ISSN 2590-9770
The Art of Discrete and Applied Mathematics 2 (2019) #P1.04
https://doi.org/10.26493/2590-9770.1246.99c
(Also available at http://adam-journal.eu)

Optimal orientations of strong products of paths

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Received 9 March 2018, accepted 13 July 2018, published online 8 August 2018

Abstract

Let $\mathrm{diam}_{\min}(G)$ denote the minimum diameter of a strong orientation of G and let $G \boxtimes H$ denote the strong product of graphs G and H. In this paper we prove that $\mathrm{diam}_{\min}(P_m \boxtimes P_n) = \mathrm{diam}(P_m \boxtimes P_n)$ for $m, n \geq 5$, $m \neq n$, and $\mathrm{diam}_{\min}(P_m \boxtimes P_n) = \mathrm{diam}(P_m \boxtimes P_n) + 1$ for $m, n \geq 5$, $m = n$. We also prove that $\mathrm{diam}_{\min}(G \boxtimes H) \leq \max\{\mathrm{diam}_{\min}(G), \mathrm{diam}_{\min}(H)\}$ for any connected bridgeless graphs G and H.

Keywords: Diameter, strong orientation, strong product.

Math. Subj. Class.: 05C12, 05C76

1 Introduction

Let $D = (V(D), A(D))$ be a directed graph. If $(u, v) \in A(D)$, we write $u \to v$. A *uv-path* is a directed path $u = u_1u_2 \dots u_n = v$ from a vertex u to a vertex v. The *length* of the path $u = u_1u_2 \dots u_n = v$ is $n - 1$. If every vertex in D is reachable from every other vertex in D, we say that directed graph D is *strong* (there is a directed uv-path in D for every $u, v \in V(D)$). The *distance* from u to v is the length of a shortest directed uv-path in D, denoted by $\mathrm{dist}_D(u, v)$. The greatest distance among all pairs of vertices in D is the diameter of D, so

$$\mathrm{diam}(D) = \max\{\mathrm{dist}_D(u, v) \mid u, v \in V(D)\}.$$

Note that the distance of two vertices u, v in undirected graph G, $\mathrm{dist}_G(u, v)$, is the length of a shortest undirected uv-path in G and the greatest distance between any two vertices in G is the diameter of G, denoted by $\mathrm{diam}(G)$.

Let G be an undirected graph. An *orientation* of G is a digraph D obtained from G by assigning to each edge in G a direction. Let $\mathcal{D}(G)$ denote the family of all strong orientations of G. In [9] it is proved that every connected bridgeless graph admits a strong orientation. We define the minimum diameter of a strong orientation of G as

$$\mathrm{diam}_{\min}(G) = \min\{\mathrm{diam}(D) \mid D \in \mathcal{D}(G)\}.$$

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The parameter $\operatorname{diam}_{\min}(G)$ was studied by many authors, because it is important from theoretical and practical points of view, as an application in traffic control problems. Orientations of graphs can be viewed as arrangements of one-way streets, if G is thought of as the system of two-way streets in a city, and we want to make every street in the city one-way and still get from every point to every other point (see [9, 10]).

For every bridgeless connected graph G of radius r it was shown, see [1], that $\operatorname{diam}_{\min}(G) \le 2r^2 + 2r$. There were also some determined values of the minimum diameter of a strong orientation of the Cartesian product of graphs. For Cartesian product of two paths it was proved that $\operatorname{diam}_{\min}(P_m \square P_n) = \operatorname{diam}(P_m \square P_n)$, for $m \ge 3$ and $n \ge 6$, see [5]. In [8] it was proved that $\operatorname{diam}_{\min}(C_m \square C_n) = \operatorname{diam}(C_m \square C_n)$ for $m, n \ge 6$. In [7] Koh and Tay proved that $\operatorname{diam}_{\min}(T_1 \square T_2) = \operatorname{diam}(T_1 \square T_2)$ for trees T_1 and T_2 with diameters at least 4. They also studied the diameter of orientations of $K_m \square K_n$, $K_m \square P_n$, $P_m \square C_n$ and $K_m \square C_n$ (see [4, 5, 6]).

In [3], the upper bound for the strong radius and the strong diameter of Cartesian product of graphs are determined.

In this article we consider the minimum diameter of strong orientations of strong products of graphs. The *strong product* of graphs G and H is the graph, denoted by $G \boxtimes H$, with the vertex set $V(G \boxtimes H) = V(G) \times V(H)$ where two distinct vertices (u, v) and (u', v') are adjacent in $G \boxtimes H$ if and only if $uu' \in E(G)$ and $v = v'$, or $u = u'$ and $vv' \in E(H)$, or $uu' \in E(G)$ and $vv' \in E(H)$. For $v \in V(H)$ we define the G-layer G_v:

$$G_v = \{(u, v) \mid u \in V(G)\}.$$

Analogously we define H-layers.

