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Fault-Hamiltonicity of Cartesian products of directed cycles*

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Abstract

Although the Cartesian product of two Hamiltonian graphs is Hamiltonian, the corresponding statement for directed graphs is not true. Indeed, it is known that it does not always hold even for the Cartesian products of two directed cycles. In this paper, we study the Cartesian product and its generalization of a directed graph G and a directed cycle. We show that if G has "strong" fault-Hamiltonicity properties, then so does $G \,\square\, C_n$, that is, the Cartesian product of G and a cycle of length n. We also discuss some related problems.

Keywords: Digraphs, fault-Hamiltonicity, Cartesian product.

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1 Introduction

The interconnection network is one of the center pieces of a parallel architecture. The underlying topology of such a parallel machine is a graph, usually referred to as an interconnection network. Depending on the applications, the graph may be undirected or directed. A *Hamiltonian cycle* in a graph is a cycle that visits every vertex of the graph (exactly once). (If the underlying graph is directed, then a cycle means a directed cycle.) A graph is *Hamiltonian* if it has a Hamiltonian cycle. A *Hamiltonian path from u to v* in a graph is a path from u to v that visits every vertex of the graph. (Again, if the underlying graph is directed, then a path means a directed path.) A graph is *Hamiltonian connected* if there exists a Hamiltonian path from u to v for every distinct ordered pair of vertices u and v. Hamiltonicity is an important issue in the study of interconnection networks and there are many papers in this area. Paper [8] contains many references in this area and we refer the readers to [8] for an extensive list of references on Hamiltonicity related problems in interconnection networks. (A small partial list of such papers is [3, 4, 5, 7, 10, 13, 14].) However, most research has been done in the undirected setting as the analysis is, in general, more complicated in the directed case. A directed graph G is *k-regular* if the in-degree and out-degree of every vertex is k. So a connected 1-regular directed graph is a directed cycle. (We will simply refer to directed cycles as *cycles* if it is clear from the context.)

The Cartesian product of two directed graphs G_1 and G_2 is the directed graph $G_1 \,\square\, G_2 = (V, E)$ where $V = V_1 \times V_2$ and $((u_1, v_1), (u_2, v_2))$ is in E if either

(1) $u_1 = u_2$ and $(v_1, v_2) \in E_2$, or

(2) $v_1 = v_2$ and $(u_1, u_2) \in E_1$.

The Cartesian product of undirected graphs can be defined similarly. (One can check that given three directed graphs G_1, G_2 and G_3, $(G_1 \,\square\, G_2) \,\square\, G_3$ is isomorphic to $G_1 \,\square\, (G_2 \,\square\, G_3)$. Thus the Cartesian product of finitely many directed graphs can be naturally defined.) Cartesian product is an important topic in the study of interconnection networks. For example, the classical hypercube is K_2^n, that is, a Cartesian product of n complete graphs on two vertices. Although the Cartesian product of two Hamiltonian graphs is always Hamiltonian, this is false for directed graphs. Trotter and Erdős [12] gave a necessary and sufficient condition for the Cartesian product of two Hamiltonian directed cycles to be Hamiltonian. To be precise, let $\gcd(m, n)$ denote the greatest common divisor of two positive integers m and n. Then the Cartesian product of two directed cycles C_m and C_n is Hamiltonian if and only if $\gcd(m, n) \geq 2$ and there exists positive integers d_1 and d_2 such that $\gcd(m, n) = d_1 + d_2$, $\gcd(m, d_1) = 1$, and $\gcd(n, d_2) = 1$. So $C_2 \,\square\, C_3$ is not Hamiltonian since $\gcd(2, 3) = 1 < 2$.

Vertices in an interconnection network represent processors and edges represent links between processors. Since processors and links may fail, it is meaningful to study such faulty networks. A graph $G = (V, E)$ is *k-Hamiltonian* if $G - F$ is Hamiltonian for every $F \subseteq V \cup E$ and $|F| \leq k$. Similarly, a graph $G = (V, E)$ is *k-Hamiltonian-connected* if $G - F$ is Hamiltonian-connected for every $F \subseteq V \cup E$ and $|F| \leq k$. Here F is the set of *faults* that represent failed processors (vertices) and failed links (edges). We note that if $G = (V, E)$ is k-Hamiltonian-connected, then G is k-Hamiltonian whenever $|V| > k + 2$. For undirected graphs, many related results on k-Hamiltonicity and k-Hamiltonian connectedness with respect to the Cartesian product are known. See, for example, [1, 6, 11].

We have already mentioned the interesting result given in [12]. It is even more interesting if one considers the Cartesian product of three directed cycles. In particular, one can check by brute force that $C_2 \,\Box\, C_3 \,\Box\, C_4$ is a 3-regular, 2-Hamiltonian and 1-Hamiltonian-connected directed graph. In fact, $C_2 \,\Box\, C_3 \,\Box\, C_5$, $C_2 \,\Box\, C_3 \,\Box\, C_6$, $C_2 \,\Box\, C_4 \,\Box\, C_5$ and $C_2 \,\Box\, C_5 \,\Box\, C_5$ are also 3-regular, 2-Hamiltonian and 1-Hamiltonian-connected directed graphs. Results similar to the one given in [12] appeared in [2, 9]. This gives an indication that Hamiltonicity problems for directed graphs are more difficult than the undirected version. In addition, it is proved in [2] that every product of more than two directed cycles is Hamiltonian.

The ultimate goal is to obtain a result on k-Hamiltonicity and k-Hamiltonian connectedness with respect to the Cartesian product of directed graphs. Given the above example, we believe that this problem is difficult. Thus we study directed graphs of the form $G \,\Box\, C_n$. We want to show that if G has "strong" Hamiltonicity property, then so does $G \,\Box\, C_n$. In fact, we will generalize the concept of Cartesian product by considering the following. Let $\mathcal{G}$ be a set of directed graphs, each with the same fixed number of vertices. We say that $\mathcal{G}$ has a certain property if every directed graph in $\mathcal{G}$ has this property. Now we take n graphs $G_0, G_1, \ldots, G_{n-1}$ from $\mathcal{G}$ with repetitions allowed. Let $f_i \colon V(G_i) \to V(G_{i+1})$, for $i = 0, 1, \ldots, n-1$, be bijections where addition is taken modulo n. We construct the directed graph $H = (V, E)$ by letting $V = \cup_{i=0}^{n-1} V(G_i)$ and $E = \left(\cup_{i=0}^{n-1} E(G_i)\right) \cup \left(\cup_{i=0}^{n-1} \{(u, f_i(u)) : u \in V(G_i)\}\right)$. We call H an *n-$\mathcal{G}$-directed graph*. So $G \,\Box\, C_n$ is an n-$\{G\}$-directed graph. For notational simplicity, we denote $EC_i = \{(u, f_i(u)) : u \in V(G_i)\}$ and we let CG_{ij} be the subgraph of H induced by $\cup_{r=i}^{j} V(G_r)$ (modulo n). Given $(u, v) \in EC_i$, we may refer to v as $f_i(u)$ and $u = f_i^{-1}(v)$. For the case $G \,\Box\, C_n$, we may simply refer to f_i as f. Whenever we refer to a range $[i, j]$, it is considered modulo n.

