, and betweenness. Recently, their scope has been extended to certain set systems associated with clustering. We characterize here the class of set systems that correspond to k-ary monotonic transit functions. Convexities form a subclass and are characterized in terms of transit functions by two additional axioms. We then focus on axiom systems associated with weak hierarchies as well as other generalizations of hierarchical set systems.

Keywords: Transit functions, convexities, weak hierarchies, axiom systems.

Math. Subj. Class.: 05C05, 05C99, 52A01

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1 Introduction

Transit functions have been introduced as unifying approach for results and ideas on intervals, convexities and betweenness in graphs and posets [25]. Formally, a *transit function* [25] on a non-empty set V is a function $R\colon V \times V \to 2^V$ satisfying the three axioms

(t1) $u \in R(u, v)$ for all $u, v \in V$.

(t2) $R(u, v) = R(v, u)$ for all $u, v \in V$.

(t3) $R(u, u) = \{u\}$ for all $u \in V$.

Transit functions capture an abstract notion of "betweenness", i.e., an element x is considered to be "between" u and v, if $x \in R(u, v)$. Several classes of interesting transit function associated with connected graphs have been studied from this perspective [15], among them the interval function [1, 24, 26], induced path function [12, 13, 23], the all paths function [11], the pre-fiber transit function [25], P_3-transit function [18]. Transit function also arose as models of recombination operators in genetics and evolutionary algorithms [16, 22, 27], where again, encapsulate the idea that offsprings are genetically "in-between" their parents.

Recently, they have also been applied to set systems naturally arising from clustering problems [9, 17]. Hierarchical structures play a key role in wide range of applications in the sciences. In [17] we considered properties of transit functions that are related to hierarchies and provided a characterization in terms of simple axioms. This begs the question whether or to what extent related systems of clusters are also within the explanatory range of transit functions. Weak hierarchies were introduced in [2, 3] as the system of "weak clusters" satisfying $s(a, b) > \min\{s(a, x), s(b, x)\}$ for all a, b within a cluster and x outside, and in [5] (with closure under pairwise intersection as additional condition). Subsequently, they have become a central structure in the theory of hierarchical clustering, see e.g. [8, 9]. They subsume a number of less general constructions such as paired hierarchies [6, 7] and pyramids [20]. In [4] systems of clusters are considered that are generated by two elements. These are, as we shall see, closely related to set systems identified by transit functions.

In particular in the context hierarchies, much of the previous work focuses the relationships of transit functions and convexities [8, 9], see also [4, 15, 25] on convexities in relation to graph-theoretical constructions. Here we take a somewhat different perspective and focus on axiom systems on transit functions giving raise to set systems such as those arising naturally in the context of clustering.

This contribution is organized as follows. We start by characterizing the set systems that are identified by transit function. Then we proceed to discussing convexities, giving a characterization of the transit functions that identify them. We then consider the transit functions arising from some hierarchy-like systems. Here we focus on weak hierarchies as well as properties discussed in our previous work [17]. We then generalize the results of Section 2 to k-ary transit functions. Finally we consider k-weak hierarchies. For $k > 2$ these are not identified by convexities but require the more general setting.

2 Set systems identified by transit functions

Throughout this contribution, V is a finite, non-empty set. Consider an arbitrary set system $\mathcal{X} \subseteq 2^V$ and the function

$$R_{\mathcal{X}}(x, y) = \bigcap \{A \in \mathcal{X} \mid x, y \in A\}. \tag{2.1}$$

We are interested here in those special classes of set systems that are *identified* by the function R, that is,

$$\mathcal{X} = \{R_{\mathcal{X}}(x,y) \mid x,y \in V\}. \tag{2.2}$$

We use the notation $\subset$ to mean proper subset, and write $\subseteq$ otherwise.

If $p,q \in R_{\mathcal{X}}(x,y)$ then by definition in every $A \in \mathcal{X}$ that contains x and y also contains p and q. Thus $R_{\mathcal{X}}$ satisfies the *monotone axiom* [13, 25]

(m) $p,q \in R(u,v)$ implies $R(p,q) \subseteq R(u,v)$.

The monotone axiom ***(m)*** can be expressed in the following equivalent form:

Lemma 2.1. *A transit function R is monotone if and only if, for all $u,v \in V$,*

$$\bigcap\{R(x,y) \mid u,v \in R(x,y)\} = R(u,v). \tag{2.3}$$

Proof. Condition ***(m)*** implies $R(u,v) \subseteq \bigcap\{R(x,y) \mid u,v \in R(x,y)\}$. By ***(t1)*** and ***(t2)*** we have $u,v \in R(u,v)$, hence $\bigcap\{R(x,y) \mid u,v \in R(x,y)\} \subseteq R(u,v)$ and thus Equation (2.3) holds. Conversely, from Equation (2.3) we immediately obtain $R(u,v) \subseteq R(x,y)$ for all x,y with $u,v \in R(x,y)$, i.e., ***(m)*** holds. □

Lemma 2.2. *A set system $\mathcal{X}$ is identified by a transit function R if and only if it satisfies the following axioms:*

(KS) *$\{x\} \in \mathcal{X}$ for all $x \in V$.*

(KR) *For every $C \in \mathcal{X}$ there are points $p,q \in C$ such that $p,q \in C'$ implies $C \subseteq C'$ for all $C' \in \mathcal{X}$.*

(KC) *For every $p,q \in V$ holds $\bigcap\{C \in \mathcal{X} \mid p,q \in C\} \in \mathcal{X}$.*

Proof. Suppose set system $\mathcal{X}$ is identified by a transit function R. Then the set system $\mathcal{X}$ satisfies ***(KS)***, since R satisfies ***(t3)***. By definition of R, $C \in \mathcal{X}$ implies $C = R(u,v) = \bigcap\{A \in \mathcal{X} \mid u,v \in A\}$ for some $u,v \in V$. Therefore, for $p,q \in V$ such that $p,q \in A$, it follows that $C \subseteq C'$, which proves that $\mathcal{X}$ satisfies ***(KR)***. Now, for every $p,q \in C \in \mathcal{X}$, $R(p,q) = \bigcap\{C \in \mathcal{X} \mid p,q \in C\}$ and hence $\mathcal{X}$ satisfies ***(KC)***.

Conversely, suppose that a set system $\mathcal{X}$ satisfies the axioms ***(KS)***, ***(KR)*** and ***(KC)***. To show that $\mathcal{X}$ is identified by a transit function, define a function $R\colon V \times V \to 2^V$ as $R_{\mathcal{X}}(x,y) = \bigcap\{A \in \mathcal{X} \mid x,y \in A\}$. Axioms ***(KS)*** and ***(KC)*** for $\mathcal{X}$ imply that this function R satisfies ***(t3)*** and ***(t1)***. Since R also satisfies ***(t2)*** by definition, we conclude that R is a transit function. Moreover, R satisfies ***(m)***. Now, we show that $\mathcal{X} = \{R_{\mathcal{X}}(x,y) \mid x,y \in V\}$. By axiom ***(KR)***, $C \in \mathcal{X}$, then there exists, $p,q \in C$ such that $p,q \in C'$ implies $C \subseteq C'$ for all $C' \in \mathcal{X}$, which implies that $C \subseteq \bigcap\{C \in \mathcal{X} \mid p,q \in C\}$, that is, $C \subseteq R_{\mathcal{X}}(p,q)$. On the other hand, $p,q \in C$ implies $R_{\mathcal{X}}(p,q) \subseteq C$, Thus every cluster C is a transit set, i.e. $\mathcal{X} \subseteq \{R_{\mathcal{X}}(p,q) \mid p,q\}$. By axiom ***(KC)***, $\{R_{\mathcal{X}}(x,y) \mid x,y \in V\} \subseteq \mathcal{X}$, which completes the proof. □

In this case R is the *canonical transit function* of $\mathcal{X}$ [25]. Condition ***(KR)*** is called 2-*arity* in [13]. If R is the *canonical transit function* of $\mathcal{X}$, then ***(KC)*** becomes

(m*) For all $u,v \in V$ there is $p,q \in V$ such that

$$\bigcap\{R(s,t) \mid s,t \in V \text{ and } u,v \in R(s,t)\} = R(p,q).$$

This can be seen as a relation of Equation (2.3), which stipulates not just the existence of $p, q \in V$ but insists that $\{p, q\} = \{u, v\}$. The example in Figure 1 shows that axiom ***(KC)*** is much weaker than closure under pairwise intersection. The set systems identified by transit functions therefore are not convexities in general. We will return to this point in the next section.