In the next section we prove that $\operatorname{diam}_{\min}(P_m \boxtimes P_n) = \operatorname{diam}(P_m \boxtimes P_n)$, for $m, n \ge 5$, $m \ne n$ and that $\operatorname{diam}_{\min}(P_m \boxtimes P_n) = \operatorname{diam}(P_m \boxtimes P_n) + 1$, for $m, n \ge 5$, $m = n$.

2 Orientations of $P_m \boxtimes P_n$

In [7] Koh and Tay proved that $\operatorname{diam}_{\min}(P_m \square P_n) = \operatorname{diam}(P_m \square P_n)$, for $m \ge 5$ and $n \ge 5$. We use some of their notations. So we will define four sections of $V(P_m \boxtimes P_n)$ and two basic orientations of $P_s \boxtimes P_t$, where $s, t \ge 3$, similarly as it was introduced in [7]. For $m, n \ge 5$ we define

(i) Southwest Section SW $= \left\{(i, j) \mid 1 \le i \le \left\lceil \frac{m}{2} \right\rceil, 1 \le j \le \left\lceil \frac{n}{2} \right\rceil \right\}$;

(ii) Northwest Section NW $= \left\{(i, j) \mid 1 \le i \le \left\lceil \frac{m}{2} \right\rceil, \left\lceil \frac{n+1}{2} \right\rceil \le j \le n \right\}$;

(iii) Southeast Section SE $= \left\{(i, j) \mid \left\lceil \frac{m+1}{2} \right\rceil \le i \le m, 1 \le j \le \left\lceil \frac{n}{2} \right\rceil \right\}$;

(iv) Northeast Section NE $= \left\{(i, j) \mid \left\lceil \frac{m+1}{2} \right\rceil \le i \le m, \left\lceil \frac{n+1}{2} \right\rceil \le j \le n \right\}$.

We define two basic orientations of $P_s \boxtimes P_t$, where $s, t \ge 3$: if $s \le t$, we define the orientation F_1 of $P_s \boxtimes P_t$ as:

(i) For $1 \le i \le s - 1$ and $2 \le j \le t$, $(i, j) \to (i + 1, j - 1)$;

(ii) For $1 \le i \le s - 1$ and $1 \le j \le t - 1$, $(i + 1, j + 1) \to (i, j)$ if $j - i \ge t - s$ and $(i, j) \to (i + 1, j + 1)$ if $j - i < t - s$;

(iii) For $1 \le i \le s - 1$ and $2 \le j \le t$, $(i, j) \to (i, j - 1)$;

(iv) For $1 \le j \le t - 1$, $(s, j) \to (s, j + 1)$;

(v) For $1 \leq i \leq s-1$ and $1 \leq j \leq t-1$, $(i,j) \to (i+1,j)$;

(vi) For $2 \leq i \leq s$, $(i,t) \to (i-1,t)$;

and if $s > t$, we define the orientation F_2 of $P_s \boxtimes P_t$ as:

(i) For $2 \leq i \leq s$ and $1 \leq j \leq t-1$, $(i,j) \to (i-1,j+1)$;

(ii) For $1 \leq i \leq s$ and $1 \leq j \leq t$, $(i+1,j+1) \to (i,j)$ if $i-j \geq s-t$ and $(i,j) \to (i+1,j+1)$ if $i-j < s-t$;

(iii) For $1 \leq i \leq s-1$ and $1 \leq j \leq t-1$, $(i,j) \to (i,j+1)$;

(iv) For $2 \leq j \leq t$, $(s,j) \to (s,j-1)$;

(v) For $2 \leq i \leq s$ and $1 \leq j \leq t-1$, $(i,j) \to (i-1,j)$;

(vi) For $1 \leq i \leq s-1$, $(i,t) \to (i+1,t)$.

The orientation F_1 of $P_3 \boxtimes P_4$ and the orientation F_2 of $P_4 \boxtimes P_3$ is shown in Figure 1.

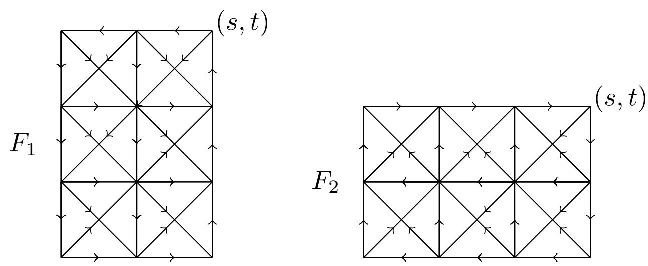

Figure 1: Orientations F_1 and F_2.

Observation 2.1. *If $s < t$, for any $(i,j) \in V(F_1)$,* $\mathrm{dist}_{F_1}((i,j),(s,t-1)) \leq t-2$.

