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Circulant matrices and mathematical juggling*

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Abstract

Circulants form a well-studied and important class of matrices, and they arise in many algebraic and combinatorial contexts, in particular as multiplication tables of cyclic groups and as special classes of latin squares. There is also a known connection between circulants and mathematical juggling. The purpose of this note is to expound on this connection developing further some of its properties. We also formulate some problems and conjectures with some computational data supporting them.

Keywords: Juggling, permutations, permanent, circulant matrices.

Math. Subj. Class.: 05A05, 05E25, 15A15

1 Introduction

Let n be a positive integer, and let $\mathbf{t} = (t_1, t_2, \ldots, t_n)$ be a sequence of n nonnegative integers. Then $\mathbf{t}$ is a *juggling sequence* of length n provided that

$$1 + t_1, 2 + t_2, \ldots, n + t_n \tag{1.1}$$

are distinct modulo n, implying, in particular, that $t_1 + t_2 + \cdots + t_n \equiv 0 \pmod{n}$. Thus if (1.1) holds and balls are juggled where, at time i, there is at most one ball that lands in the juggler's hand and is immediately tossed so that it lands in t_i time units $(1 \leq i \leq n)$[1], then there are no collisions; that is, juggling balls *with one hand* according to these rules is possible (for a talented juggler!). The number of balls juggled equals $(t_1+t_2+\cdots+t_n)/n$. If we extend $\mathbf{t}$ to a two-way infinite sequence $(t_i : i \in \mathbb{Z})$ where $t_i = t_{i \bmod n}$, then a ball

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[1] If $t_i = 0$, then there is no ball to toss at time i.

caught at time i is tossed so that it lands at time $i + t_i$. This defines certain *orbits* of the balls being juggled determined by the times at which a specified ball is caught and then tossed.

The sequence $\mathbf{t}$ is a *minimal juggling sequence* provided that the integers t_i have been reduced modulo n to $0, 1, \ldots, n-1$. In particular, $t_i = n$ (a ball is caught and tossed at time i to land in n time units) is equivalent to $t_i = 0$ (no ball is caught and tossed at time i). For some references on mathematical juggling and related work, see e.g. [1, 4, 10].

We now briefly summarize the contents of this paper. In the next section we introduce many examples and discuss some basic properties of juggling sequences and show how they correspond to decompositions of all 1's matrices. We also show how palindromic juggling sequences correspond to a special graph property. In Section 3, we elaborate on the connection between juggling sequences and circulant matrices as discussed in [3], and relate juggling sequences to the permanent of circulants defined in terms of n indeterminates. In Section 4, we present some calculations concerning the coefficients of the distinct terms in the permanents of these circulants and discuss certain questions and conjectures. Finally, in Section 5 we discuss the existence of juggling sequences with additional properties. Part of the purpose of this paper is to draw attention to a number of directions, questions, and conjectures concerning juggling sequences and the permanent expansion of circulants.

2 Juggling sequences

In this section we introduce some of the basic ideas of juggling sequences with many examples and, in the case of palindromic juggling sequences, establish a connection with matchings in complete graphs.

A theorem of M. Hall, Jr. [8] for abelian groups when restricted to cyclic groups yields the following result concerning juggling sequences.

Theorem 2.1. *Let $U = \{u_1, u_2, \ldots, u_n\}$ be a multiset of n integers. Then there is at least one permutation π of $\{1, 2, \ldots, n\}$ such that $\mathbf{u}_\pi = (u_{\pi(1)}, u_{\pi(2)}, \ldots, u_{\pi(n)})$ is a juggling sequence, that is, for which*

$$1 + u_{\pi(1)}, 2 + u_{\pi(2)}, \ldots, n + u_{\pi(n)}$$

are distinct modulo n, if and only if

$$u_1 + u_2 + \cdots + u_n \equiv 0 \pmod{n}. \tag{2.1}$$

In this theorem there is no loss in generality in assuming that $0 \leq u_1, u_2, \ldots, u_n \leq n-1$.

