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Reaction graphs of double Fano planes

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Dedicated to Mario Gionfriddo on the occasion of his 70th birthday.

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Abstract

We consider various reaction graphs on the set of distinct double Fano planes.

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The concept of a *reaction graph*, which has its origin in chemistry, has been explored in several papers, for example, [5, 7, 8, 9, 10, 11].

The reaction graph(s) of the Fano plane (i.e. projective plane of order 2, Steiner triple system of order 7, or $\mathrm{BIBD}(7,3,1)$) are considered in detail in [8, 9], with some additional comments provided in [7]. The vertices of such reaction graph are the 30 distinct Fano planes (on a fixed 7-element set). The reaction graph is of degree $14, 8$, and 7, respectively, according to how adjacency is defined: namely, whenever two vertices (Fano planes) have one, zero, or three triples in common, respectively. The graph of degree 14 is actually isomorphic to $2K_{15}$, that is, two disjoint complete graphs K_{15} as components. Each of these corresponds to a maximal set of MAD $\mathrm{STS}(7)$s (mutually almost disjoint STSs, cf. [6]).

It is well known that the *simple* $\mathrm{BIBD}(7,3,2)$ (i.e. with no repeated blocks) is unique up to an isomorphism and consists of two disjoint Fano planes. It contains 14 blocks (triples) and its automorphism group is of order 42. The blocks (triples) of one such design can be represented as $\{0,1,3\},\{0,2,3\} \bmod 7$. We shall call any simple $\mathrm{BIBD}(7,3,2)$ a *double Fano plane*. Thus there are $\frac{7!}{42} = 120$ distinct double Fano planes on any 7-element set. A double Fano plane will be denoted (a,b) provided a and b are the two disjoint Fano planes that constitute it.

Let $(a,b),(c,d)$ be two distinct double Fano planes. Due to the structure of the reaction graphs of the single Fano plane, whenever $|\{a,b,c,d\}| = 4$, two of the 4 intersections

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between a and c, a and d, b and c, and b and d must contain exactly one triple. Without loss of generality we may assume that the intersection between a and c, and also between b and d both contain one triple. The remaining two intersections, namely between a and d, and between b and c, may

(i) both contain zero triples, or

(ii) both contain three triples, or

(iii) one contains zero and the other contains three triples.

The edges of our reaction graph on K_{120} can now be one of four kinds: either $|\{a, b, c, d\}| = 3$, or it is one of the three types above (see Figure 1).

We shall use the following "colourful" terminology.

A *green* edge joins two double Fano planes $(a, b), (c, d)$ when $|\{a, b, c, d\}| = 3$, that is, one of a, b equals one of c, d. A *yellow, blue* or *red* edge, respectively, joins two double Fano planes $(a, b), (c, d)$ when $|\{a, b, c, d\}| = 4$ and case (i), (ii) or (iii), respectively, occurs.

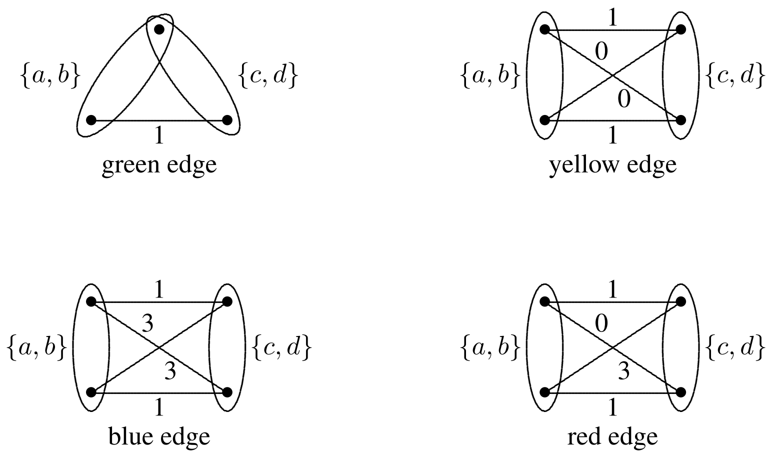

Figure 1: Pairs of double Fano planes.