In this paper, we consider deleting vertices and arcs. As mentioned before, these deleted elements correspond to failed processors and links in an interconnection network, and we refer them as faults. Let G be an r-regular directed graph. Clearly the best one can hope for is for G to be $(r-1)$-Hamiltonian and $(r-2)$-Hamiltonian connected. As pointed out earlier, there exist directed graphs that achieve such optimal properties when $r = 3$. In this paper, we show that if G has such optimal properties, then so does $G \,\Box\, C_n$. In fact, our result covers the more general n-$\mathcal{G}$-directed graph. At first glance, one may wonder whether this is consistent with the necessary and sufficient condition given by Trotter and Erdős for $C_n \,\Box\, C_m$ to be Hamiltonian. After all, C_m is 1-regular and Hamiltonian but $C_m \,\Box\, C_n$ may not be Hamiltonian. One may argue that in this case, the condition "-1"-Hamiltonian connected is meaningless. As we shall see, our main result requires the regularity of G to be at least 3.

2 The main result

In this section, we present our main result. We want to show that if $\mathcal{G}$ has good Hamiltonian properties, then so does an n-$\mathcal{G}$-directed graph. We start with the following lemma.

Lemma 2.1. *Let $k \geq 2$ and $N \geq k+5$. Let $\mathcal{G}$ be a class of $(k+1)$-regular and $(k-1)$-Hamiltonian-connected graphs on N vertices. Let H be an n-$\mathcal{G}$-directed graph obtained from $G_0, G_1, \ldots, G_{n-1}$ in $\mathcal{G}$ with the corresponding bijections $f_0, f_1, \ldots, f_{n-1}$. Let $[i, j]$ be a range. Let $F_r \subseteq V(G_r) \cup E(G_r)$ for every r in the range $[i, j]$. Let $F_{r,r+1} \subseteq EC_r$. Let s and t be vertices in $G_i - F_i$ and $G_j - F_j$, respectively. Suppose*

1. *$|F_r| \leq k-1$ for every r in the range $[i, j]$ and*

2. $|F_r| + |F_{r+1}| + |F_{r,r+1}| \leq k + 2$ *for every* r *in the range* $[i, j-1]$.

Then there is a Hamiltonian path from s *to* t *in* $CG_{i,j} - (\cup_{r=i}^{j} F_r) - (\cup_{r=i}^{j-1} F_{r,r+1})$.

Proof. If $i = j$, then there is nothing to prove as G_i is $(k-1)$-Hamiltonian-connected. For notational simplicity, we may assume that $i = 1$. We consider two cases.

Case 1: $j = 2$. We want to find an arc $(u_1, v_2) \in EC_1 - F_{1,2}$ where $u_1 \in V(G_1) - (F_1 \cup \{s\})$ and $v_2 \in V(G_2) - (F_2 \cup \{f_1(s)\})$ and $(u_1, v_2) \neq (f_1^{-1}(t), t)$. Such an arc exists if

$$N > |F_1| + |F_2| + |F_{1,2}| + |\{(s, f_1(s))\}| + |\{(f_1^{-1}(t), t)\}|.$$

But $|F_1| + |F_2| + |F_{1,2}| \leq k + 2$. Thus we are done as $N > k + 2 + 2 = k + 4$. We now obtain a desired Hamiltonian path by using a Hamiltonian path from s to u_1, the arc (u_1, v_2) and a Hamiltonian path from v_2 to t.

Case 2: $j \geq 3$. We first find an arc $(u_1, v_2) \in EC_1 - F_{1,2}$ where $u_1 \in V(G_1) - (F_1 \cup \{s\})$ and $v_2 \in V(G_2) - (F_2 \cup \{f_1(s)\})$. Such an arc exists if

$$N > |F_1| + |F_2| + |F_{1,2}| + |\{(s, f_1(s)\}|.$$

But $|F_1| + |F_2| + |F_{1,2}| \leq k + 2$. Thus we are done as $N > k + 2 + 1 = k + 3$.

Similarly, we can obtain an arc $(u_2, v_3) \in EC_2 - F_{2,3}$ where $u_2 \neq v_2$, and so on, via an inductive argument, in obtaining (u_i, v_{i+1})'s, until we obtain an arc

$$(u_{j-2}, v_{j-1}) \in EC_{j-2} - F_{j-2,j-1}$$

where $u_{j-2} \in V(G_{j-2}) - (F_{j-2} \cup \{v_{j-2}\})$. Now, we need to find an arc

$$(u_{j-1}, v_j) \in EC_{j-1} - F_{j-1,j}$$

where $u_{j-1} \in V(G_{j-1}) - (F_{j-1} \cup \{v_{j-1}\})$ and $v_j \in V(G_j) - (F_j \cup \{t\})$ which can be guaranteed since $N > |F_{j-1}| + |F_j| + |F_{j-1,j}| + 2$ (as $|F_{j-1}| + |F_j| + |F_{j-1,j}| \leq k + 2$ and $N \geq k + 5$). Now since G_r is $(k-1)$-Hamiltonian-connected, we have a Hamiltonian path from v_r to u_r in G_r for every r in $[i, j]$ with $v_1 = s$ and $u_j = t$. These paths together with the arcs (u_r, v_{r+1})'s give a desired Hamiltonian path. □

We remark that if we replace (2) by $|F_r| + |F_{r+1}| + |F_{r,r+1}| \leq k + 1$ for every r in the range $[i, j-1]$ in Lemma 2.1, then the assumption that $N \geq k + 5$ can be replaced with the weaker assumption that $N \geq k + 4$.

Theorem 2.2. *Let* $k \geq 2$ *and* $n \geq 3$. *Let* $\mathcal{G}$ *be a class of* $(k+1)$*-regular,* k*-Hamiltonian and* $(k-1)$*-Hamiltonian connected graphs on* N *vertices. Let* H *be an* n*-*$\mathcal{G}$*-directed graph. Then* H *is* $(k+2)$*-regular. Moreover* H *is* $(k+1)$*-Hamiltonian if* $N \geq k + 4$ *and* k*-Hamiltonian connected if* $N \geq k + 5$ *and* $k \geq 3$.

Proof. We first prove that H is $(k+1)$-Hamiltonian. Let F be a set of faults with $|F| \leq k + 1$. We let F_i be the set of faults in G_i. We consider two cases.

Case 1: $|F_i| = k+1$ for some i. Without loss of generality, we may assume that $|F_0| = k + 1$. Let $x \in F_0$ and define $F_0' = F_0 - \{x\}$. By assumption, there is a Hamiltonian cycle C_0' in $G_0 - F_0'$. Regardless of whether x is a vertex or an arc, $C_0' - \{x\}$ is a Hamiltonian path P_0' from u to v for some u and v. Now let $y = f_0(v)$ and $z = f_{n-1}^{-1}(u)$. By Lemma 2.1, there is a Hamiltonian path from y to z in the $CG_{1,n-1} - (\cup_{r=1}^{n-1} F_r) = CG_{1,n-1}$. (Note that equality holds as $F_r = \emptyset$ for $r \in \{1, 2, \dots, n-1\}$.) This together with P'_0 gives a Hamiltonian cycle in $H - F$.