In view of Theorem 2.1, we call a multiset $U = \{u_1, u_2, \ldots, u_n\}$ of n integers satisfying (2.1) a *juggleable set* of size n. If $u_1, u_2, \ldots, u_n$ have been reduced modulo n, then we have a *minimal juggleable set*. It follows from Theorem 2.1 that $U = \{0, 1, 2, \ldots, n-1\}$ is a (minimal) juggleable set if and only if n is odd.

Given $U = \{u_1, u_2, \ldots, u_n\}$, whether or not U is a juggleable set is independent of which representatives of the equivalence classes modulo n determined by the u_i have been chosen, in particular, whether or not the integers u_i have been reduced modulo n. But if $\mathbf{t} = (t_1, t_2, \ldots, t_n)$ is a juggling sequence for the juggleable set U, the number of balls that are juggled depends on which representatives of the equivalence classes modulo n have been chosen, in particular, on whether or not the integers in U have been reduced modulo n.

A juggling sequence $(t_1, t_2, \ldots, t_n)$ is determined by a unique permutation of $\{1, 2, \ldots, n\}$ and conversely any permutation of $\{1, 2, \ldots, n\}$ determines a unique juggling sequence.

Example 2.2. Let $n = 7$ and consider the permutation σ of $\{1, 2, 3, 4, 5, 6, 7\}$ whose cycle decomposition is $(1, 5, 6)(2, 4, 7, 3)$. (Thus in σ, $1 \to 5 \to 6 \to 1$ and $2 \to 4 \to 7 \to 3 \to 2$). For each $i = 1, 2, \ldots, 7$, define $t_i = \sigma(i) - i \bmod 7$, then $\mathbf{t} = (4, 2, 6, 3, 1, 2, 3)$ is a minimal juggling sequence.

Reversing this procedure, let $n = 9$ and consider the juggling sequence $\mathbf{t} = (1, 5, 3, 4, 8, 3, 3, 6, 3)$. We obtain a permutation σ of $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ by calculating and reducing modulo 9:

$$\begin{array}{lll} \sigma(1) = 1 + 1 = 2, & \sigma(2) = 5 + 2 = 7, & \sigma(3) = 3 + 3 = 6, \\ \sigma(4) = 4 + 4 = 8, & \sigma(5) = 8 + 5 = 4, & \sigma(6) = 3 + 6 = 9, \\ \sigma(7) = 3 + 7 = 1, & \sigma(8) = 6 + 8 = 5, & \sigma(9) = 3 + 9 = 3. \end{array}$$

Thus σ is the permutation with cycle decomposition $(1, 2, 7)(3, 6, 9)(4, 8, 5)$. $\Diamond$

Example 2.3. Let $n = 3$ and consider $\mathbf{t} = (4, 4, 1)$. Then to juggle according to $\mathbf{t}$ requires three balls and the balls determine three *orbits* of $\mathbb{Z}$:

$$\begin{array}{c} \cdots \to 1 \to 5 \to 9 \to 10 \to 14 \to 18 \to 19 \to \cdots, \\ \cdots \to 2 \to 6 \to 7 \to 11 \to 15 \to 16 \to 20 \to \cdots, \\ \cdots \to 3 \to 4 \to 8 \to 12 \to 13 \to 17 \to 21 \to \cdots. \end{array}$$

(Here, for instance, $2 \to 6$ represents the fact that at time unit 2, a ball is tossed so that it lands in 4 time units in the future, that is, at time unit 6; then the ball is tossed to land in 1 time unit in the future, that is at time unit 7.) Reducing $\mathbf{t} \bmod 3$ to $(1, 1, 1)$ results in only one ball and only one orbit:

$$\cdots \to 1 \to 2 \to 3 \to 4 \to 5 \to 6 \to \cdots.$$

Let $J_{m,n}$ denote the $m \times n$ matrix of all 1's. Juggling using the juggling sequence $(4, 4, 1)$ gives a decomposition of the matrix $J_{3,3}$ of all 1's whereby any three consecutive matrices sum to $J_{3,3}$. (The first subscript '3' in $J_{3,3}$ represents the number of balls juggled, the second '3' represents the number of terms in the juggling sequence. The ordering of the rows is arbitrary.) This is indicated by