Proof. Let $(i,j) \in V(F_1)$. We shall consider four cases.

(i) If $j \neq t$ and $j \geq i+t-s-1$, then $(i,j) \to (i+1,j) \to \cdots \to (j-(t-s)+1,j) \to (j-(t-s)+2,j+1) \to \cdots \to (s,t-1)$ is a path of length at most $s-1 \leq t-2$.

(ii) If $j \neq t$ and $j < i+t-s-1$, then $(i,j) \to (i+1,j+1) \to \cdots \to (s,j+s-i) \to (s,j+s-i+1) \to \cdots \to (s,t-1)$ is a path of length at most $t-2$.

(iii) If $j = t$ and $i \neq s$, then $(i,t) \to (i+1,t-1) \to (i+2,t-1) \to \cdots \to (s,t-1)$ is a path of length at most $s-1 \leq t-2$.

(iv) If $j = t$ and $i = s$, then $(s,t) \to (s-1,t-1) \to (s,t-1)$ is a path of length two. □

Observation 2.2. *If $s < t$, for any $(i,j) \in V(F_1)$,* $\mathrm{dist}_{F_1}((i,j),(s,t)) \leq t-1$.

Proof. Since $(s,t-1) \to (s,t)$, the claim follows by Observation 2.1:

$$\mathrm{dist}_{F_1}((i,j),(s,t)) = \mathrm{dist}_{F_1}((i,j),(s,t-1)) + 1 \leq s-1+1 \leq t-1. \qquad \square$$

Observation 2.3. *If $s < t$, for any $(i,j) \in V(F_1)$,* $\mathrm{dist}_{F_1}((s-1,t),(i,j)) \leq t-1$.

Proof. Let $(i, j) \in V(F_1)$. We shall consider four cases.

(i) If $i \neq s$ and $j > i + t - s$, then $(s-1, t) \to (s-2, t) \to \cdots \to (i + (t-j), t) \to (i + (t-j) - 1, t-1) \to \cdots \to (i, j)$ is a path of length at most $s - 2 \leq t - 2$.

(ii) If $i \neq s$ and $j \leq i + t - s$, then $(s-1, t) \to (s-1, t-1) \to (s-2, t-2) \to \cdots \to (i, i+t-s) \to (i, i+t-s-1) \to \cdots \to (i, j)$ is a path of length at most $t - 1$.

(iii) If $i = s$ and $j \neq t$, $(s-1, t) \to (s-1, t-1) \to (s-1, t-2) \to \cdots \to (s-1, j+1) \to (s, j)$ is a path of length at most $t - 1$.

(iv) If $i = s$ and $j = t$, then $(s-1, t) \to (s, t-1) \to (s, t)$ is a path of length two. □

Observation 2.4. *If $s < t$, for any $(i, j) \in V(F_1)$, $\mathrm{dist}_{F_1}((s, t), (i, j)) \leq t - 1$.*

Proof. Since $(s, t) \to (s-1, t)$ and $(s, t) \to (s-1, t-1)$, the proof is similar as the proof of Observation 2.3. □

Observation 2.5. *If $s = t$, for any $(i, j) \in V(F_1)$, $\mathrm{dist}_{F_1}((i, j), (s, s)) \leq s$.*

Proof. Let $(i, j) \in V(F_1)$. We shall consider three cases.

(i) If $j \neq t$ and $j \geq i-1$, then $(i, j) \to (i+1, j) \to \cdots \to (j+1, j) \to (j+2, j+1) \to \cdots \to (s, s-1) \to (s, s)$ is a path of length at most s.

(ii) If $j \neq t$ and $j < i - 1$, then $(i, j) \to (i+1, j+1) \to \cdots \to (s, j+s-i) \to (s, j+s-i+1) \to \cdots \to (s, s)$ is a path of length at most $s - 1$.

(iii) If $j = s$ and $i \neq s$, then $(i, s) \to (i+1, s-1) \to (i+2, s-1) \to \cdots \to (s, s-1) \to (s, s)$ is a path of length at most s. □

Observation 2.6. *If $s = t$, for any $(i, j) \in V(F_1)$, $\mathrm{dist}_{F_1}((s, s), (i, j)) \leq s - 1$.*

Proof. Let $(i, j) \in V(F_1)$. We shall consider three cases.

(i) If $i \neq s$ and $j > i$, then $(s, s) \to (s-1, s) \to \cdots \to (i + (s-j), s) \to (i + (s-j) - 1, t-1) \to \ldots \to (i, j)$ is a path of length at most $s - 1$.

(ii) If $i \neq s$ and $j \leq i$, then $(s, s) \to (s-1, s-1) \to \cdots \to (i, i) \to (i, i-1) \to \ldots \to (i, j)$ is a path of length at most $s - 1$.