Case 2: $|F_i| \leq k$ for every i. We first note that $2k > k+1$ as $k \geq 2$. Thus there is at most one i with $|F_i| = k$. Therefore we may assume that $|F_0|$ is the largest and $|F_i| \leq k-1$ for $i \neq 0$. Now, by assumption, there is a Hamiltonian cycle C_0 in $G_0 - F_0$. We want to find an arc (v, u) in C_0 such that

$$(v, f_0(v)), (f_{n-1}^{-1}(u), u), f_0(v), f_{n-1}^{-1}(u) \notin F.$$

Here $|F_r| + |F_{r+1}| + |F_{r,r+1}| \leq k+1$ for $r \in \{0, 1, 2, \dots, n-2\}$ as $|F| \leq k+1$. So we only require $N \geq k+4$ from the remark after Lemma 2.1. Now C_0 has at least $N - |F_0|$ arcs. Since $N - |F_0| > |F| - |F_0|$, such (v, u) exists. Now the argument in Case 1 applies, and we are done.

This completes the proof for H being $(k+1)$-Hamiltonian. The case for H being k-Hamiltonian connected is much more difficult. We assume $N \geq k+5$. (We will see later why $k+5$ is needed.) Let F be a set of faults with $|F| \leq k$ and we define the F_i's as before. Let s and t be two fault-free vertices and our goal is to construct a Hamiltonian path from s to t in $H - F$. We consider two main cases. (Unfortunately, subcases are needed here.)

Case 1: $|F_i| = k$ for some i. Without loss of generality, we may assume that $|F_0| = k$. So all the faults are in F_0. We have to consider subcases depending on the locations of s and t.

Subcase 1.1: s and t are in $G_0 - F_0$. Let $x \in F_0$ and define $F'_0 = F_0 - \{x\}$. By assumption, there is a Hamiltonian path P'_0 from s to t in $G_0 - F'_0$. Regardless of whether x is a vertex or an arc, $P'_0 - \{x\}$ contains the following two disjoint paths that span $G_0 - F_0$: Q_0 from s to u and Q'_0 from v to t for some u and v. (It is possible that $s = u$ or $v = t$.) Moreover, Q_0 and Q'_0 cover all the vertices in $G_0 - F_0$. Now let $y = f_0(u)$ and $z = f_{n-1}^{-1}(v)$. By Lemma 2.1, there is a Hamiltonian path from y to z in $CG_{1,n-1} - (\cup_{r=1}^{n-1} F_r) = CG_{1,n-1}$. This, together with the edges (u, y) and (z, v), Q_0 and Q'_0, gives a Hamiltonian path from s to t in $H - F$.

Subcase 1.2: s is in $G_0 - F_0$ and t is in $G_i - F_i = G_i$ where $i \neq 0$. If $i = n-1$, then it is straightforward as $G_0 - F_0$ is a Hamiltonian. (Since $G_0 - F_0$ is Hamiltonian, there is a Hamiltonian path Q_0 from s to y in $G_0 - F_0$ for some y. Now apply Lemma 2.1 to obtain a Hamiltonian path from $f_0(y)$ to t in $CG_{1,n-1} - (\cup_{r=1}^{n-1} F_r) = CG_{1,n-1}$. This, together with the edge $(y, f_0(y))$ and Q_0, gives a Hamiltonian path from s to t in $H - F$.) Thus we may assume that $i \neq n-1$. By assumption, $G_0 - F_0$ has a Hamiltonian cycle C_0. Since $N \geq k+5$, there exists a vertex u_0 on C_0 such that (u_0, s) is not an arc in C_0 and $u_i \neq t$ where $u_1 = f_0(u_0), u_2 = f_1(u_1), \dots, u_i = f_{i-1}(u_{i-1})$. (It is possible that $u_0 = s$.) Now C_0 contains the following two disjoint paths that span $G_0 - F_0$: Q_0 from s to u_0 and Q'_0 from v to x for some v and x as determined by C_0 and Q_0. (It is possible that $v = x$.) We note that since (u_0, s) is not an arc, Q'_0 is not empty. Now, apply Lemma 2.1 to obtain a Hamiltonian path P_1 from $f_i(u_i)$ to $f_{n-1}^{-1}(v)$ in $CG_{i+1,n-1} - (\cup_{r=i+1}^{n-1} F_r) = CG_{i+1,n-1}$. We apply Lemma 2.1 again, this time to obtain a Hamiltonian path P_2 from $f_0(x)$ to t in $CG_{1,i} - (\cup_{r=1}^{i} (F_r \cup \{u_r\})) = CG_{1,i} - (\cup_{r=1}^{i} \{u_r\})$. For the moment, assume that $f_0(x) \neq t$. Then

$$Q_0, (u_0, u_1, \dots, u_i, f_i(u_i)), P_1, (f_{n-1}^{-1}(v), v), Q'_0, (x, f_0(x)), P_2$$

is a desired Hamiltonian path from s to t in $H - F$. (See Figure 1.)

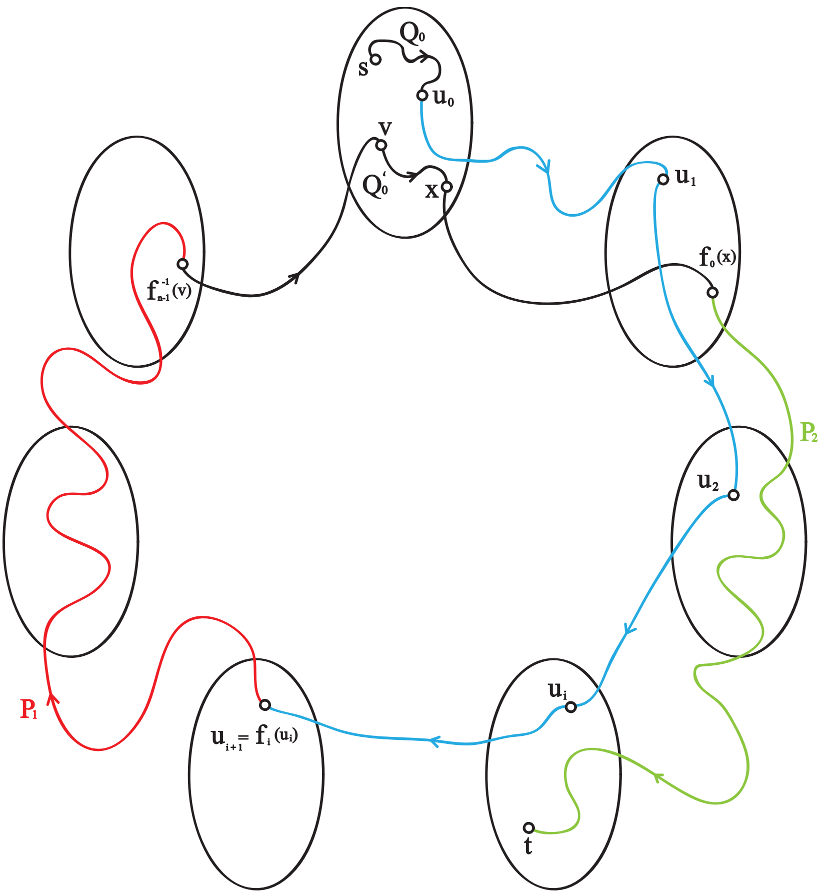

Figure 1: The Hamiltonian path of Subcase 1.2.