…	4	4	1	4	4	1	4	4	1	…
…	1				1				1	…,
		1				1	1			
			1	1				1		

giving

$$J_{3,3} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} + \begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix} + \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix}.$$

Using the $\bmod 3$ reduction $(1, 1, 1)$ of $(4, 4, 1)$ gives the trivial decomposition

$$J_{1,3} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}.$$

$\Diamond$

Example 2.4. Let $n = 5$ and consider $\mathbf{t} = (3, 3, 4, 4, 1)$. Then juggling (with three balls) using this juggling sequence is indicated by

3	3	4	4	1	3	3	4	4	1	3	3	4	4	1
1			1				1				1			1
		1				1			1	1			1	
	1			1	1			1				1		

$\ldots \quad \ldots,$

giving the decomposition

$$J_{3,5} = \left[\begin{array}{c|c|c|c|c} 1 & & & 1 & \\ \hline & & 1 & & \\ \hline & 1 & & & 1 \end{array}\right] + \left[\begin{array}{c|c|c|c|c} & & 1 & & \\ \hline & 1 & & & 1 \\ \hline 1 & & & 1 & \end{array}\right] + \left[\begin{array}{c|c|c|c|c} & 1 & & & 1 \\ \hline 1 & & & 1 & \\ \hline & & 1 & & \end{array}\right].$$

The juggling sequence $\mathbf{t} = (2, 4, 2, 3, 4)$ corresponds to

2	4	2	3	4	2	4	2	3	4	2	4	2	3	4
1		1		1				1			1			
			1			1				1		1		1
	1				1		1		1				1	

$\ldots \quad \ldots,$

and gives a different decomposition of $J_{3,5}$. ♢

We call a juggling sequence $\mathbf{t} = (t_1, t_2, \ldots, t_n)$ *decomposable* provided the permutation associated with $\mathbf{t}$ has at least two nontrivial cycles in its cycle decomposition. Equivalently, $\mathbf{t}$ is decomposable provided $\mathbf{t} = \mathbf{r} + \mathbf{s}$ where $\mathbf{r} = (r_1, r_2, \ldots, r_n)$, $\mathbf{s} = (s_1, s_2, \ldots, s_n)$ are juggling sequences such that $\{r_i, s_i\} = \{0, t_i\}$ for $i = 1, 2, \ldots, n$, and $\mathbf{r}, \mathbf{s} \neq \mathbf{t}$. Any juggling sequence can be uniquely written as a sum of indecomposable juggling sequences arising from the unique cycle decomposition of the associated permutation.

Example 2.5. With $n = 9$, $\mathbf{t} = (1, 5, 3, 4, 8, 3, 3, 6, 3)$ is a juggling sequence (4 balls). The corresponding decomposition is not that of $J_{3,9}$ but, after permutation of columns, is

$$J_{1,3} \oplus J_{1,3} \oplus \left(\left[\begin{array}{c|c|c} 1 & 1 & \\ \hline & & 1 \end{array}\right] + \left[\begin{array}{c|c|c} & & 1 \\ \hline 1 & 1 & \end{array}\right] \right).$$ ♢

We summarize this discussion with the following theorem.

Theorem 2.6. *Let $\mathbf{t} = (t_1, t_2, \ldots, t_n)$ be a sequence of n integers. Then $\mathbf{t}$ is a (minimal) juggling sequence if and only if $\sigma \colon \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ defined by*

$$\sigma(i) \equiv t_i + i \pmod{n}$$

is a permutation of $\{1, 2, \ldots, n\}$. There is a one-to-one correspondence between minimal juggling sequences of length n and permutations of $\{1, 2, \ldots, n\}$.

Notice that Theorem 2.6 provides an algorithm to determine whether a sequence is a juggling sequence.