The independence of the axioms ***(KS)***, ***(KR)***, and ***(KC)*** is established by the following examples.

Example 2.3 (***(KS)*** but not ***(KR)*** or ***(KC)***)**.** For $V = \{a, b, c, d, e\}$ consider

$$\mathcal{X} = \big\{\{a\}, \{b\}, \{c\}, \{d\}, \{e\}, \{a, b, c\}, \{b, c, d\}, \{a, b, d\}, \{a, c, e\}\big\}.$$

For every pair of points $p, q \in \{a, b, c\}$ there is another set containing p and q that is not a superset of $\{a, b, c\}$:

$$a, b \in \{a, b, d\}, \quad a, c \in \{a, c, e\}, \quad \text{and} \quad b, c \in \{b, c, d\}.$$

Thus ***(KR)*** does not hold. Since $\{a, b, c\} \cap \{a, b, d\} = \{a, b\} \notin \mathcal{X}$, ***(KC)*** is not satisfied. On the other hand, all singletons are in $\mathcal{X}$, i.e., the $\mathcal{X}$ satisfies ***(KS)***.

Example 2.4 (***(KR)*** but not ***(KS)*** or ***(KC)***)**.** For $V = \{a, b, c, d, e\}$ consider

$$\mathcal{X} = \big\{\{a, b, c\}, \{a, b, d\}\big\}.$$

We can easily see that $\mathcal{X}$ satisfies ***(KR)***. Since no singleton is in $\mathcal{X}$, we can see that the set system fails to satisfy ***(KS)***. Now $\{a, b, c\}, \{a, b, d\} \in \mathcal{X}$, but $\{a, b, c\} \cap \{a, b, d\} = \{a, b\} \notin \mathcal{X}$. Hence $\mathcal{X}$ does not satisfy ***(KC)***.

Example 2.5 (***(KC*** but not ***(KS)*** or ***(KR)***)**.** For $V = \{a, b, c, d, e\}$ consider

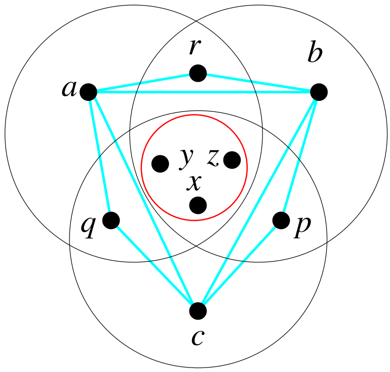
$$\mathcal{X} = \big\{\{a, b, c\}, \{b, c, d\}, \{a, b, d\}, \{a, c, e\}, \{a, b\}, \{b, c\}, \{a, c\}, \{b, d\}, \{a\}, \{b\}, \{c\}\big\}.$$

We can see that $\mathcal{X}$ satisfies ***(KC)***. For any pair of elements in $\{a, b, c\}$, there are the sets $\{b, c, d\}$, $\{a, b, d\}$, $\{a, c, e\}$ that contain a pair of $\{a, b, c\}$ but are not supersets of $\{a, b, c\}$. Hence $\mathcal{X}$ does not satisfy ***(KR)***. Also since no singleton is in $\mathcal{X}$, the set system fails to satisfy ***(KS)***.

Theorem 2.6. *There is a 1-1 correspondence between monotone transit functions R on V and set systems $\mathcal{X}$ satisfying **(KS)**, **(KR)**, and **(KC)** on V. This bijection is given by $\mathcal{X} \mapsto R_{\mathcal{X}}$ defined in Equation (2.1) and $\{R(p, q) \mid p, q \in V\} \mapsto \mathcal{X}$, respectively.*

Proof. Suppose $\mathcal{X}$ is a set system satisfying ***(KS)***, ***(KR)***, and ***(KC)*** with transit function R. By Lemma 2.2, $\mathcal{X} = \{R(u, v) \mid u, v \in V\}$. The transit function defined by Equation (2.1) satisfies ***(m)*** as argued above.

Now suppose R is a monotone transit function. Then by construction $\{R(p, q) \mid p, q \in V\}$ satisfies ***(KS)*** and ***(KR)***. Furthermore ***(m)*** implies ***(m*)***, which can be rewritten as ***(KC)*** using the fact that $\mathcal{X} = \{R(u, v) \mid u, v \in V\}$. □



$\mathcal{X}$ consists of V, $\{u\}$ for $u \in V$, the nine pairs $\{a,r\}$, $\{a,q\}$, $\{b,r\}$, $\{b,p\}$, $\{c,p\}$, $\{c,q\}$, $\{a,b\}$, $\{b,c\}$, $\{a,c\}$ as well and the 18 pairs in $\{a,b,c,p,q,r\} \times \{x,y,z\}$, $A = \{a,r,q,x,y,z\}$, $B = \{b,p,r,x,y,z\}$, $C = \{c,p,q,x,y,z\}$, and $Z = \{x,y,z\}$.

Figure 1: In addition to the 27 defining pairs we have $R(a,p) = R(b,q) = R(c,r) = V$, $R(q,r) = A$, $R(r,p) = B$, $R(p,q) = C$, and $R(x,y) = R(x,z) = R(y,z) = Z$, accounting for all pairs in V. Hence it satisfies ***(KR)***. For each pair of points i,j, furthermore, the non-empty intersection of all sets containing them is again a member of $\mathcal{X}$. This is obvious for all intersections involving two-element sets in $\mathcal{X}$. In the remaining cases yield the transit sets listed above. The pairwise intersections $A \cap B$, $A \cap C$, and $B \cap C$, however, are not members of $\mathcal{X}$: any pair in these sets has as intersection of its containing sets either that pairs or Z.

Definition 2.7. A set system $\mathcal{X}$ satisfying ***(KS)***,

(K0) $\emptyset \notin \mathcal{X}$, and

(K1) $V \in \mathcal{X}$

is called a *clustering system* [4].

In the terminology of [4], a *clustering system* is *pre-binary* if, for every $u,v \in V$ the set system $\{C \in \mathcal{X} \mid u,v \in C\}$ has exactly one inclusion-minimal element, i.e., in our language, if every transit set is a cluster. The proof of Lemma 2.2 shows that this condition is equivalent to ***(KR)***. A cluster system is called *binary* if it is pre-binary and every cluster is a transit set. The latter condition is equivalent to ***(KC)***. We can summarize this discussion as

Corollary 2.8. *A set system $\mathcal{X}$ is a binary clustering system if and only if it satisfies **(K0)**, **(K1)**, **(KS)**, **(KR)**, and **(KC)**.*

Axiom ***(K1)*** can trivially be translated into the language of transit functions as

(a') There is $u,v \in V$ such that $R(u,v) = V$.