(iii) If $i = s$ and $j \neq s-1$, $(s, s) \to (s-1, s-1) \to (s-1, s-2) \to \cdots \to (s-1, j+1) \to (s, j)$ is a path of length at most $s - 1$.

(iv) If $i = s$ and $j = s - 1$, then $(s, s) \to (s-1, s-1) \to (s, s-1)$ is a path of length two. □

Similarly as above, we can prove next Observations 2.7–2.10.

Observation 2.7. *If $s > t$, for any $(i, j) \in V(F_2)$, $\mathrm{dist}_{F_2}((s, t-1), (i, j)) \leq s - 1$.*

Observation 2.8. *If $s > t$, for any $(i, j) \in V(F_2)$, $\mathrm{dist}_{F_2}((s, t), (i, j)) \leq s - 1$.*

Observation 2.9. *If $s > t$, for any $(i, j) \in V(F_2)$, $\mathrm{dist}_{F_2}((i, j), (s-1, t)) \leq s - 2$.*

Observation 2.10. *If $s > t$, for any $(i, j) \in V(F_2)$, $\mathrm{dist}_{F_2}((i, j), (s, t)) \leq s - 1$.*

In [7], Koh and Tay also introduced a key-vertex $v \in V(F)$ of digraph F. Let $F \in \mathcal{D}(P_s \boxtimes P_t)$. We say that a vertex $v \in V(F)$ is a *key-vertex* of F if

$$\operatorname{dist}_F(u,v) \le \max\{t,s\} \quad \text{and} \quad \operatorname{dist}_F(v,u) \le \max\{t,s\}$$

for all $u \in V(F)$. Note that (s,t) is a key-vertex of F_1 and of F_2.

Analogously as F_1 and F_2, we define 6 other isomorphic orientations F_i, $3 \le i \le 8$ of $P_s \boxtimes P_t$ as shown in Figures 2 and 3.

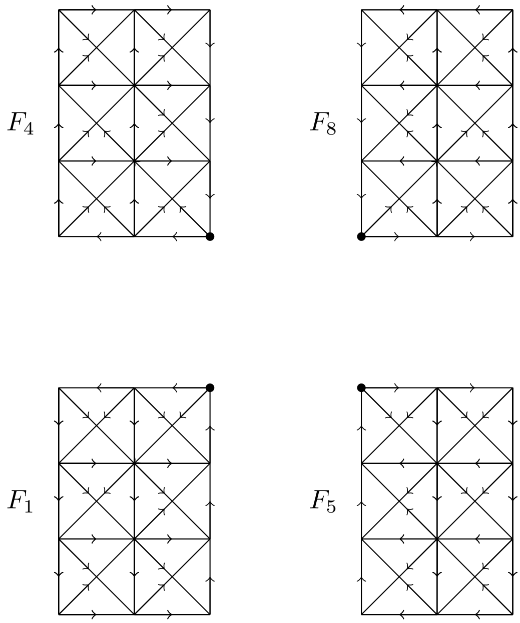

Figure 2: Orientations F_1, F_4, F_5 and F_8.

Obviously vertices denoted by black dots in Figures 2 and 3 are key-vertices of F_i for $i = 1, \ldots, 8$ (similar arguments as in Observations 2.1–2.6).

Lemma 2.11. *Let $m, n \ge 5$, $m \ne n$ and $m, n \equiv 1 \pmod 2$. Then*

$$\operatorname{diam}_{\min}(P_m \boxtimes P_n) \le \max\{m-1, n-1\}.$$

Proof. Let $m < n$. We define the orientation D of $P_m \boxtimes P_n$ by F_1, F_4, F_5 and F_8:

(a) orient the section NW as F_4;

(b) orient the section NE as F_8;

(c) orient the section SW as F_1;

(d) orient the section SE as F_5.

As an illustration, the orientation of $P_5 \boxtimes P_7$ is shown in Figure 4. The vertex $z = (\frac{m+1}{2}, \frac{n+1}{2})$ is the key-vertex of each F_i, for $i = 1, 4, 5, 8$. For any $u, v \in V(D)$,

$$\operatorname{dist}_D(u,v) \le \operatorname{dist}_D(u,z) + \operatorname{dist}_D(z,v).$$

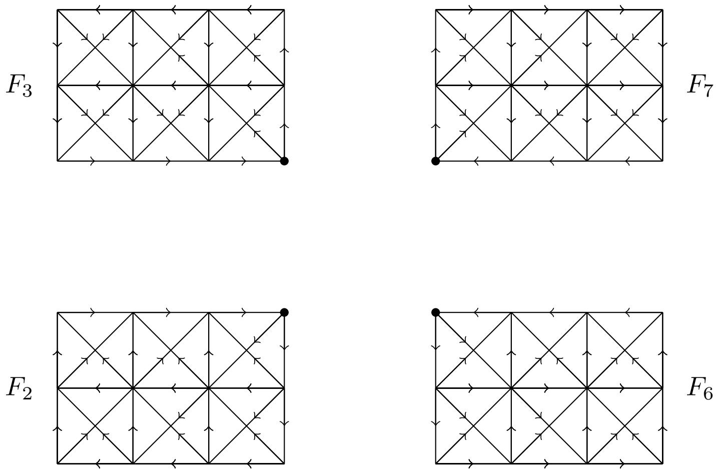

Figure 3: Orientations F_2, F_3, F_6 and F_7.