Knutson (see [10]) showed how to generate all juggling sequences of length n with k balls ($1 \leq k \leq n$) from the constant juggling sequence $(k, k, \ldots, k)$ of length n. There are two transformations used in the algorithm:

I. Given a juggling sequence $(k_1, k_2, \ldots, k_n)$, the *cyclic shift* $(k_n, k_1, k_2, \ldots, k_{n-1})$ is also a juggling sequence.

II. Given a juggling sequence $(k_1, \ldots, k_i, \ldots, k_j, \ldots, k_n)$, then the *swap* $(k_1, \ldots, (j-i)+k_j, \ldots, -(j-i)+k_i, \ldots, k_n)$ is also a juggling sequence:

$$i + ((j-i)+k_j) = j + k_j \quad \text{and} \quad j + (-(j-i)+k_i) = i + k_i$$

where the balls thrown at times i and j swap landing times.

Theorem 2.7 ([10]). *Any juggling sequence of length n with k balls can be generated from the constant juggling sequence $(k, k, \ldots, k)$ by cyclic shifts and swaps.*

We now consider a special property of juggling sequences that are *palindromic*. In the following argument, we use that a sequence $(t_0, t_1, \ldots, t_{n-1})$ is a juggling sequence of length n if $t_i + i \not\equiv t_j + j \pmod n$ for each $i \in \{0, 1, \ldots, n-1\}$, which is a direct consequence of the original definition. That is, if $\mathbf{p}$ is a juggling sequence of length n, then, modulo n, $\mathbf{p} + (0, 1, 2, \ldots, n-1)$ will be a permutation of $\{0, 1, 2, \ldots, n-1\}$.

Let n be odd and $n = 2m+1$. Let $\mathbf{p} = (p_m, p_{m-1}, \ldots, p_1, p_0, p_1, \ldots, p_{m-1}, p_m)$ be a minimal palindromic juggling sequence. For each $i \in \{1, 2, \ldots, m\}$, define

$$x_i = p_i + i \bmod n \quad \text{and} \quad y_i = p_i - i \bmod n.$$

Since $\mathbf{p}$ is a juggling sequence, we have that, modulo n,

$$\begin{aligned}(y_m, \ldots, y_2, y_1, p_0, x_1, x_2, \ldots, x_m) = \\ (p_m - m, \ldots, p_2 - 2, p_1 - 1, p_0, p_1 + 1, p_2 + 2, \ldots, p_m + m) = \\ \mathbf{p} + (0, 1, \ldots, n-1) - (m, m, \ldots, m).\end{aligned}$$

Hence $\{p_0, x_1, \ldots, x_m, y_1, \ldots, y_m\}$ is a set of distinct values.

Construct a digraph $G(V, E)$ with vertex set $V = \{0, 1, 2, \ldots, n-1\}$ and edge set $E = \{e_1, \ldots, e_m\}$, where $e_i = (x_i, y_i)$ for each $i \in \{1, 2, \ldots, m\}$. We define the *length* of edge e_i to be $y_i - x_i \bmod n$. Hence each e_i has length $n - 2i$; thus G is a directed near 1-factor whose set of edge lengths is $\{1, 3, 5, \ldots, n-2\}$.

Conversely, let $V = \{0, 1, 2, \ldots, n-1\}$ and suppose $G(V, E)$ is a directed near 1-factor whose set of edge lengths is $\{1, 3, 5, \ldots, n-2\}$. Then we may assume $E = \{e_1, \ldots, e_m\}$, where e_i is the directed edge of length $n - 2i$ with $e_i = (x_i, y_i)$. Let p_0 denote the vertex in G not incident to any edge, and for each $i \in \{1, 2, \ldots, m\}$, let $p_i = x_i - i \bmod n$. Then $p_i = y_i + i \bmod n$ for each $i \in \{1, 2, \ldots, m\}$. Define $\mathbf{p} = (p_m, p_{m-1}, \ldots, p_1, p_0, p_1, \ldots, p_{m-1}, p_m)$. Then modulo n we have

$$\mathbf{p} + (0, 1, \ldots, n-1) = (y_m, \ldots, y_2, y_1, p_0, x_1, x_2, \ldots, x_m) + (m, m, \ldots, m).$$

Since all values in $\{p_0, x_1, \ldots, x_m, y_1, \ldots, y_m\}$ are distinct, $\mathbf{p}$ is a juggling sequence.