Thus we also have

Corollary 2.9. *A transit function identifies as a binary clustering system if and only if it satisfies **(m)** and **(a')**.*

3 Transit functions of convexities

A systems of sets $\mathcal{K} \subset 2^V$ that contains V and is closed under intersection is called a *convexity*. Usually, one requires the empty set $\emptyset$ to be part of a convexity. However, $\emptyset$ cannot be a transit set according to ***(t1)***. It is convenient in our context, therefore, to use a slightly modified definition, excluding the empty set and restricting closure to non-empty intersection. Furthermore, ***(t3)*** implies that $\{x\}$ is a transit set for every $x \in V$. Thus we are only interested in set systems that contain the singletons.

Definition 3.1. A *convexity* is a set system $\mathcal{X} \subseteq 2^V$ satisfying ***(K0)***, ***(K1)***, and

(K2) if $A, B \in \mathcal{X}$ and $A \cap B \neq \emptyset$ then $A \cap B \in \mathcal{X}$.

A set system is *closed (under pairwise intersection)* if it satisfies ***(K2)***. A convexity is called *grounded* if it satisfies ***(KS)***. Thus grounded convexities are the same as closed clustering systems. Closure under non-empty intersections, ***(K2)***, immediately implies ***(KC)***, but the converse is not true, as we have seen in Figure 1.

The following result from [14] is now a direct consequence of Lemma 2.2:

Proposition 3.2. *A convexity $\mathcal{X}$ is identified by a monotone transit function R if and only if it is grounded (that is, $\mathcal{X}$ satisfies **(KS)**), and satisfies **(KR)**.*

It is now easy to characterize the transit functions that identify convexities:

Lemma 3.3. *Let R be a monotone transit function. Then the transit sets $\{R(p,q) \mid p,q \in V\}$ form a convexity if and only R satisfies **(a')** and*

(m') *For all $u, v, x, y \in V$ with $R(u,v) \cap R(x,y) \neq \emptyset$, there is $p, q \in V$ such that $R(u,v) \cap R(x,y) = R(p,q)$.*

Proof. Note that by assumption there is a 1-1 correspondence between transit sets and the sets $C \in \mathcal{X}$. Thus ***(a')*** and ***(m')*** directly translate to ***(K1)*** and ***(K2)***, respectively. □

Axiom ***(m')*** was introduced in [17] in the context of hierarchies. It is clear that ***(m')*** implies ***(m*)***, it does not imply ***(m)***, however. The example in Figure 1 also shows that ***(m*)*** does not imply ***(m')***.

Figure 2 gives a smaller counterexample. The interval function of a graph G with vertex set V, defined by $I(x,y) = \{z \in V \mid z \text{ lies on some shortest path between } x \text{ and } y\}$, is not monotone in general [26]. In other words, the collection of intervals is in general not sufficient to determine the "end points" of the interval.

4 Transit functions of hierarchy-like systems

Consider the following axioms for a set system $\mathcal{X} \subseteq 2^V$.

(H) $A, B \in \mathcal{X}$ implies $A \cap B \subseteq \{A, B, \emptyset\}$.

(PH) $A \in \mathcal{X}$ properly intersects at most one $B \in \mathcal{X}$.

(WH) $A, B, C \in \mathcal{X}$ implies $A \cap B \cap C \subseteq \{A \cap B, A \cap C, B \cap C\}$.

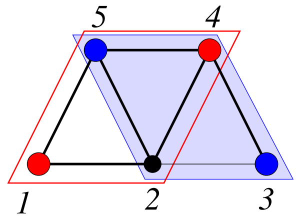


Figure 2: The interval function I of the graph G consists of singletons, a pairs with the exception of $I(1,3) = \{1,2,3\}$ (a path), $I(1,4) = \{1,2,4,5\}$ (red), and $I(3,5) = \{2,3,4,5\}$. One easily checks that I satisfies ***(m)***. For the transit sets thus ***(K0)***, ***(KS)***, ***(KR)***, and ***(KC)*** are satisfied. The pairwise intersection $I(1,4) \cap I(3,5) = \{2,4,5\}$, however is not an interval, and hence ***(K2)*** fails. Furthermore, ***(a')*** fails since $\{1,2,3,4,5\}$ is not a transit set.

A set system is called *paired hierarchical* [7] if it satisfies ***(K0)***, ***(K1)***, and ***(PH)***. If it satisfies in addition ***(WH)***, it is called a *weak hierarchy*, if ***(H)*** holds, it is a *hierarchy*. It is well known (and easy to see) that ***(H)*** implies ***(PH)*** implies ***(WH)***. Furthermore ***(H)*** implies ***(K2)***. In the following we will sometimes write $\mathfrak{W}$ instead of $\mathcal{X}$ for set systems that are weak hierarchies.

Among several hierarchy-like set systems considered in the literature, see e.g. [8, 9], the paired hierarchies are the most restrictive one, while the weak hierarchies are the most general model. In the following we show that weak hierarchies, and hence also all its more restrictive subclasses such as the paired hierarchies, are identified by transit functions if and only if they form a convexity.

Proposition 4.1 ([21])**.** *(**WH**) implies (**KR**).*

Lemma 4.2. *The canonical transit function R of a weak hierarchy $\mathcal{X}$ satisfies **(m')**.*

Proof. By construction $R(p,q) = \bigcap\{C \in \mathcal{X} \mid p,q \in C\} = \bigcap_{i=1}^{h} C_i$ with $C_i \in \mathcal{X}$ for $1 \leq i \leq h$ and some $h \geq 1$. By axiom ***(WH)***, the intersection of any three sets C_i can be replaced by a pair, hence $R(p,q) = C' \cap C''$ for two not necessarily distinct sets $C', C'' \in \mathcal{X}$.

It was shown in [2] that $\mathcal{X}$ is a weak hierarchy if and only if its closure under pair-wise intersection is a weak hierarchy. Denote this weak hierarchy by $\overline{\mathcal{X}}$. Since ***(KR)*** holds for $\mathcal{X}$ it follows that $C \in \mathcal{X}$ implies $C \in \overline{\mathcal{X}}$, since every $C \in \mathcal{X}$ has points r, s that otherwise appear only in supersets of C, and thus any intersection of sets in $\mathcal{X}$ that contains r, s also contains C. Since $\mathcal{X}$ is again a weak hierarchy, all pairwise intersections $C' \cap C'' \in \overline{\mathcal{X}}$ also contain points u, v such that every set $u, v \in D \in \overline{\mathcal{X}}$ satisfies $C' \cap C'' \subseteq D$. Since all sets D are intersections of elements of $\mathcal{X}$ containing u, v, we have $C' \cap C'' = R(u,v)$. Thus all elements of $\mathcal{X}$ and their pairwise intersections are transit sets of R. In other words, $\{R(p,q) \mid p,q \in \mathcal{X}\} = \overline{\mathcal{X}}$. Thus R satisfies ***(m')***. ☐

Since R satisfies ***(m)***, ***(m')***, and by construction also ***(a')***, Lemma 3.3 and Lemma 4.2 together imply

Corollary 4.3. *The canonical transit function of a weak hierarchy is a convexity.*

A similar result was shown with a different approach in [9]. This is in particular also true of hierarchies [17] and paired hierarchies [9].