Since $\text{dist}_D(u, z) \leq \frac{n-1}{2}$ and $\text{dist}_D(z, v) \leq \frac{n-1}{2}$ (similarly as in Observation 2.2 and Observation 2.4), we have

$$\text{dist}_D(u, v) \leq \frac{n-1}{2} + \frac{n-1}{2} = n - 1.$$

If $m > n$ we define the orientation D of $P_m \boxtimes P_n$ by F_2, F_3, F_6 and F_7. Similarly as above, we have

$$\text{dist}_D(u, v) \leq \text{dist}_D(u, z) + \text{dist}_D(z, v) \leq \frac{m-1}{2} + \frac{m-1}{2} = m - 1$$

(see Observation 2.10 and Observation 2.8). □

Lemma 2.12. *Let* $m, n \geq 6$*,* $m \neq n$ *and* $m, n \equiv 0 \pmod 2$*. Then*

$$\text{diam}_{\min}(P_m \boxtimes P_n) \leq \max\{m - 1, n - 1\}.$$

Proof. Let $m < n$. Denote $z_1 = (\frac{m}{2}, \frac{n}{2})$, $z_4 = (\frac{m}{2}, \frac{n}{2} + 1)$, $z_5 = (\frac{m}{2} + 1, \frac{n}{2})$ and $z_8 = (\frac{m}{2} + 1, \frac{n}{2} + 1)$. We define the orientation D of $P_m \boxtimes P_n$ by F_1, F_4, F_5 and F_8 as follows:

(a) orient the section NW as F_4;

(b) orient the section NE as F_8;

(c) orient the section SW as F_1;

(d) orient the section SE as F_5;

(e) Orient $z_1 \to (\frac{m}{2}-1, \frac{n}{2}+1)$, $(\frac{m}{2}+1, \frac{n}{2}-1) \to z_1$, $z_4 \to (\frac{m}{2}-1, \frac{n}{2})$, $(\frac{m}{2}+1, \frac{n}{2}+2) \to z_4$, $z_5 \to (\frac{m}{2} + 2, \frac{n}{2} + 1)$, $(\frac{m}{2}, \frac{n}{2} - 1) \to z_5$, $z_8 \to (\frac{m}{2} + 2, \frac{n}{2})$, $(\frac{m}{2}, \frac{n}{2} + 2) \to z_8$, and orient all other edges arbitrarily.

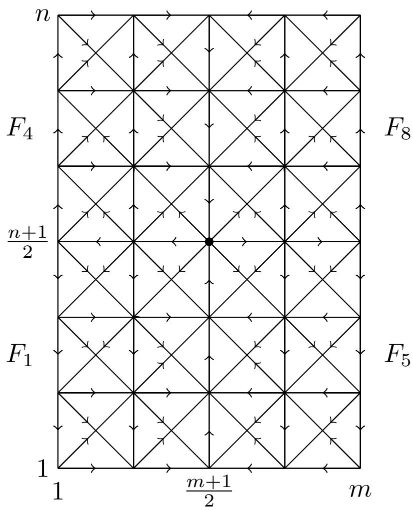

Figure 4: The orientation D of $P_5 \boxtimes P_7$.

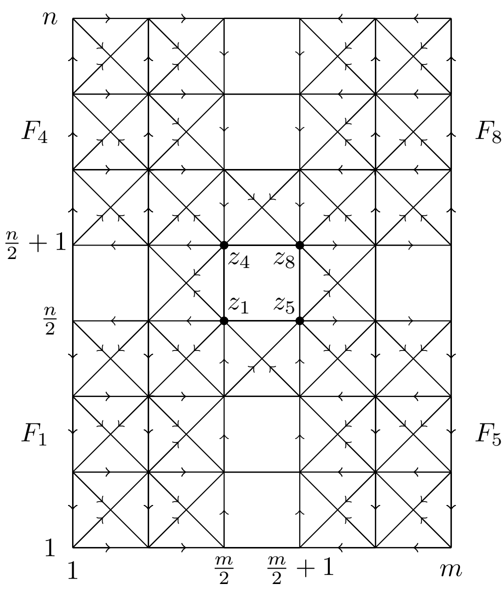

Figure 5: The orientation D of $P_6 \boxtimes P_8$.

The orientation D is shown in Figure 5. Note that vertices z_1, z_4, z_5 and z_8 are key-vertices of F_i, for $i = 1, 4, 5, 8$.