The remaining possibility is $f_0(x) = t$. Then $i = 1$. This case is actually simpler as we can obtain a desired Hamiltonian path by using Q_0, a Hamiltonian path from u_1 to $f_{n-1}^{-1}(v)$ in $CG_{1,n-1} - \{t\}$ (via Lemma 2.1), the edge $(f_{n-1}^{-1}(v), v)$, the path Q_0', and the edge (x, t).

Subcase 1.3: t is in $G_0 - F_0$ and s is in $G_i - F_i$ where $i \neq 0$. This is similar to Subcase 1.2 by observing instead of going from G_0 to G_i via $G_1, G_2, \ldots, G_{i-1}$ to obtain a directed path from s to t, we can "trace backward" from to t to s via $G_{n-1}, G_{n-2}, \ldots, G_{i+1}$. To be precise, we let G^R be the directed graph obtained from $G - F$ by reversing the direction on every arc. Then a directed path from s to t in $G - F$ can be obtained from a directed path from t to s in G^R, whose existence is proved in Subcase 1.2.

Subcase 1.4: s and t are in $G_i - F_i$ where $i \neq 0$. We have to consider several scenarios. We first assume that $i = n - 1$. By assumption, there is a Hamiltonian path P from s to t in $G_{n-1} - F_{n-1} = G_{n-1}$. Choose any (u, v) on P such that $f_{n-1}(u) \notin F_0 = F$. (Again such u exists as $N \geq k + 5$. Henceforth, we will not explicitly mention this when choosing an appropriate vertex.) Now P contains the following two disjoint paths that span $G_{n-1} - F_{n-1} = G_{n-1}$: Q from s to u and Q' from v to t. (It is possible that $s = u$ or $v = t$.) By assumption, there is a Hamiltonian cycle in $G_0 - F_0$, which implies that there is a Hamiltonian path P from $f_{n-1}(u)$ to w in $G_0 - F_0$ for some w. Now apply Lemma 2.1 to obtain a Hamiltonian path R from $f_0(w)$ to $f_{n-2}^{-1}(v)$ in $CG_{1,n-2} - (\cup_{r=1}^{n-2} F_r) = CG_{1,n-2}$. Now

$$Q, (u, f_{n-1}(u)), P, (w, f_0(w)), R, (f_{n-2}^{-1}(v), v), Q'$$

is a desired Hamiltonian path from s to t in $H - F$.

We now assume $i = 1$. By assumption, there is a Hamiltonian cycle in $G_0 - F_0$, which implies that there is a Hamiltonian path P from u to v in $G_0 - F_0$ for some u and v. We may choose v such that $f_0(v) \notin \{s, t\}$. By assumption, there is a Hamiltonian path Q from $f_0(v)$ to t in $G_1 - (F_1 \cup \{s\}) = G_1 - \{s\}$. Now by Lemma 2.1, there is a Hamiltonian path R from $f_1(s)$ to $f_{n-1}^{-1}(u)$ in $CG_{2,n-1} - (\cup_{r=2}^{n-1} F_r) = CG_{2,n-1}$. Now

$$(s, f_1(s)), R, (f_{n-1}^{-1}(u), u), P, (v, f_0(v)), Q$$

is a desired Hamiltonian path from s to t in $H - F$.

We may now assume that $2 \leq i \leq n - 2$. By assumption, there is a Hamiltonian cycle in $G_0 - F_0$, which implies that there is a Hamiltonian path P_0 from u to v in $G_0 - F_0$ for some u and v. By assumption, there is a Hamiltonian path P_i from s to t in $G_i - F_i = G_i$. Pick any (y, z) on P_i such that $f_i(y) \neq f_{n-1}^{-1}(u)$ and $f_{i-1}^{-1}(z) \neq f_0(v)$. Now P_i contains two disjoint paths that span $G_i - F_i = G_i$: Q_i from s to y and Q_i' from z to t. Apply Lemma 2.1 to get a Hamiltonian path R from $f_0(v)$ to $f_{i-1}^{-1}(z)$ in $CG_{1,i-1} - (\cup_{r=1}^{i-1} F_r) = CG_{1,i-1}$. Apply Lemma 2.1 to get a Hamiltonian path R' from $f_i(y)$ to $f_{n-1}^{-1}(u)$ in $CG_{i+1,n-1} - (\cup_{r=i+1}^{n-1} F_r) = CG_{i+1,n-1}$. Now

$$Q_i, (y, f_i(y)), R', (f_{n-1}^{-1}(u), u), P_0, (v, f_0(v)), R, (f_{i-1}^{-1}(z), z), Q_i'$$

is a desired Hamiltonian path from s to t in $H - F$. (See Figure 2.)

Subcase 1.5: s is in $G_i - F_i$ and t is in $G_j - F_j$ where $1 \leq i < j \leq n - 1$. By assumption, there is a Hamiltonian cycle in $G_0 - F_0$, which implies that there is a Hamiltonian path P_0 from u to v in $G_0 - F_0$ for some u and v. We first assume that $i \neq 1$. We may assume that $f_0(v) \neq s$. By Lemma 2.1, we obtain a Hamiltonian path P' from $f_0(v)$ to w

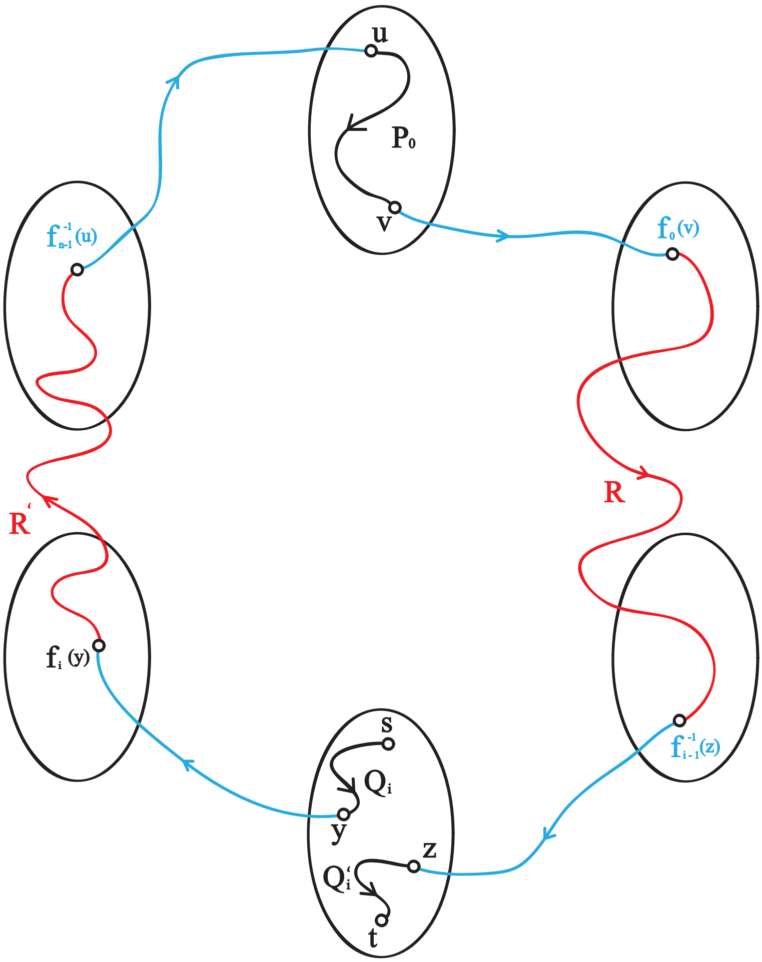

Figure 2: The Hamiltonian path of Subcase 1.4.