According to the aforementioned intersections, one may define the following four reaction graphs:

I. The *green* graph (the subgraph of K_{120} induced by the green edges).

This graph is regular of degree 14, and is quasi-strongly regular of grade 3 (cf. [4]) with parameters $(120, 14, 6, (0, 1, 2))$.

II. The *yellow* graph (the subgraph of K_{120} induced by the yellow edges).

This graph is regular of degree 21, and is quasi-strongly regular of grade 2 (cf. [4]) with parameters $(120, 21, 0, (3, 6))$.

III. The *blue* graph (the subgraph of K_{120} induced by the blue edges).

This graph is regular of degree 28 and is quasi-strongly regular of grade 3 with parameters $(120, 28, 6, (4, 6, 12))$.

IV. The *red* graph (the subgraph of K_{120} induced by the red edges).

This graph is regular of degree 56. and is strongly regular with parameters $(120, 56, 28, 24)$. A strongly regular graph with these parameters and automorphism group of order $348\,364\,800$ is known to exist (cf. [3]; see also [12, 13]).

The four coloured graphs together form a 4-class association scheme. The intersection numbers for this scheme can be found at `http://home.agh.edu.pl/~meszka/reaction_graphs.html`.

Next we want to investigate the structure of so-called neighbourhood graphs.

For a vertex $\{a, b\}$ of the reaction graph, a vertex $\{c, d\}$ joined to it by a green edge is called a *green neighbour*, and similarly for yellow, blue or red edges we have *yellow, blue*, or *red* neighbours.

Given a vertex of the reaction graph, the *green neighbourhood graph* is the complete graph K_{14} on its green neighbours. Its edges are coloured green, yellow or red – there are no blue edges. The green edges induce graph consisting of two disjoint K_7's, the yellow edges induce the Heawood graph (cf. [1]) , and the red edges induce the bipartite complement of the Heawood graph. It is well-known that the automorphism group of the Heawood graph is $\mathrm{PGL}(2, 7)$ of order 336. The coloured edges form a 3-class association scheme.

The *yellow neighbourhood graph* of a vertex is the complete graph K_{21} on its yellow neighbours. Its edges are 3-coloured: green, blue and red; there are no yellow edges. The graph induced by the green edges is regular of degree 4 and is distance-transitive with intersection array $[4, 2, 2; 1, 1, 2]$, with automorphism group $\mathrm{PGL}(2, 7)$ of order 336. The graphs induced by the blue and red edges, respectively, both have degree 8, and the same automorphism group as the graph induced by green edges. In this case too the coloured edges form a 3-class association scheme.

The *blue neighbourhood graph* of a vertex is the complete graph K_{28} on its blue neighbours. Its edges are 4-coloured. The graph induced by green and blue edges, respectively, is regular of degree 6, while the graph induced by the yellow edges is cubic, and is actually isomorphic to the Coxeter graph (cf. [2]). The automorphism group of each of these three graphs is again $\mathrm{PGL}(2, 7)$. The graph induced by the red edges is the so-called 8-triangular graph; it is regular of degree 12, and is distance-transitive with intersection array $[12, 5; 1, 4]$. Its automorphism group is S_8 of order $40\,320$. The coloured edges form a 4-class association scheme.

Finally, the *red neighbourhood graph* of a vertex is the complete graph K_{56} on its red neighbours. Its edges are 4-coloured. The graph induced by green edges is of degree 6, and its automorphism group has order $225\,792$. Those induced by the yellow, blue and red edges, respectively, are of degree $9, 12$, and 28, respectively, where the first two of these have automorphism group $\mathrm{PGL}(2, 7)$ of order 336, while the last one has large automorphism group of order $2\,903\,040$. In this case, the coloured edges *do not* form an association scheme.

Let us remark that the graph induced by the union of the green and blue edges is of degree 42, and turns out to be a quasi-strongly regular graph of grade 2 (cf. [4]) with parameters $(120, 42, 18, (6, 15))$. Of course, the graph induced by the union of green, yellow and blue edges is complementary to the red graph, and so is strongly regular with parameters $(120, 63, 30, 36)$.

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