Because of the 1-1 correspondence between the clusters of a closed weak hierarchy $\mathcal{W}$ and the transit sets of R, we can directly translate ***(KW)*** to the language of transit sets

(w) For any six points $p, q, r, s, t, u \in V$ holds

$$R(p,q) \cap R(r,s) \cap R(t,u) \in \{R(p,q) \cap R(r,s), R(p,q) \cap R(t,u), \\ R(r,s) \cap R(t,u)\}.$$

Summarizing our discussion so far we have

Corollary 4.4. *R identifies as a closed weak hierarchy if and only if it satisfies **(m)**, **(m')**, **(a')**, and **(w)**.*

Using a different route, [9] showed that a convexity is a weak hierarchy if and only if its transit function satisfies

(w') There are not three distinct points $x_1, x_2, x_3 \in V$ such that for all $\{h, i, j\} = \{1, 2, 3\}$ holds $x_h \notin R(x_i, x_j)$.

In Section 6 we prove in the more general setting of k-ary transit functions that for a monotone transit function that ***(w')*** implies ***(a')*** (Lemma 6.4) and ***(a')*** $\wedge$ ***(w)*** implies ***(w)*** (Lemma 6.7).

Corollary 4.5. *R identifies as a closed weak hierarchy if and only if it satisfies **(m)**, **(m')**, and **(w')**.*

The independence of the axioms is established by the following examples.

Example 4.6 (***(m)***, ***(m')***, and ***(a')***, but not ***(w)***)**.** For $V = \{a, b, c, d, e\}$ consider

$$R(x,x) = \{x\} \text{ for all } x \in V, \qquad R(a,b) = V, \quad \text{and} \\ R(x,y) = \{x,y\} \text{ for all other pairs.}$$

We can easily see that R satisfies ***(m)***, ***(m')***, and ***(a')***. We have

$$R(b,c) \cap R(c,d) \cap R(b,d) = \emptyset$$

but

$$R(b,c) \cap R(c,d) = \{c\}, \\ R(b,c) \cap R(b,d) = \{b\}, \text{ and} \\ R(c,d) \cap R(b,d) = \{d\}.$$

Therefore ***(w)*** does not hold.

Example 4.7 (***(m')***, ***(a')***, and ***(w)***, but not ***(m)***)**.** For $V = \{a, b, c, d, e\}$, let

$$R(x,x) = \{x\} \text{ for all } x \in V, \qquad R(a,b) = \{a,b,c,d\}, \\ R(b,c) = \{b,c,d\}, \qquad R(b,d) = \{b,d\},$$

and for all other pair $R(x, y) = V$. We observe that R satisfies ***(m')***, ***(a')***, and ***(w)***. However, ***(m)*** fails since $c, d \in R(a, b)$ and $e \in R(c, d)$, but $e \notin R(a, b)$.

Example 4.8 (***(m)***, ***(a')***, and ***(w)***, but not ***(m')***)**.** For $V = \{a, b, c, d, e\}$ consider

$$\begin{aligned} &R(a,c) = V, && R(a,b) = \{a,b\}, \\ &R(a,d) = V - \{c\}, && R(a,e) = \{a,e\}, \\ &R(b,x) = \{b,x\} \text{ for all } x \in V, && R(c,d) = \{c,d\}, \\ &R(c,e) = V - \{a\}, \text{ and} && R(d,e) = \{d,e\}. \end{aligned}$$

It can be verified that R satisfies ***(m)***, ***(a')***, and ***(w)***. But

$$R(a,d) \cap R(c,e) = \{b,d,e\} \neq R(x,y)$$

for each pair of elements $x, y \in V$. Hence R does not satisfy ***(m')***.

Example 4.9 (***(m)***, ***(m')***, and ***(w)***, but not ***(a')***)**.** For $V = \{a, b, c, d\}$ consider

$$\begin{aligned} &R(a,b) = \{a,b\}, && R(a,c) = \{a,b,c\}, \\ &R(a,d) = \{a,b,d\}, && R(b,c) = \{b,c\}, \\ &R(b,d) = \{b,d\}, \text{ and} && R(c,d) = \{c,d\}. \end{aligned}$$

We can easily see that R satisfies ***(m)***, ***(m')***, ***(w)***. However, there is no pair of points $x, y \in V$ such that $R(x,y) = V$; hence R does not satisfy ***(a')***.

Barthélemy and Brucker [4] call a clustering system *strongly binary* if it satisfies ***(KR)*** and

(ST) For each $S \subseteq V$, $S \neq \emptyset$, there exist $u, v \in S$ such that $S \subseteq \bigcap\{C \mid u, v \in C\}$.

and show that the strongly binary clustering systems are exactly the closed weak hierarchies. We note that ***(ST)*** implies ***(KC)*** by restricting the sets S to at most two elements.

In [9] it is shown that R is the transit function of paired hierarchy if it satisfies

(ph) For all $u, v \notin R(x,y)$ holds both $x \notin R(u,y)$ implies $y \in R(v,x)$; and $v \notin R(u,y)$ implies $x \in R(v,y)$.

We note that in [9] this condition is mistakenly stated for "pairwise distinct u, v, x, y". It is also necessary to check the implications for $u = v$, however. In addition, if $x = y$, the first precondition is always false, while the second conditions implies $x \in R(u,x)$, which is always true by ***(t1)***. This it suffices to drop the qualifier "pairwise distinct".

Corollary 4.10. *R identifies a closed paired hierarchy if and only if it satisfies **(m)**, **(m')**, **(a')**, and **(ph)**.*

4.1 Independence of *(m)*, *(m')*, *(a')* and *(ph)*

Example 4.11 (***(m)***, ***(m')***, and ***(a')***, but not ***(ph)***)**.** Let $V = \{a, b, c, d, e\}$ and define R on V as follows: $R(a,b) = V$, and for all other pair of elements $x, y \in V$ we set $R(x,y) = \{x,y\}$. It is not difficult to check that R satisfies ***(m)***, ***(m')***, and ***(a')***. For $R(c,d)$ we have $a, b \notin R(c,d)$ but $a \notin R(c,b)$ and $b \notin R(a,d)$; $d \notin R(c,b)$ and $a \notin R(d,b)$. Thus R does not satisfy the axiom ***(ph)***.

Example 4.12 (***(m)***, ***(m')***, and ***(ph)***, but not ***(a')***)**.** Let $V = \{a, b, c, d\}$ and define R on V as follows:

$$R(x, x) = \{x\} \text{ for all } x \in V, \qquad R(a, b) = \{a, b, c\},$$
$$R(a, c) = \{a, c\}, \qquad R(b, c) = \{b, c\},$$
$$R(a, d) = \{a, c, d\} = R(c, d), \text{ and} \qquad R(b, d) = \{b, d\}.$$

It is easy to verify that R satisfies ***(m)***, ***(m')*** and ***(ph)***, but there is no $x, y \in V$ such that $R(x, y) = V$.