Let $u, v \in V(D)$. We claim that $\mathrm{dist}_D(u, v) \le n - 1$. There are four cases.

(i) If u and v are in the same section, then we have

$$\mathrm{dist}_D(u, v) \le \mathrm{dist}_D(u, z_i) + \mathrm{dist}_D(z_i, v) \le \frac{n}{2} - 1 + \frac{n}{2} - 1 = n - 2$$

as in Observation 2.2 and Observation 2.4.

(ii) If $u \in$ NW and $v \in$ SW, then (see Observation 2.2 and Observation 2.3):

$$\begin{aligned}\mathrm{dist}_D(u, v) &\le \mathrm{dist}_D(u, z_4) + \mathrm{dist}_D\left(z_4, (\tfrac{m}{2} - 1, \tfrac{n}{2})\right) + \mathrm{dist}_D\left((\tfrac{m}{2} - 1, \tfrac{n}{2}), v\right) \\ &\le \frac{n}{2} - 1 + 1 + \frac{n}{2} - 1 = n - 1.\end{aligned}$$

The argument is similar if $u \in$ SW and $v \in$ NW, or $u \in$ NE and $v \in$ SE, or $u \in$ SE and $v \in$ NE.

(iii) If $u \in$ SW and $v \in$ SE, then the claim follows from Observation 2.1 and Observation 2.4, similarly as above. Also, if $u \in$ SE and $v \in$ SW, or $u \in$ NW and $v \in$ NE, or $u \in$ NE and $v \in$ NW, then the argument is analogous.

(iv) If $u \in$ SW and $v \in$ NE, then (see Observation 2.1 and Observation 2.3) we have

$$\begin{aligned}\mathrm{dist}_D(u, v) \le\ & \mathrm{dist}_D\left(u, (\tfrac{m}{2}, \tfrac{n}{2} - 1)\right) + \mathrm{dist}_D\left((\tfrac{m}{2}, \tfrac{n}{2} - 1), z_5\right) + \\ & + \mathrm{dist}_D\left(z_5, (\tfrac{m}{2} + 2, \tfrac{n}{2} + 1)\right) + \mathrm{dist}_D\left((\tfrac{m}{2} + 2, \tfrac{n}{2} + 1), v\right) \\ \le\ & \frac{n}{2} - 2 + 1 + 1 + \frac{n}{2} - 1 = n - 1.\end{aligned}$$

The argument is similar for $u \in$ NE and $v \in$ SW, or $u \in$ NW and $v \in$ SE, or $u \in$ SE and $v \in$ NW.

Analogously if $m > n$, we have $\mathrm{dist}_D(u, v) \le m - 1$ for any $u, v \in V(D)$. □

Lemma 2.13. *Let* $m \ge 5$, $n \ge 6$, $m \equiv 1 \pmod 2$ *and* $n \equiv 0 \pmod 2$. *Then*

$$\mathrm{diam}_{\min}(P_m \boxtimes P_n) \le \max\{m - 1, n - 1\}. \tag{2.1}$$

Proof. Let $m < n$. Denote $z_1 = (\frac{m+1}{2}, \frac{n}{2})$ and $z_4 = (\frac{m+1}{2}, \frac{n}{2} + 1)$. We define the orientation D of $P_m \boxtimes P_n$ by F_1, F_4, F_5 and F_8 as follows:

(a) orient the section NW as F_4;

(b) orient the section NE as F_8;

(c) orient the section SW as F_1;

(d) orient the section SE as F_5;

(e) orient $z_4 \to (\frac{m+1}{2} - 1, \frac{n}{2})$, $z_1 \to z_4$, $z_4 \to (\frac{m+1}{2} + 1, \frac{n}{2})$, and orient all other edges arbitrarily.

The orientation D is shown in Figure 6. Note that vertex z_1 is a key-vertex of F_1 and F_5 and that vertex z_4 is a key-vertex of F_4 and F_8.

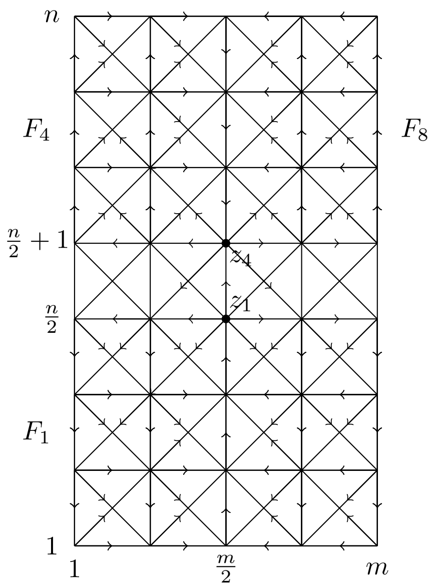

Figure 6: The orientation D of $P_5 \boxtimes P_8$.