in $CG_{1,i-1}$ for some w in $G_{i-1} - F_{i-1} = G_{i-1}$ to be determined. By assumption, there is a Hamiltonian path P_i from s to y in $G_i - F_i = G_i$ for some y such that $y \neq s$ and $f_i(y) \neq t$. Let choose (x, z) on P_i such that

$$f_{j-1}(\cdots(f_{i+1}(f_i(x))) \neq t \text{ and } f_{i-1}^{-1}(z) \neq f_0(v).$$

(If $n - 1 = j + 1$, we further require that $f_j(f_{j-1}(\cdots(f_{i+1}(f_i(x)))) \neq f_{n-1}^{-1}(u)$.) Now we choose $w = f_{i-1}^{-1}(z)$. Now let $A = (x, f_i(x), \ldots, f_{j-1}(\cdots(f_{i+1}(f_i(x))))$. Let $y' = f_i(y)$ if $j = i + 1$ and y' be any vertex in $G_j - \{t, f_{j-1}(\cdots(f_{i+1}(f_i(x)))\}$ otherwise. (If $j = i + 2$, we further require $f_i(y) \neq f_{j-1}^{-1}(y')$.) For notational convenience, let $x' = f_{j-1}(\cdots(f_{i+1}(f_i(x)))$. Let A' be (y, y') if $j = i + 1$ and let A' be a Hamiltonian path from $f_i(y)$ to $f_{j-1}^{-1}(y')$ in $CG_{i+1,j-1} - (\cup_{r=i+1}^{j-1}\{f_{r-1}(\cdots(f_{i+1}(f_i(x)))\})$. Now P_i contains two disjoint paths that span $G_i - F_i = G_i$: Q_i from s to x and Q'_i from $z = f_{i-1}(w)$ to y.

Now let R be a Hamiltonian path from y' to t in $G_j - \{x'\}$ and R' be a Hamiltonian path from $f_j(x')$ to $f_{n-1}^{-1}(u)$ in $CG_{j+1,n-1}$. Then

$$Q_i, A, (x', f_j(x')), R', (f_{n-1}^{-1}(u), u), P_0, (v, f_0(v)), P', (w, z), Q'_i,$$
$$(y, f_i(y)), A', (f_{j-1}^{-1}(y'), y'), R$$

is a desired Hamiltonian path from s to t in $H - F$. (See Figure 3.) If $i = 1$, then a small adjustment is needed for P'. We note that there is freedom in choosing v, y and z. However, once v is chosen, u is forced; similarly, once z is chosen, x is forced. Here we reduce the degree of freedom by one and choose v such that $z = f_0(v)$ in addition to the other restrictions. It is not difficult to see that such v can be chosen. So P' is the arc $(v, f_0(v))$.

Subcase 1.6: s is in $G_j - F_j$ and t is in $G_i - F_i$ where $1 \leq i < j \leq n-1$. We construct a desired Hamiltonian path in several steps. (This is similar to Subcase 1.5 but it is slightly more complicated.) By assumption, there is a Hamiltonian cycle C_0 in $G_0 - F_0$. We want to find two arcs (v, u) and (x, y) on C_0 to delete so that C_0 contains two disjoint paths that span $G_0 - F_0$: Q_0 from u to x and Q'_0 from y to v. It is possible that $u = x$ or $y = v$. (But both cannot occur at the same time.) There are only a few restrictions on the candidacies of (v, u) and (x, y). We call the path

$$(v, f_0(v), f_1(f_0(v)), \ldots, f_{i-1}(\cdots f_1(f_0(v))), f_i(f_{i-1}(\cdots f_1(f_0(v)))),$$

R_1; and the requirement is $f_{i-1}(\cdots f_1(f_0(v))) \neq t$. For convenience, let $w = f_i(f_{i-1}(\cdots f_1(f_0(v))))$. (For the case $i + 1 = j$, then w is in G_j and we further require $w \neq s$.) We will call the path

$$(f_j^{-1}(\cdots f_{n-2}^{-1}(f_{n-1}^{-1}(u))), \ldots, f_{n-2}^{-1}(f_{n-1}^{-1}(u)), f_{n-1}^{-1}(u), u),$$

R_2; moreover, the requirement is $f_j^{-1}(\cdots f_{n-2}^{-1}(f_{n-1}^{-1}(u))) \neq s$. For convenience, let $w' = f_j^{-1}(\cdots f_{n-2}^{-1}(f_{n-1}^{-1}(u)))$. (For the case $i + 1 = j$, then w is in G_j and we further require $w' \neq w$.) Now let R_3 be a Hamiltonian path from w to w' in $CG_{i+1,j} - \{s\}$. It turns out that the case $j = n - 1$ and $f_{n-1}(s) \in F$ requires modification of our construction. So for now, we assume that this is not the case. (Note that $j = n - 1$ and $f_{n-1}(s) \in F$ implies $(s, f_{n-1}(s))$ is fault-free as $F = F_0$.) We have more freedom for (x, y) in most

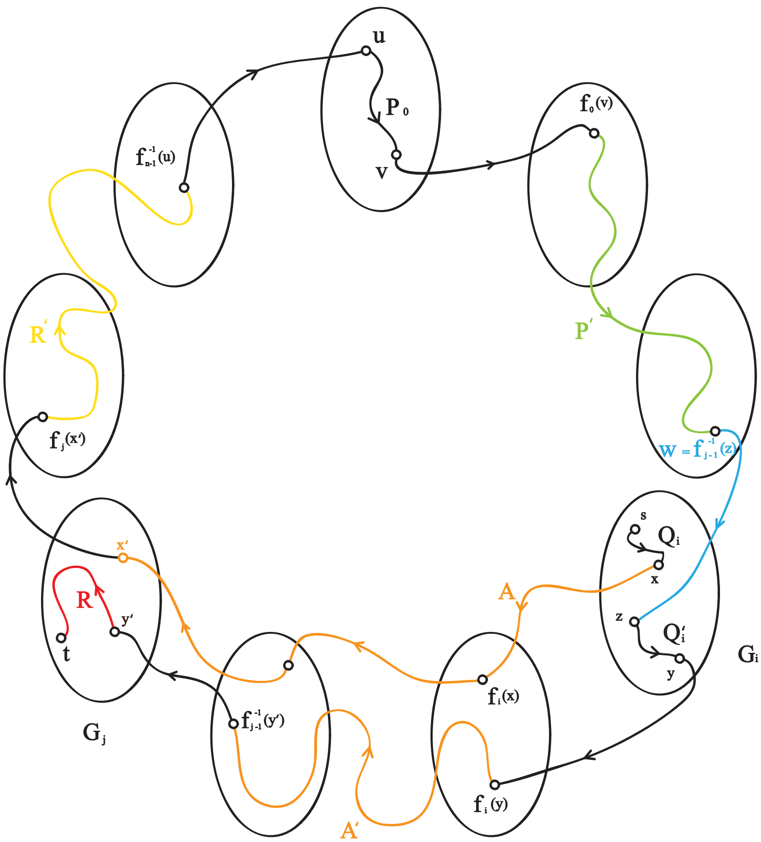

Figure 3: The Hamiltonian path of Subcase 1.5.