Example 4.13 (***(m)***, ***(a')***, and ***(ph)***, but not ***(m')***)**.** For $V = \{a, b, c, d, e\}$ define $R\colon V \times V \to 2^V$ as follows:

$$R(x, x) = \{x\} \text{ for all } x \in V, \qquad R(a, b) = \{a, b\},$$
$$R(a, c) = \{a, c\}, \qquad R(a, d) = R(c, d) = \{a, b, c, d\},$$
$$R(a, e) = R(b, e) = R(c, e) = \{a, b, c, e\}, \qquad R(b, c) = \{b, c\},$$
$$R(b, d) = \{b, d\}, \text{ and} \qquad R(d, e) = V.$$

We have $R(a, d) \cap R(a, e) = \{a, b, c\}$ and $R(x, y) \neq \{a, b, c\}$ for all $x, y \in V$. Therefore R does not satisfy ***(m')***. The transit sets $R(a, b)$, $R(a, c)$, $R(b, c)$, and $R(b, d)$ are subsets of $R(a, d) = R(c, d)$. Likewise, $R(a, b)$, $R(a, c)$, and $R(b, c)$ are subsets of $R(a, e) = R(b, e) = R(c, e)$. Thus R satisfies the axiom ***(m)***. Furthermore, R satisfies ***(a')*** because $R(d, e) = V$. In order to verify ***(ph)*** we observe the following:
$c, d \notin R(a, b)$: we check that $a \notin R(b, c) \Rightarrow b \in R(a, d)$ and $d \notin R(b, c) \Rightarrow a \in R(b, d)$.
$c, e \notin R(a, b)$: we check that $a \notin R(b, c) \Rightarrow b \in R(a, e)$ and $d \notin R(b, c) \Rightarrow a \in R(b, e)$.
$d, e \notin R(a, b)$: we check that $a \in R(b, d)$ and $e \notin R(b, d) \Rightarrow a \in R(b, e)$.
$b, d \notin R(a, c)$: we check that $a \notin R(b, c) \Rightarrow c \in R(a, d)$ and $d \notin R(b, c) \Rightarrow a \in R(c, d)$.
$b, e \notin R(a, c)$: we check that $a \notin R(b, c) \Rightarrow c \in R(a, e)$ and $e \notin R(b, c) \Rightarrow a \in R(c, e)$.
$d, e \notin R(a, c)$: we check that $a \in R(c, d)$ and $e \notin R(c, d) \Rightarrow c \in R(d, e)$.

Example 4.14 (***(m')***, ***(a')***, and ***(ph)***, but not ***(m)***)**.** Let $V = \{a, b, c, d\}$,

$$R(a, c) = \{a, b, c\}, \qquad R(x, x) = \{x\}$$

and $R(x, y) = V$ for all other pairs. Since $b \in R(a, c)$ and $R(a, b) = V \nsubseteq R(a, c)$, R does not satisfy axiom ***(m)***. We can easily prove that the other axioms hold.

Let us now return to the convexities identified by a transit function. Condition ***(a')*** can be seen as a weak version of the axiom of antipodality

(a) For every $x \in V$ there is $\bar{x} \in V$ such that $R(x, \bar{x}) = V$.

It is satisfied in particular by hierarchies. Since the restriction of a (weak) hierarchy to one of its clusters is again a (weak) hierarchy, the following axiom also holds for hierarchies:

(a") For every $u, v \in V$ and $x \in R(p, q)$ there is $\bar{x} \in V$ such that $R(x, \bar{x}) = R(u, v)$.

Axiom ***(a)***, and thus ***(a")*** does not hold for weak hierarchies in general, as the following counter-example shows. Suppose $C_1, C_2 \in \mathcal{W}$ are proper subsets of $V \in \mathcal{W}$ such that $C_1 \cup C_2 = V$ and $C_1 \cap C_2 = C_3 \in \mathcal{W}$. Choose $x \in C_3$, then $R(x, y) \subseteq C_1$ for

$y \in C_1$ and $R(x, y) \subseteq C_2$ for $y \in C_2$. Since $C_1 \cup C_2 = V$, there is no $\bar{x} \in V$ such that $R(x, \bar{x}) = V$. In fact, the set system $\mathcal{W}$ is even a paired hierarchy, since the only proper intersection is $C_1 \cap C_2$. Hence paired hierarchies, the most restrictive relaxation of a hierarchy studied in some detail in the literature [7], also do not satisfy ***(a")***.

Axiom ***(a")*** can be translated into conditions of the clusters of $\mathcal{W}$

Lemma 4.15. *Let $\mathcal{X}$ be a set system identified by a transit function R. Then R satisfies **(a")** if and only if $\mathcal{X}$ satisfies*

(KA) *For every $C \in \mathcal{X}$ and every collection $\mathcal{Q}_C \subseteq \{C' \in \mathcal{X} \mid C' \subset C\}$ with non-empty intersection $\bigcap_{C' \in \mathcal{Q}_C} C' \neq \emptyset$ holds $\bigcap_{C' \in \mathcal{Q}_C} C' \neq C$.*

Proof. Due to the correspondence of transit sets and members of $\mathcal{X}$, ***(a")*** is equivalently expressed as: For all $C \in \mathcal{X}$ and all $x \in C$, there is $\bar{x} \in C$ such that $\{x, \bar{x}\} \not\subseteq C'$ for any $C' \subset C$. Equivalently, the union of all $C' \subset C$ that contain any given x is a proper subset of C, which in turn is equivalent to ***(KA)***. □

In [17] we considered the axioms

(h') $x \in R(u, v) \Rightarrow R(u, x) = R(u, v)$ or $R(x, v) = R(u, v)$.

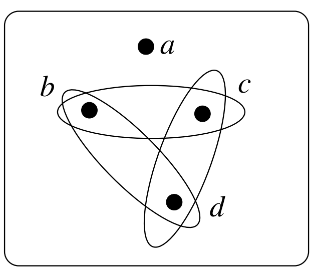
(h") $x \in R(u, v) \Rightarrow R(u, v) = R(u, x) \cup R(x, v)$.

Lemma 4.16. ***(h')*** $\wedge$ ***(m)*** *is equivalent to* ***(h")*** $\wedge$ ***(a")***.

Proof. We showed in [17] that ***(h")*** follows from ***(h')*** and monotone axiom. Furthermore, ***(h')*** implies ***(a")***: for given u, v and x we may choose $\bar{x} = u$ or $\bar{x} = v$.

Conversely, by ***(a")***, for every $x \in R(u, v)$ there is $\bar{x}$ in $R(u, v)$ such that $R(x, \bar{x}) = R(u, v)$. By ***(h")***, $R(u, x) \cup R(x, v) = R(u, v)$, hence $\bar{x} \in R(u, x)$ or $\bar{x} \in R(x, v)$, i.e., $R(u, v) \subseteq R(x, \bar{x}) \subseteq R(u, x)$ or $R(u, v) \subseteq R(x, \bar{x}) \subseteq R(x, v)$, i.e., ***(h')*** is satisfied. It was shown in [17] that ***(h")*** implies ***(m)***. □

The example in Figure 3 shows that ***(h")*** and ***(a")*** together are still insufficient to turn the transit sets into even a weak hierarchy.



R(a,b)=R(a,c)=R(a,d)={a,b,c,d}

R(b,c)={b,c}

R(c,d)={c,d}

R(b,d)={b,d}

Figure 3: The transit function R satisfies ***(m)***, ***(m')***, ***(a")***, and ***(h')***. It is not a weak hierarchy since $R(b, c) \cap R(b, d) \cap R(c, d) = \emptyset$ but the three pairwise intersection each contain a point, and thus also not a hierarchy.

5 Transit functions with arity > 2

Transit function or 2-ary functions have been generalized to k arguments to generalize convexities generated by k-ary functions [14]. For n-ary convexities, also refer [28].