These two operations which map between minimal palindromic juggling sequences of length n and directed near 1-factors on n vertices whose set of edge lengths is $\{1, 3, 5, \ldots, n-2\}$ are inverses of one another, which leads to the following theorem.

Theorem 2.8. *Let n be an odd positive integer. Then there is a one-to-one correspondence between minimal palindromic juggling sequences of length n and directed near 1-factors on the vertex set $\{0, 1, \ldots, n-1\}$ whose set of edge lengths is $\{1, 3, 5, \ldots, n-2\}$.*

A similar construction gives a result for all positive even integers n.

Theorem 2.9. *Let n be a positive even integer. Then there is a one-to-one correspondence between minimal palindromic juggling sequences of length n and directed 1-factors on the vertex set $\{0, 1, \ldots, n-1\}$ whose set of edge lengths is $\{1, 3, 5, \ldots, n-1\}$.*

Proof. Let $n = 2m$. The proof method is similar to that given for the argument to Theorem 2.8, so in what follows we give only the construction for the correspondence.

Let $\{(x_i, y_i) \mid i \in \{1, 2, \ldots, m\}\}$ be a 1-factor on $\{0, 1, 2, \ldots, n-1\}$ with (x_i, y_i) having length $2i - 1$ for each $i \in \{1, 2, \ldots, m\}$. For each $i \in \{1, 2, \ldots, m\}$, let $p_i = x_i - m + i \bmod n$. Then $p_i = y_i - m - i + 1 \bmod n$. So modulo n,

$$(x_m, \ldots, x_2, x_1, y_1, y_2, \ldots, y_m) - (0, 1, \ldots, n-1) = \\ (p_m, \ldots, p_2, p_1, p_1, p_2, \cdots, p_m).$$

Therefore $(p_m, \ldots, p_2, p_1, p_1, p_2, \ldots, p_m)$ is a minimal palindromic juggling sequence.

Conversely, if $(p_m, \ldots, p_2, p_1, p_1, p_2, \ldots, p_m)$ is a minimal palindromic juggling sequence, then we may define $x_i = p_i + m - i \bmod n$ and $y_i = p_i + m + i - 1 \bmod n$ and have that $\{(x_i, y_i) \mid i \in \{1, 2, \ldots, m\}\}$ is the edge set of a directed 1-factor in which (x_i, y_i) has length $2i - 1$ for each $i \in \{1, 2, \ldots, m\}$. □

Example 2.10. For $n = 6$, $(2, 5, 2, 2, 5, 2)$ is the minimal palindromic juggling sequence corresponding to the directed 1-factor with edge set $\{(4, 5), (0, 3), (2, 1)\}$. Note the edges have lengths 1, 3, and 5, respectively. Similarly for $n = 7$, $(2, 5, 3, 1, 3, 5, 2)$ is the minimal palindromic juggling sequence corresponding to the directed near 1-factor with unused vertex 1 and edge set $\{(4, 2), (0, 3), (5, 6)\}$. In this case, the edges have lengths 5, 3, and 1, respectively. ♢

3 Juggleable sets and circulants

Let $\mathcal{P}_n$ be the set of minimal juggleable sets of size n. For $U \in \mathcal{P}_n$, let $\mathcal{J}_n(U)$ be the set of juggling sequences of length n with U as juggleable set. It follows from [2] that the number of minimal juggleable sets of size n is given by

$$|\mathcal{P}_n| = \frac{1}{n} \sum_{d|n} \binom{2d-1}{d} \phi\left(\frac{n}{d}\right) \tag{3.1}$$

where ϕ is Euler's totient function and the summation extends over all positive integers d dividing n. The number of minimal juggling sequences of length n is $n!$, since for each permutation $(i_1, i_2, \ldots, i_n)$ of $\{1, 2, \ldots, n\}$, we have

$$(i_1 - 1) + (i_2 - 2) + \cdots + (i_n - n) = \sum_{i=1}^{n} i - \sum_{i=1}^{n} i = 0,$$

and hence the multiset $\{i_1 - 1, i_2 - 2, \ldots, i_n - n\}$ of integers taken modulo n, is a juggleable set.