instances. If $i = 1$, we require $f_0(x) \neq t$. If $j = n-1$ and $f_{n-1}(s) \notin F$, then we let $y = f_{n-1}(s)$. (Recall that the case $j = n-1$ and $f_{n-1}(s) \in F$ is deferred.) We further note that if $j = n-1$, there is only one choice for y and hence there is only one choice of (x, y), so we should pick (x, y) first and then (v, u). If $j \neq n-1$, let A be the path consisting of $(s, f_j(s))$, the Hamiltonian path from $f_j(s)$ to $f_{n-1}^{-1}(y)$ in $CG_{j+1,n-1} - \cup_{r=j+1}^{n-1}\{f_{r-1}(f_{r-2}(\cdots f_j(w)))\}$ and $(f_{n-1}^{-1}(y), y)$; otherwise (that is, $j = n-1$ and $f_{n-1}(s) \notin F$), let $A = (s, y)$. (Note that if $j \neq n-1$, then $(s, f_j(s))$ is fault-free as $F = F_0$.) Let B be the path consisting of $(x, f_0(x))$ and the Hamiltonian path from $f_0(x)$ to t in $CG_{1,i} - \cup_{r=1}^{i}\{f_{r-1}(f_{r-2}(\cdots f_0(v)))\}$. Then

$$A, Q_0', R_1, R_3, R_2, Q_0, B$$

is a desired Hamiltonian path from s to t in $H - F$. (See Figure 4.) Now we consider the case when $j = n-1$ and $f_{n-1}(s) \in F$. Then we consider the $k+1$ arcs $(s, s^1), (s, s^2), \ldots, (s, s^{k+1})$ in G_j that start at s. Since $|F| = k$, we may assume, without loss of generality, that $f_{n-1}(s^1) \notin F$. So we let $y = f_{n-1}(s^1)$, and the path A will be (s, s^1, y). So we pick (x, y) first, then we pick (v, u) as before but we now need to include the restriction that $w' \neq s^1$. We note that then R_3 needs to be a Hamiltonian path from w to w' in $CG_{i+1,n-1} - \{s, s^1\}$. Since $k \geq 3$, such a path exists via the usual explanation. We remark that one can actually adjust the proof to give a construction for $k = 2$.

Case 2: $|F_i| \leq k-1$ for every i. We consider two subcases depending on the locations of s and t.

Subcase 2.1: s and t belong to the same $G_i - F_i$. Without loss of generality, we may assume that s and t belong to $G_0 - F_0$. By assumption, there is a Hamiltonian path P_0 from s to t in $G_0 - F_0$. Choose (u, v) on P_0 such that $f_0(u), (u, f_0(u)) \notin F$ and $f_{n-1}^{-1}(v), (f_{n-1}^{-1}(v), v) \notin F$. Now apply Lemma 2.1 to obtain a Hamiltonian path from $f_0(u)$ to $f_{n-1}^{-1}(v)$ in $CG_{1,n-1} - (\cup_{r=1}^{n-1} F_r)$. Now, the usual argument gives a desired Hamiltonian path.

Subcase 2.2: s and t belong to different $G_i - F_i$'s. We may assume that s belong to $G_0 - F_0$ and t belong to $G_j - F_j$ where $j \neq 0$. If $j = n-1$, then this result follows directly from Lemma 2.1. So we may assume that $j \leq n-2$.

Subsubcase 2.2.1: $|F_i| \leq k-2$ for every i. Find x in $G_0 - F_0$ such that $f_0(x) \neq t$ (if $j = 1$). We remark that the argument in this subcase requires only $|F_1|, |F_2|, \ldots, |F_j| \leq k-2$. By assumption, there is a Hamiltonian path P_0 from s to x in $G_0 - F_0$. We want to find (u, v) on P_0 to delete so that P_0 contains two disjoint paths that span $G_0 - F_0$: Q_0 from s to u and Q_0' from v to x. There are only a few restrictions on the candidacy of (u, v). We want $f_{n-1}^{-1}(v), (f_{n-1}^{-1}(v), v) \notin F$, the path

$$(u, f_0(u), f_1(f_0(u)), \ldots, y = f_{j-1}(f_{j-2}(\cdots f_1(f_0(u)))), f_j(y))$$

be a fault-free path, and $y \neq t$. (We note that there is a definition embedded in the path. The penultimate vertex is $f_{j-1}(f_{j-2}(\cdots f_1(f_0(u))))$ which we call y. Thus the last vertex is $f_j(y)$.) It is easy to see that such an edge (u, v) exists. Let R_1 be $(f_0(u), f_1(f_0(u)), \ldots, y)$. Let R_2 be the Hamiltonian path from $f_0(x)$ to t in $CG_{1,j} - (\cup_{r=1}^{j} F_r) - \{f_0(u), f_1(f_0(u))), \ldots, y\}$. (Such a path exists by Lemma 2.1 since $|F_r| \leq k-2$ and we delete at most one additional vertex from each G_r so $|F_r|' + |F_{r+1'}| + |F_{r,r+1}| \leq k+2$ where F_r' is F_r. Here we need $N \geq k+5$.) Let R_3 be the Hamiltonian path from $f_j(y)$ to $f_{n-1}^{-1}(v)$

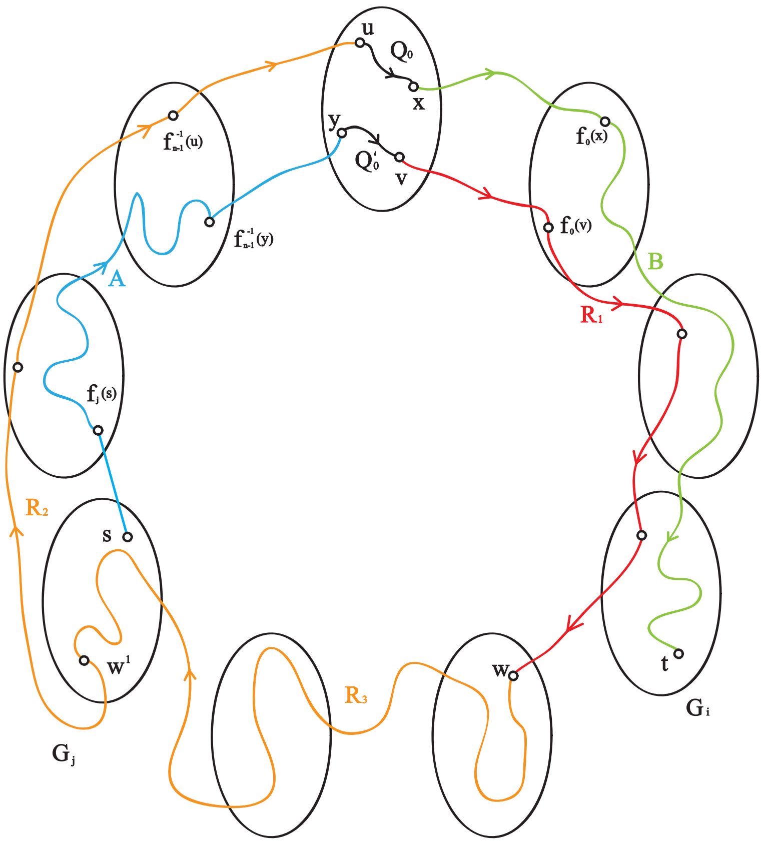

Figure 4: The Hamiltonian path of Subcase 1.6.

in $CG_{j+1,n-1} - (\cup_{r=j}^{n-1} F_r)$. Then

$$Q_0, (u, f_0(u)), R_1, (y, f_j(y)), R_3, (f_{n-1}^{-1}(v), v), Q'_0, (x, f_0(x)), R_2$$

is a desired Hamiltonian path from s to t in $H - F$. (See Figure 5.)