Definition 5.1. A function $R\colon \underbrace{V \times V \ldots \times V}_{k \text{ times}} \to 2^V$ is a transit function of arity k (or k-ary transit function) on V if R satisfies the following axioms:

(kt1) $u_1 \in R(u_1, u_2, \ldots, u_k)$;

(kt2) $R(u_1, u_2, \ldots, u_k) = R(\pi(u_1, u_2, \ldots, u_k))$ for all $u_i \in V$, where $\pi(u_1, u_2, \ldots, u_k)$ is any permutation of $(u_1, u_2, \ldots, u_k)$;

(kt3) $R(u, u, \ldots, u) = \{u\}$ for all $u \in V$.

For an arbitrary set system $\mathcal{X}$ we define the k-ary function $R_{\mathcal{X}}\colon V^k \to 2^V$ by

$$R_{\mathcal{X}}(u_1, u_2, \ldots, u_k) = \bigcap \{A \in \mathcal{X} \mid u_1, u_2, \ldots, u_k \in A\}. \tag{5.1}$$

As in the case $k = 2$, the function R is monotone by construction, i.e., it satisfies

(km) For every $x_1, \ldots, x_k \in R(u_1, \ldots, u_k)$ holds $R(x_1, \ldots, x_k) \subseteq R(u_1, \ldots, u_k)$,

which in turn implies

(km*) For very $T = \{t_1, \ldots, t_k\}$ there is $Q = \{q_1, \ldots, q_k\}$ such that

$$\bigcap \{R(u_1, \ldots, u_k) \mid u_1, \ldots, u_k \in V \text{ and } T \in R(u_1, \ldots, u_k)\} = R(q_1, \ldots, q_k).$$

We are again interested in the case that the transit sets of R identify the set system $\mathcal{X}$, i.e.,

$$\mathcal{X} = \{R_{\mathcal{X}}(u_1, u_2, \ldots, u_k) \mid u_1, u_2, \ldots, u_k \in V\}. \tag{5.2}$$

Furthermore we consider the following generalizations of axioms ***(KR)*** and ***(KC)***:

(kKR) For all $C \in \mathcal{X}$ there is a set $T \subseteq C$ with $|T| \leq k$ such that $T \subseteq C'$ implies $C \subseteq C'$ for all $C' \in \mathcal{X}$.

(kKC) For every $T \subseteq V$ with $|T| \leq k$ holds $\bigcap \{C \in \mathcal{X} \mid T \subseteq C\} \in \mathcal{X}$.

Condition ***(kKR)*** was introduced in [14]. A necessary condition for R to explain $\mathcal{X}$ is that every set $C \in \mathcal{X}$ is identified by at most k distinct points. The following statement is a generalization of Lemma 2.2 and Theorem 2.6.

Theorem 5.2. *A set system $\mathcal{X} \subseteq 2^V$ is identified by a k-ary transit function if and only if $\mathcal{X}$ satisfies **(KS)**, **(kKC)**, and **(kKR)**. Conversely, a k-ary transit function identifies a set system if and only if R satisfies **(km)**.*

Proof. We argue in parallel with the proof of Lemma 2.2. First we note that ***(kt3)*** and ***(KS)*** ensure that $\mathcal{X}$ and the collection of transit sets both contain all singletons. As shown in [14], ***(kKR)*** is equivalent to $\mathcal{X} \subseteq \{R(u_1, \ldots, u_k) \mid u_1, \ldots, u_k \in V\}$. Property ***(km*)*** and the definition R together imply that ***(kKC)*** is equivalent to $\{R(u_1, \ldots, u_k) \mid u_1, \ldots, u_k \in V\} \subseteq \mathcal{X}$. The k-ary transit function defined by $\mathcal{X}$ is monotone. Conversely, one easily checks that the transit sets of a monotone k-ary transit function satisfy ***(KS)***, and ***(kKC)*** (by rewriting ***(km*)***), and ***(kKR)*** holds by construction. □

For a given set system $\mathcal{X}$, the minimal value of k for which ***(kKR)*** holds is called the *arity* of $\mathcal{X}$ [14].

Let $\mathcal{X}$ be a convexity on V. For a subset $S \subseteq V$, the smallest convex set containing S is the *convex hull* of S, denoted by

$$\langle S \rangle_{\mathcal{X}} = \bigcap \{C \in \mathcal{X} \mid S \subseteq C\}. \tag{5.3}$$

Basic concept from the theory of convexities can be generalized to arbitrary set systems:

Definition 5.3. Let $\mathcal{X} \subseteq 2^V$ be a set system. For every subset $S \subseteq V$, the closure of S with respect to a subset T (T-closure of S) is the set

$$\langle S \rangle_T = \bigcap \{C \in \mathcal{X} \mid S \subseteq C \text{ and } T \subseteq C\}. \tag{5.4}$$

Note that in general $\langle S \rangle_T \notin \mathcal{X}$. Interesting set systems arise by requiring $\langle S \rangle_T \in \mathcal{X}$ for all $S \subseteq V$ and certain sets T.

Definition 5.4. Let $\mathcal{X} \subseteq 2^V$. A set $S \subseteq V$ containing a subset T is *$\mathcal{X}_T$-independent* (T-independent) if $x \notin \bigcap\{C \in \mathcal{X} \mid T \subseteq (S \setminus \{x\}) \subseteq C\}$ holds for all $x \in S$. Otherwise S is T-dependent. In other words S is T-dependent if $x \in \langle S \setminus \{x\} \rangle_T$, for all $x \in S$. The T-rank $r(\mathcal{X})_T$ is the maximum cardinality of an $\mathcal{X}_T$-independent set S in V.

We can generalize the definition of well-known convexity invariants such as Carathéodory number with respect to the T-closure in a set system $\mathcal{X}$ identified with a k-ary transit function as follows.

Definition 5.5. The *T-Carathéodory number* c of a set system $\mathcal{X}$ is the smallest integer c (if it exists) such that for any finite subset F of V containing T, we have

$$\langle F \rangle_{\mathcal{X}_T} = \bigcup \{\langle S \rangle_{\mathcal{X}} \mid T \subseteq S \subseteq F, |S| \leq c\}. \tag{5.5}$$

The usual definition of the *Carathéodory number* is recovered by dropping the restrictions on T.

The *arity of a convexity* [14] is the smallest integer (which exists since V is finite in our setting) such that

$$\mathcal{X} = \{C \subseteq V \mid F \subset C, |F| \leq c \text{ implies } \langle F \rangle_{\mathcal{C}} \subseteq C\}. \tag{5.6}$$

By axiom ***(K2)***, $\langle S \rangle_{\mathcal{X}} \in \mathcal{X}$. Furthermore, ***(K2)*** implies ***(kKC)*** and every set system satisfies ***(kKR)*** for sufficiently large k. Every grounded convexity $\mathcal{X}$ is therefore identified by a transit function with sufficient arity. More precisely we have

Proposition 5.6 ([14]). *$\mathcal{X}$ is a grounded convexity on V of arity k if and only if $\mathcal{X}$ is identified by a k-ary transit function R on V.*

Lemma 5.7. *The k-ary transit function R identifies a convexity if and only if it satisfies **(km)** and the following two conditions:*

(ka') *There are vertices $u_1, u_2, \dots, u_k$ such that $R(u_1, u_2, \dots, u_k) = V$.*

(km') *If $R(u_1, u_2, \dots, u_k) \cap R(v_1, v_2, \dots, v_k) \neq \emptyset$ then there is $x_1, x_2, \dots, x_k$ such that $R(u_1, u_2, \dots, u_k) \cap R(v_1, v_2, \dots, v_k) = R(x_1, x_2, \dots, x_k)$.*

Proof. Monotonicity follows directly from the construction of R in Equation (5.1). Condition ***(km')*** states closure w.r.t. to intersection, i.e., is equivalent to ***(K2)***, and axiom ***(ka')*** is equivalent to ***(K1)***. □

6 k-weak hierarchies

We now turn to k-weak hierarchies [3, 10, 19] as a generalization of weak hierarchies.