Let $u, v \in V(D)$. There are three cases.

(i) If $u \in \mathrm{NW} \cup \mathrm{NE}$ and $v \in \mathrm{NW} \cup \mathrm{NE}$, then we have

$$\mathrm{dist}_D(u,v) \le \mathrm{dist}_D(u,z_4) + \mathrm{dist}_D(z_4,v) \le \frac{n}{2} - 1 + \frac{n}{2} - 1 = n - 2$$

(see Observation 2.2 and Observation 2.4). The case that $\{u,v\} \subseteq \mathrm{SW} \cup \mathrm{SE}$ is similar.

(ii) If $u \in \mathrm{SW} \cup \mathrm{SE}$ and $v \in \mathrm{NW} \cup \mathrm{NE}$, then (see Observation 2.2 and Observation 2.4):

$$\begin{aligned}\mathrm{dist}_D(u,v) &\le \mathrm{dist}_D(u,z_1) + \mathrm{dist}_D(z_1,z_4) + \mathrm{dist}_D(z_4,v)\\ &\le \frac{n}{2} - 1 + 1 + \frac{n}{2} - 1 = n - 1.\end{aligned}$$

(iii) If $u \in \mathrm{NW} \cup \mathrm{NE}$ and $v \in \mathrm{SW}$, then from Observation 2.2 and Observation 2.3:

$$\begin{aligned}\mathrm{dist}_D(u,v) &\le \mathrm{dist}_D(u,z_4) + \mathrm{dist}_D\left(z_4, \left(\tfrac{m+1}{2} - 1, \tfrac{n}{2}\right)\right) + \\ &\quad + \mathrm{dist}_D\left(\left(\tfrac{m+1}{2} - 1, \tfrac{n}{2}\right), v\right)\\ &\le \frac{n}{2} - 1 + 1 + \frac{n}{2} - 1 = n - 1.\end{aligned}$$

The case that $u \in \mathrm{NW} \cup \mathrm{NE}$ and $v \in \mathrm{SE}$ is similar.

Let $m > n$. Denote $z_2 = (\frac{m+1}{2}, \frac{n}{2})$ and $z_3 = (\frac{m+1}{2}, \frac{n}{2} + 1)$. We define the orientation D of $P_m \boxtimes P_n$ by F_2, F_3, F_6 and F_7 as follows:

(a) orient the section NW as F_3;

(b) orient the section NE as F_7;

(c) orient the section SW as F_2;

(d) orient the section SE as F_6;

(e) orient $(\frac{m+1}{2}-1, \frac{n}{2}) \to z_3$, $z_3 \to z_2$, $(\frac{m+1}{2}+1, \frac{n}{2}) \to z_3$ and all other edges oriented arbitrarily.

The rest of the proof is analogously as above. □

Note that if $m \geq 5$ and $n \geq 6$, $m \equiv 0 \pmod 2$ and $n \equiv 1 \pmod 2$, we also have (2.1).

Lemma 2.14. *Let* $m \geq 5$, $m \equiv 1 \pmod 2$. *Then*

$$\mathrm{diam}_{\min}(P_m \boxtimes P_m) \leq m.$$

Proof. Denote $z = (\frac{m+1}{2}, \frac{m+1}{2})$. We define the orientation D of $P_m \boxtimes P_m$ by F_1, F_4, F_5 and F_8 as follows:

(a) orient the section NW as F_4;

(b) orient the section NE as F_8;

(c) orient the section SW as F_1;

(d) orient the section SE as F_5.

Note that z is a key-vertex of F_i, for $i = 1, 4, 5, 8$. For any $u, v \in D$ we have

$$\mathrm{dist}_D(u, v) \leq \mathrm{dist}_D(u, z) + \mathrm{dist}_D(z, v) \leq \frac{m+1}{2} + \frac{m-1}{2} = m$$

as in Observation 2.5 and Observation 2.6. □

Lemma 2.15. *Let* $m \geq 6$, $m \equiv 0 \pmod 2$. *Then*

$$\mathrm{diam}_{\min}(P_m \boxtimes P_m) \leq m.$$

Proof. The proof is similarly as the proof of Lemma 2.12 (it follows from Observations 2.1, 2.3, 2.5 and 2.6). □

In [2], it is proved that if (u, v) and (u', v') are vertices of a strong product $G \boxtimes H$, then

$$\mathrm{dist}_{G \boxtimes H}((u, v), (u', v')) = \max\{\mathrm{dist}_G(u, u'), \mathrm{dist}_H(v, v')\}.$$