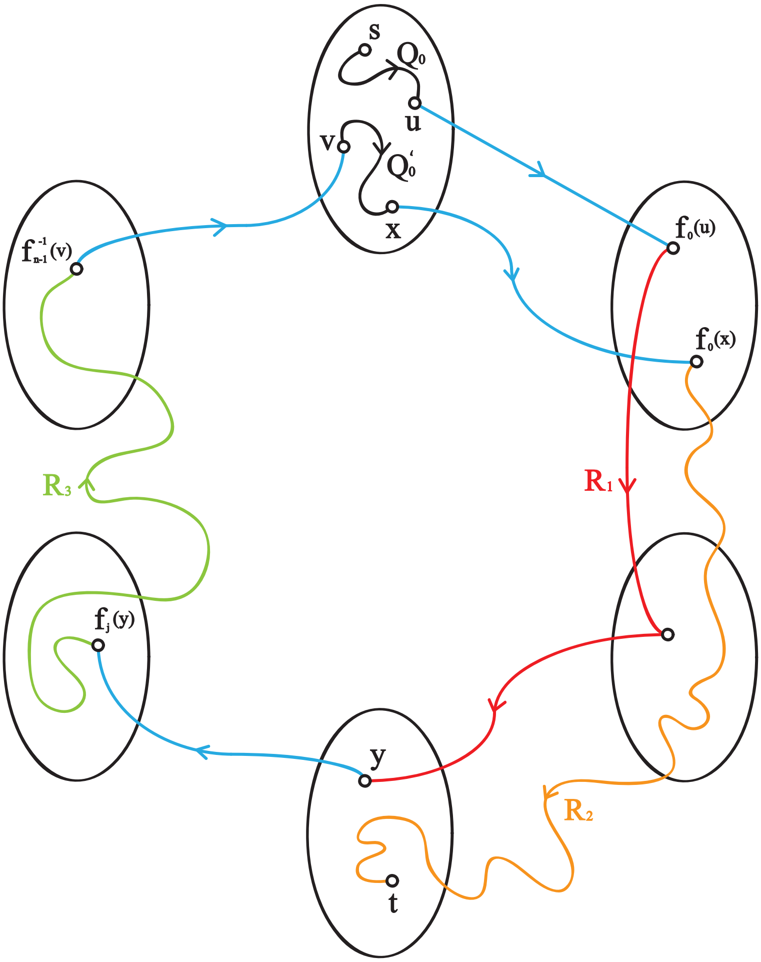

Figure 5: The Hamiltonian path of Subsubcase 2.2.1.

Henceforth, $|F_i| = k - 1$ for some i. Since $k \geq 3$, such an i is unique.

Subsubcase 2.2.2: $|F_0| = k - 1$ or $|F_q| = k - 1$ for some $q = j+1, j+2, \ldots, n-1$. Then the argument of Subsubcase 2.2.1 applies by the remark given at the start of its argument.

Subsubcase 2.2.3: $|F_j| = k - 1$. We will adjust the construction given in Subsubcase 2.2.1. We note that there is at most one fault not in F_j. We find a vertex $y \neq t$ in $G_j - F_j$ such that $(f_0^{-1}(\cdots f_{j-1}^{-1}(y)), \ldots, f_{j-1}^{-1}(y), y)$ is fault-free. Now for this chosen y, let P_j be a Hamiltonian path from y to t in $G_j - F_j$. We find an arc (w, w') on P_j such that $(w, f_j(w))$ and $(f_{j-1}^{-1}(w'), w')$ are fault-free. (We allow $y = w$ or $w' = t$.) If

$j = 1$, we further require $f_0^{-1}(w') \neq s$. Let P_j^y and P_j^t be the subpaths of P_j from y to w and from w' to t, respectively. Let $u = f_0^{-1}(\cdots f_{j-1}^{-1}(y))$ and R_1 be $(u, \ldots, f_{j-1}^{-1}(y), y)$, followed by P_j^y and $(w, f_j(w))$. If $j = 1$, then let $x = f_0^{-1}(w')$. Otherwise, we pick x in $G_0 - (F_0 \cup \{s, u\})$ such that $(x, f_0(x))$ is fault-free. (If $j = 2$, then we further require $f_0(x) \neq f_1^{-1}(w')$.) We can now construct R_2 (similar to Subsubcase 2.2.1) by taking a fault-free Hamiltonian path from $f_0(x)$ to $f_{j-1}^{-1}(w')$ in

$$CG_{1,j-1} - (\cup_{r=1}^{j} F_r) - \{f_0(u), f_1(f_0(u)), \ldots, f_{j-1}^{-1}(y)\},$$

followed by P_j^t. Now consider $G_0 - F_0$. Recall that $|F_0| \leq 1$. If there is a z such that (u, z) is an arc in G_0 and $\{f_{n-1}^{-1}(z), (f_{n-1}^{-1}(z), z)\} \cap F \neq \emptyset$, then set $F_0' = F_0 \cup \{(u, z)\}$; otherwise, $F_0' = F_0$. Now we find a Hamiltonian path P_0 from s to x in $G_0 - F_0'$. Let (u, v) on P_0. By construction, $f_{n-1}^{-1}(v), (f_{n-1}^{-1}(v), v) \notin F$. We can now construct

$$R_1, R_3, Q_0, Q_0'$$

as in Subsubcase 2.2.1 with the following extra condition for choosing v when $j = n - 2$: $f_{n-2}(w) \neq f_{n-1}^{-1}(v)$. We also note that R_3 starts at $f_j(w)$ rather than $f_j(y)$. (See Figure 6.)

Subsubcase 2.2.4: $|F_q| = k - 1$ for some $q = 1, 2, \ldots, j - 1$. One can adapt the construction in Subsubcase 2.2.3 to cover this case. For completeness, we describe the procedure. We note that there is at most one fault not in F_q. We pick two (distinct) vertices a and b in $G_q - F_q$ such that $(f_0^{-1}(\cdots f_{q-1}^{-1}(a)), \ldots, f_{j-1}^{-1}(a), a)$ is fault-free and $(b, f_q(b))$ is fault-free. If $q = j - 1$, we further require that $f_q(b) \neq t$. Let P_q be a Hamiltonian path from a to b in $G_q - F_q$. We find an arc (w, w') on P_q such that $(w, f_q(w))$ and $(f_{q-1}^{-1}(w'), w')$ are fault-free. (We allow $a = w$ or $w' = b$.) If $j = 1$, we further require $f_0^{-1}(w') \neq s$. Let P_q^y and P_q^t be the subpaths of P_j from a to w and from w' to b, respectively. Let $u = f_0^{-1}(\cdots f_{q-1}^{-1}(a))$ and R_1 be $(u, \ldots, f_{q-1}^{-1}(a), a)$, followed by P_q^a and $(w, f_j(w))$. If $j = 1$, then let $x = f_0^{-1}(w')$. Otherwise, we pick x in $G_0 - (F_0 \cup \{s\})$ such that $(x, f_0(x))$ is fault-free. We can now construct R_2 by taking a fault-free Hamiltonian path from $f_0(x)$ to $f_{j-1}^{-1}(w')$ in