Definition 6.1. A *k-weak hierarchy* on V is a set system $\mathcal{X} \in 2^V$ so that ***(K0)***, ***(K1)***, ***(KS)*** and one of the two equivalent conditions

(kWH) $A_1, A_2, A_3, \ldots, A_{k+1} \in \mathcal{W}$ implies $\bigcap_{i=1}^{k+1} A_i \subseteq \left\{ \bigcap_{i=1, i\neq j}^{k} A_i \,\middle|\, 1 \leq j \leq k+1 \right\}$.

(kWH') There are no $k+1$ elements $x_1, \ldots, x_{k+1}$ such that $x_i \in A_j$ iff $i \neq j$.

is satisfied.

The equivalence of ***(kWH)*** and ***(kWH')*** is established in [3, 19]. A k-weak hierarchy is closed (w.r.t. intersection) if in addition ***(K2)*** is satisfied. It is clear that a closed k-weak hierarchy is a convexity.

In the following we write $x_1, \ldots, \widehat{x_i}, \ldots, x_{k+1}$ for sequence that leaves out x_i, i.e., $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{k+1}$.

Lemma 6.2. *If the set system $\mathcal{X}$ is a closed k-weak hierarchy, then its rank satisfies $r(\mathcal{X}) \leq k$.*

Proof. Let $x_1, \ldots, x_{k+1}$ be distinct $k+1$ convexly independent points. Hence $x_i \notin \langle x_1, x_2, \ldots, \widehat{x_i}, \ldots, x_{k+1}\rangle_{\mathcal{X}}$. Let $C_i = \langle x_1, x_2, \ldots, \widehat{x_i}, \ldots, x_{k+1}\rangle_{\mathcal{X}}$. Then $x_i \notin \cap C_j$ for $i = 1, \ldots, k+1$. But some $x_i \in \bigcap_{i\neq j}^{k+1} C_j$. This is a contradiction to $\mathcal{X}$ being a closed k-weak hierarchy. □

It can be verified that if $\mathcal{X}$ is a closed k-weak hierarchy, then the Helly number and Carathéodory number of $\mathcal{X}$ is at most k.

Without recourse to the theory of convexities we can prove directly

Lemma 6.3. *A k-weak hierarchy satisfies **(kKR)**.*

Proof. First consider an arbitrary set system $\mathcal{X} \subseteq 2^V$ and fix a set $C \in \mathcal{X}$. Let $T \subseteq C$ be a set of minimum cardinality with the property that $T \subseteq C'$ implies $C \subseteq C'$ for all $C' \in \mathcal{X}$. Then for each $a \in T$ there is a set C_a such that $a \notin C_a$ and $T \setminus \{a\} \subseteq C_a$. This statement is a simple consequence of the minimality of T: For every $a \in T$, there must be sets $C' \in \mathcal{X}$ not containing C that do not contain a. Otherwise a could be removed from T, contradicting minimality. Now consider the set $\mathcal{X}_a$ of clusters that do not contain a. Suppose none of them contain $T \setminus \{a\}$. Then $C' \in \mathcal{X}_a$ also lacks some other element of $a' \in T$ and hence is recognizable as not containing C by virtue of a'. Thus a can be removed from T, contradicting minimality of T.

Now let $\mathcal{X}$ be a k-weak hierarchy.

Suppose there is a $C \in \mathcal{X}$ such that the minimal set T has cardinality $|T| \geq k+1$. Then there are at least $k+1$ distinct points a_i and corresponding clusters $C_i \in \mathcal{W}$ with $a_i \notin C_i$ and $T \setminus \{a_i\} \subseteq C_i$. The intersection $\bigcap_{i=1}^{k+1} C_i$ contains none of the a_i. By Axiom ***(kWH)***, however, this intersection can be written as the intersection of at most k of these clusters, and thus must contain at least one of the a_i, a contradiction. Thus the cardinality of T is at most k and the lemma follows. □

The set system defined by the transit sets is not necessarily a convexity. We can conclude, however, that every transit set of a k-weak hierarchy is the intersection of at most k others. We can generalize the notion of convex hulls and convex independence using the concept of a weak closure, we can show that for k-weak hierarchies, the axiom ***(kw')*** can be extended.

Now consider the following axioms:

(kw) For any $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{k+1} \in V^k$ holds

$$\bigcap_{i=1}^{k+1} R(\mathbf{x}_i) \subseteq \left\{ \bigcap_{i=1, i\neq j}^{k+1} R(\mathbf{x}_i) \,\middle|\, 1 \leq j \leq k+1 \right\}.$$

(kw') For every set of $k+1$ distinct points $x_1, \dots, x_{k+1} \in V$ holds

$$x_i \in R(x_1, x_2, \dots \widehat{x_i}, \dots, x_{k+1}).$$

Lemma 6.4. *Axioms **(kw')** and **(km)** imply **(a')**.*

Proof. Consider a set V with at least $k+1$ elements and let R be any k-ary transit function on V. If we have $R(x_1, \dots, x_k) = V$, we are done. Otherwise, there exists an element $x_{k+1} \in V$ that is not in $R(x_1, \dots, x_k)$. By axiom ***(kw')*** there exists i such that $x_i \in R(x_1, \dots, \widehat{x_i}, \dots, x_{k+1})$. Since R satisfies ***(km)***, we have $R(x_1, \dots, x_k) \subseteq R(x_1, \dots, \widehat{x_i}, \dots, x_{k+1})$. If $R(x_1, \dots, \widehat{x_i}, \dots, x_{k+1}) = V$, we are done. Otherwise we can find an element, say $x_{k+2} \in V \setminus R(x_1, \dots, \widehat{x_i}, \dots, x_{k+1})$. Again by the axiom ***(kw')***, there is some $x_j \in R(x_1, \dots, \widehat{x_i}, \dots, \widehat{x_j}, \dots, x_{k+1}, x_{k+2})$, and ***(km)*** implies

$$x_j \in R(x_1, \dots, \widehat{x_i}, \dots, x_j, \dots, x_{k+1}, x_{k+2}) \subseteq R(x_1, \dots, \widehat{x_i}, \dots, \widehat{x_j}, \dots, x_{k+1}, x_{k+2}).$$

If $R(x_1, \dots, \widehat{x_i}, \dots, \widehat{x_j}, \dots, x_{k+1}, x_{k+2}) = V$ then we are done. Otherwise we can repeat the argument. Since V contains only a finite number of elements, we eventually find a set of k elements, say $u_1, \dots, u_k$ from V so that $R(u_1, u_2, \dots, u_k) = V$, which proves the axiom ***(a')***. □

Lemma 6.5. *Let $\mathcal{X}$ be a k-weak hierarchy identified by a k-ary transit function R. Then the T-rank, $r(\mathcal{X}_T) \leq k$.*

Proof. Let $S = \{x_1, \dots, x_{k+1}\}$ be a T-independent set, i.e., for every $x_i \in S$ we have $x_i \notin \bigcap\{C \in \mathcal{X} \mid T \subseteq (S \setminus \{x_i\}) \subseteq C\}$. Let $C_i \in \mathcal{X}$ contains T and the set $S \setminus \{x_i\}$ for every i. Then $x_i \notin \cap C_j$ for $i = 1, \dots, k+1$. But some $x_i \in \bigcap_{i\neq j}^{k+1} C_j$. This is a contradiction to $\mathcal{X}$ being a k-weak hierarchy. □

Remark 6.6. If the set system $\mathcal{X}$ identified with a k-ary transit function is a k-weak hierarchy, then it can be shown easily that the T-Carathéodory number of $\mathcal{X}$ is at most k, as any set S with $|F| > k+1$ is T-dependent and hence any $x \in \langle F \rangle_{\mathcal{X}_T}$ belongs to $\langle F_i \rangle_{\mathcal{X}_T}$, where F_i is subset of F with $|F_i|$ at most k.