Since $\mathrm{diam}(P_m) = m - 1$, we get $\mathrm{diam}(P_m \boxtimes P_n) = \max\{m - 1, n - 1\}$. Since $\mathrm{diam}(P_m \boxtimes P_n) = \mathrm{dist}_{P_m \boxtimes P_m}((1, 1), (m, m)) = m - 1$ and there is only one path from $(1, 1)$ to (m, m) in $P_m \boxtimes P_m$ possessing the length $m - 1$, it follows that

$$\mathrm{dist}_D((1, 1), (m, m)) > m - 1 \quad \text{or} \quad \mathrm{dist}_D((m, m), (1, 1)) > m - 1$$

for any $D \in \mathcal{D}(P_m \boxtimes P_n)$. To combine these two observations with Lemmas 2.11–2.15, we obtain the following theorem:

Theorem 2.16. *If* $m, n \geq 5$, *then*

$$\operatorname{diam}_{\min}(P_m \boxtimes P_n) = \begin{cases} \operatorname{diam}(P_m \boxtimes P_n), & \text{if } m \neq n; \\ \operatorname{diam}(P_m \boxtimes P_n) + 1, & \text{if } m = n. \end{cases}$$

At the end of this section, we give the bounds of $\operatorname{diam}_{\min}(P_n \boxtimes P_m)$ for $m < 5$. From Figure 7, we see that $n - 1 \leq \operatorname{diam}_{\min}(P_n \boxtimes P_2) = n$ for $n > 2$, $n - 1 \leq \operatorname{diam}_{\min}(P_n \boxtimes P_3) = n$ for $n > 3$ and $n - 1 \leq \operatorname{diam}_{\min}(P_n \boxtimes P_4) = n + 1$ for $n > 4$.

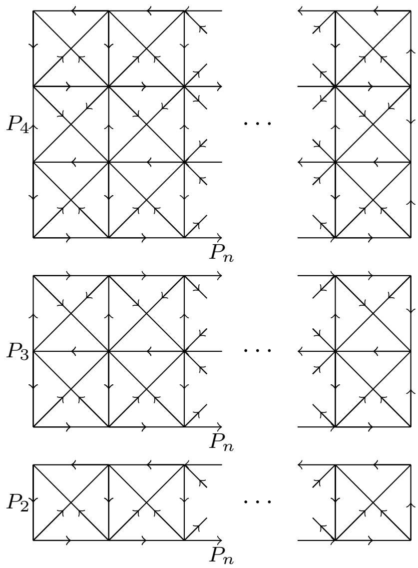

Figure 7: Orientations of $P_n \boxtimes P_2$, $P_n \boxtimes P_3$ and $P_n \boxtimes P_4$.

3 Strong orientation of graphs

In this section we shall prove the next theorem.

Theorem 3.1. *Let* G *and* H *be connected bridgeless graphs. Then*

$$\operatorname{diam}_{\min}(G \boxtimes H) \leq \max\{\operatorname{diam}_{\min}(G), \operatorname{diam}_{\min}(H)\}.$$

Proof. Let D_G be a strong orientation of G such that $\operatorname{diam}(D_G) = \operatorname{diam}_{\min}(G) = d_1$ and let D_H be a strong orientation of H such that $\operatorname{diam}(D_H) = \operatorname{diam}_{\min}(H) = d_2$. We define the orientation $D_{G\boxtimes H}$ of $G \boxtimes H$ as:

(a) Every edge with endvertices in layers G_v, $v \in V(H)$ gets the orientation D_G.

(b) Every edge with endvertices in layers H_u, $u \in V(G)$ gets the orientation D_H.

(c) If $u \to u'$ in G and $v \to v'$ in H, then $(u, v) \to (u', v')$, all other edges are oriented arbitrarily.

We have to prove that for every pair of vertices (u,v), (u',v') in $G \boxtimes H$ there is a directed path P from (u,v) to (u',v') in $D_{G\boxtimes H}$, such that the length of P is at most $\max\{d_1, d_2\}$.

If (u,v) and (u',v) are vertices in the same G-layer or if (u,v) and (u,v') are vertices in the same H-layer, then there is a directed path from (u,v) to (u',v) in $D_{G\boxtimes H}$ of length at most d_1 or a directed path from (u,v) to (u,v') of length at most d_2.

Now let (u,v) and (u',v') be arbitrary vertices in $D_{G\boxtimes H}$. There is a directed path $u = u_1u_2\dots u_m = u'$ in G of length at most d_1 and there is a directed path $v = v_1v_2\dots v_n = v'$ in H of length at most d_2. Without loss of generality we can assume $m \geq n$. We have

$$(u,v) \to (u_2,v_2) \to (u_3,v_3) \to \cdots \to (u_n,v_n) \to \\ (u_{n+1},v_n) \to \cdots \to (u_m,v_n) = (u',v')$$

is a path of length at most d_1. □

Since $\operatorname{diam}_{\min}(C_3) = 2$ and $\operatorname{diam}_{\min}(C_3 \boxtimes C_3) = 2$, the bound is tight.

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