$$CG_{1,q-1} - (\cup_{r=1}^{j} F_r) - \{f_0(u), f_1(f_0(u))), \ldots, f_{j-1}^{-1}(a)\},$$

followed by P_q^t. The rest is the same as Subsubcase 2.2.3. □

We remark that the main reason that the argument given in Subsubcase 2.2.3 is not valid for $k = 2$ is because two vertices may be removed from a G_i and hence R_2 cannot be constructed as G_i is only 1-Hamiltonian connected. The same problem occurs for the other subcases. In fact, we did not notice this gap and gave this proof for $k \geq 2$ in an earlier draft. Fortunately for us, the anonymous referee noticed the error. We do not see an easy way to repair this gap. We note that for Subsubcase 2.2.3, the path R_1 and R_2 together span several G_i's, and in general, R_1 only covers one vertex of such G_i's. One idea is to be less restrictive in using the two paths covering such G_i's. (For example, the vertices in $G_j - F_j$ in Subsubcase 2.2.3 are spanned by two paths R_2 and R_3, with each path covering possibly more than one vertex of $G_i - F_i$.) Due to the distribution of s and t and the two vertices in F, there are 8 cases to consider with additional "boundary" subcases in each. We feel that a full discussion adds minimal value. So we choose to present the result for $k \geq 3$ only.

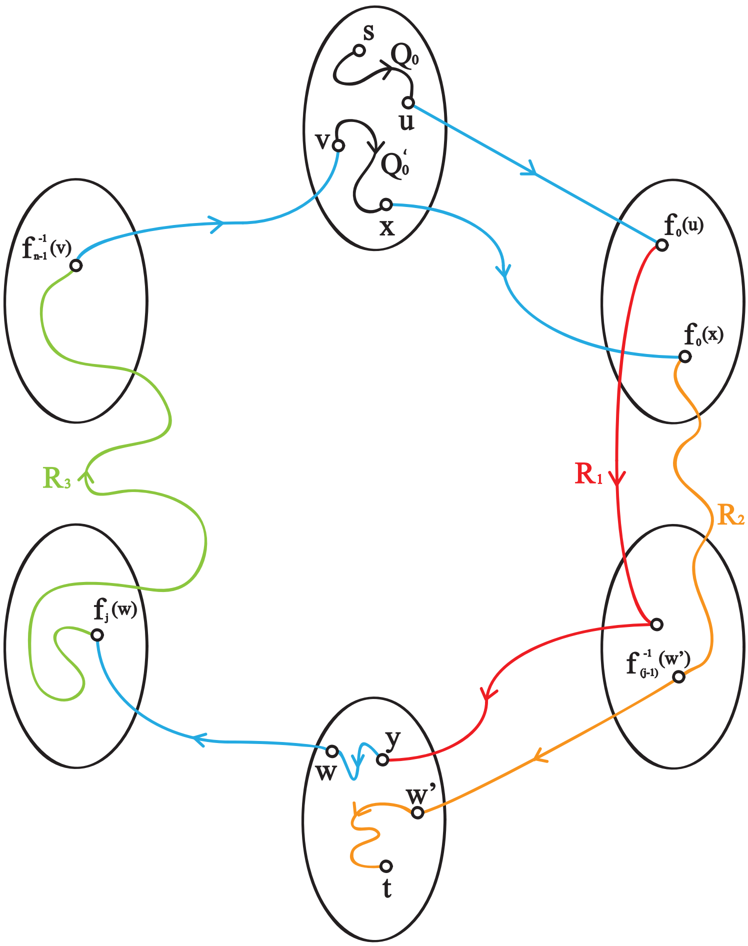

Figure 6: The Hamiltonian path of Subsubcase 2.2.3.

3 Analyzing the conditions in Theorem 2.2

In Theorem 2.2, one of the conditions is the requirement that $N \geq k+4$ for $(k+1)$-Hamiltonicity and $N \geq k+5$ for k-Hamiltonian connectedness. We are unsure whether this condition can be relaxed. However, we do know that the result for k-Hamiltonian connectedness does not hold if $N = k+2$. We choose $G = K_{k+2}$, the complete directed graph on $k+2$ vertices. Clearly it is $(k+1)$-regular and one can check that it is k-Hamiltonian and $(k-1)$-Hamiltonian connected. Then consider $H = G \square C_3$. Let F be k vertices in G_0, s be a vertex in $G_0 - F$ and $t = f(s)$. Then it is clear that there is no Hamiltonian path from s to t. One may wonder whether there is a counterexample $N = k+3$? An obvious choice is to let G be the directed graph obtained from the complete graph K_{k+3}, where k is even, by deleting a perfect matching (and treat the resulting graph as a directed graph). So G is $(k+1)$-regular. If G is k-Hamiltonian and $(k-1)$-Hamiltonian connected, then we have a counterexample. Unfortunately, this is not true as if $k = 2i-1$, then by deleting appropriate $k-1$ vertices from G, we have a 4-cycle which is not Hamiltonian connected.

We now consider the condition on k. As we pointed out earlier that $k = 0$ is not applicable, that is, G needs to be at least 2-regular. We have the condition $k \geq 2$ (that is, G is at least 3-regular) in the statement. In the proof, we did use this assumption; for example, we used it in Case 2 in proving that H is $(k+1)$-Hamiltonian. This is not to say that the result is not true for $k = 1$. On the other hand, we know of no 2-regular, 1-Hamiltonian and Hamiltonian-connected directed graphs. We have already commented on the condition of $k \geq 3$ for the k-Hamiltonian connectedness portion of the theorem.

Finally, there is the condition on n. In an undirected graph, a cycle must have at least three vertices. By the same convention, one usually requires a directed cycle in a directed graph to have at least three vertices; thus the condition $n \geq 3$. However, some authors do consider the two arcs (x, y) and (y, x) to form a directed cycle of length two. In any case, one may consider two directed graphs G and H with the same number of vertices and construct a new directed graph by two set of matchingss that match the vertices of G with the vertices of H and orient the edges in the first set from G to H and vice versa for the second set. One can ask if both G and H have "strong" Hamiltonian properties, does the resulting graph have "strong" Hamiltonian properties. One can apply similar analysis as in the proof of Theorem 2.2 for this problem.

We further remark that Theorem 2.2 seeks the strongest possible property, that is, for a $(k+1)$-regular graph G to be k-Hamiltonian and $(k-1)$-Hamiltonian connected, and then consider an n-$\mathcal{G}$-directed graph. The proof of Theorem 2.2 mainly relies on G being k-Hamiltonian and $(k-1)$-Hamiltonian connected, and not G being k-regular. So our proof is applicable to establish the following: Let $k \geq 2$ and $n \geq 3$. Let $1 \leq r \leq k$. Suppose the class of directed graphs $\mathcal{G}$ is $(k+1)$-regular, r-Hamiltonian, $(r-1)$-Hamiltonian connected and of order N. Let H be an n-$\mathcal{G}$-directed graph. Then H is $(k+2)$-regular, $(r+1)$-Hamiltonian if $N \geq k+4$ and r-Hamiltonian connected if $N \geq k+5$.

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