Lemma 6.7. *Let R be a k-ary transit function satisfying **(km)** and **(a')**. Then the axioms **(kw)** and **(kw')** are equivalent.*

Proof. Let R satisfies ***(kw)***. Then the set system $\mathcal{X}$ identified by R is a k-weak hierarchy. By definition of R, $R(x_1, \ldots, x_k) = \bigcap\{C_i \in \mathcal{X} \mid x_1, \ldots, x_k \in C_i\} = \langle x_1, \ldots, x_k \rangle_{\mathcal{X}}$. Now by Lemma 6.5, any distinct $k+1$ points, $x_1, \ldots, x_{k+1}$ are dependent with respect to some subset T contained in S. That is, $x_i \in R(x_1, \ldots, \widehat{x_i}, \ldots, x_{k+1})$ for some i. Hence R satisfies ***(kw')***.

Suppose R satisfies ***(kw')***. Proving that R satisfies ***(kw)*** is equivalent to showing that $\mathcal{X}$ is a k-weak hierarchy. Suppose this is not the case. Then we can find $k+1$ distinct elements $x_1, \ldots, x_{k+1} \in S$ and $C_1, \ldots, C_{k+1} \in \mathcal{X}$ such that $x_i \in C_j$ if and only if $i \neq j$. By axiom ***(kw')***, $x_i \in R(x_1, \ldots, \widehat{x_i}, \ldots, x_k)$ for some x_i, say x_{k+1}. Then $x_{k+1} \in C_i$ for $i = 1, \ldots, k$ and $x_{k+1} \notin C_{k+1}$. Since $x_1, \ldots, x_k \in C_{k+1}$, by definition of R, $R(x_1, \ldots, x_k) \subseteq C_{k+1}$. But $x_{k+1} \notin C_{k+1}$ and $x_{k+1} \in R(x_1, \ldots, x_k)$, a contradiction. Hence R satisfies ***(kw)***. □

Theorem 6.8. *The set system $\mathcal{X}$ induced by the k-ary transit function R on V is a closed k-weak hierarchy if and only R satisfies the axioms* **(m)***,* **(m')***, and* **(kw')***.*

Proof. Suppose $\mathcal{X}$ be a closed k-weak hierarchy induced by the k-ary transit function R on V. Since a closed k weak hierarchy is a convexity, R satisfies ***(m)***, ***(m')***. By Lemma 6.7 R satisfies ***(kw')***.

Conversely suppose R satisfies ***(m)***, ***(m')***, and ***(kw')***. Since R satisfies ***(m)*** and ***(kw')***, by Lemma 6.4, R satisfies (a'). Hence the transit sets form a convexity $\mathcal{X}$. By Lemma 6.7, R satisfies ***(kw)***. Hence $\mathcal{X}$ is a closed k-weak hierarchy. □

Remark 6.9. Axiom ***(kw)*** or ***(kw')*** is a direct translation of ***(kWH)***. The condition is necessary and sufficient given that ***(m)***, ***(m')***, ***(a')*** are equivalent to the transit sets forming a convexity.

We briefly discuss the mutual dependencies between the axioms ***(km)***, ***(km')***, ***(kw)***, ***(kw')***, and ***(a')***. We already know that ***(km)*** and ***(kw')*** implies ***(a')***. Furthermore, if ***(km)*** and ***(a')*** are satisfied, then ***(kw)*** and ***(kw')*** are equivalent.

Example 6.10 (***(km)***, ***(a')***, and ***(km')***, but not ***(kw)*** and ***(kw')***)**.** Let $V = \{x_1, x_2, \ldots, x_{k+1}\}$. Define R on V as follows:

$$R(a_1, a_2, \ldots, a_k) = V,$$

for all other k-tuples

$$R(x_1, x_2, \ldots, x_k) = \{x_1, x_2, \ldots, x_k\}.$$

Example 6.11 (***(km)***, ***(a')***, ***(kw)***, and ***(kw')***, but not ***(km')***)**.** Let $V = \{x_1, x_2, \ldots, x_{k+1}\}$. Let

$$\begin{aligned}
R(x_1, x_3, x_3, \ldots, x_3) &= V, \\
R(x_1, x_4, x_4, \ldots, x_4) &= V - \{x_3\}, \\
R(x_3, x_5, x_5, \ldots, x_5) &= V - \{x_1\}, \\
R(x_1, x_2, x_2, \ldots, x_2) &= \{x_1, x_2\}, \\
R(x_1, x_5, x_5, \ldots, x_5) &= \{x_1, x_5\}, \\
R(x_2, x_2, x_2, \ldots, y) &= \{x_2, y\} \text{ for all } y \in V
\end{aligned}$$

and set $R(x_1, x_2, x_3, \ldots, x_k) = V$ for all other k-tuples. We can see that

$$R(x_1, x_4, x_4, \ldots, x_4) \cap R(x_3, x_5, x_5, \ldots, x_5) = \{x_2, x_4, \ldots, x_{k+1}\}$$

but there is no k-tuple whose R-image is $\{x_2, x_4, \ldots, x_{k+1}\}$.

Example 6.12 (***(km')***, ***(a')***, ***(kw)***, and ***(kw')***, but not ***(km)***)**.** Consider $V = \{x_1, x_2, \ldots, x_{k+1}\}$ and set

$$\begin{aligned}
R(x_1, x_2, \ldots, x_2) &= \{x_1, x_2, x_3, x_4\}, \\
R(x_2, x_3, \ldots, x_3) &= \{x_2, x_3, x_4\}, \\
R(x_2, x_4, \ldots, x_4) &= \{x_2, x_4\},
\end{aligned}$$

and $R(x_1, x_2, \ldots x_k) = V$ for all other k-tuples. Since $R(x_3, x_4, \ldots x_4) = V$ we can see that R does not satisfy the axiom ***(km)***.

Example 6.13 (***(km)***, ***(km')***, and ***(kw)***, but not ***(a')***)**.** $V = \{x_1, x_2, \ldots, x_{k+1}\}$, consider R on V by

$$\begin{aligned}
R(x_1, x_2, \ldots, x_2) &= \{x_1, x_2\}, \\
R(x_1, x_3, \ldots, x_3) &= \{x_1, x_2, x_3\}, \\
R(x_1, x_4, \ldots, x_4) &= \{x_1, x_2, x_4\}, \\
R(x_2, x_3, \ldots, x_3) &= \{x_2, x_3\}, \\
R(x_2, x_4, \ldots, x_4) &= \{x_2, x_4\}, \\
R(x_3, x_4, \ldots, x_4) &= \{x_3, x_4\},
\end{aligned}$$

and set $R(u_1, u_2, \ldots u_k) = \{u_1, u_2, \ldots u_k, x_3\}$ for all other k-tuples $(u_1, u_2, \ldots u_k)$.

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