Let n, k, and ν be positive integers. In [7] it is proved that the number of nonnegative integer solutions of

$$u_1 + u_2 + \cdots + u_n = k \quad \text{and} \quad \sum_{i=1}^{n} i u_i \equiv \nu \pmod{n} \tag{3.2}$$

equals the number of nonnegative integer solutions of

$$v_1 + v_2 + \cdots + v_k = n \quad \text{and} \quad \sum_{i=1}^{k} i v_i \equiv \nu \pmod{k}. \tag{3.3}$$

Taking $\nu = 0$, we get the following duality result.

Theorem 3.1. *The number of minimal juggleable sets* $\{u_1, u_2, \ldots, u_n\}$ *with* $u_1 + u_2 + \cdots + u_n = k$ *equals the number of minimal juggleable sets* $\{v_1, v_2, \ldots, v_k\}$ *with* $v_1 + v_2 + \cdots + v_k = n$.

The above discussion gives a characterization of the number of juggling sequences corresponding to each minimal juggleable set.

Theorem 3.2. *Let* $U = \{u_1, u_2, \ldots, u_n\}$ *be a minimal juggleable set. The number* $|\mathcal{J}(U)|$ *of juggling sequences with* U *as juggleable set equals the number of permutations* $(j_1, j_2, \ldots, j_n)$ *of* $\{1, 2, \ldots, n\}$ *such that* $j_i \equiv i + r \pmod{n}$ *has* u_i *solutions for each* $r = 0, 1, \ldots, n-1$.

Another viewpoint (see [2]) is the following. Consider the $n \times n$ circulant matrix

$$C(x_0, x_1, \ldots, x_{n-1}) = \begin{bmatrix} x_0 & x_1 & \cdots & x_{n-2} & x_{n-1} \\ x_{n-1} & x_0 & \cdots & x_{n-3} & x_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_2 & x_3 & \cdots & x_0 & x_1 \\ x_1 & x_2 & \cdots & x_{n-1} & x_0 \end{bmatrix}. \tag{3.4}$$

Thus

$$C(x_0, x_1, \ldots, x_{n-1}) = x_0 I_n + x_1 P_n + x_2 P_n^2 + \cdots + x_{n-1} P_n^{n-1},$$

where P_n is the $n \times n$ permutation matrix corresponding to the cyclic permutation $(2, 3, \ldots, n, 1)$ (thus $P_n^0 = P_n^n = I_n$). The book [6] contains a thorough discussion of circulants.

Recall that the permanent of an $n \times n$ matrix $A = [a_{ij} : 0 \le i, j \le n]$ is

$$\operatorname{per}(A) = \sum_{(i_1, i_2, \ldots, i_n)} a_{1i_1} a_{2i_2} \cdots a_{n,i_n}$$

where the summation extends over all the permutations $(i_1, i_2, \ldots, i_n)$ of $\{1, 2, \ldots, n\}$. Each term $a_{1i_1} a_{2i_2} \cdots a_{n,i_n}$ in the permanent of $C(x_0, x_1, \ldots, x_{n-1})$ is of the form

$$x_0^{k_0} x_1^{k_1} \cdots x_{n-1}^{k_{n-1}}$$

where $k_0, k_1, \ldots, k_{n-1}$ are integers such that

$$0 \le k_i \le n, \ (0 \le i \le n-1) \quad \text{and} \quad k_0 + k_1 + \cdots + k_{n-1} = n,$$

and k_i is the number of integers r with $0 \le r \le n-1$ such that $i_r - r \equiv k_i \pmod{n}$ and

$$k_0 \cdot 0 + k_1 \cdot 1 + \cdots + k_{n-1} \cdot (n-1) \equiv 0 \pmod{n}.$$

Thus the number of distinct terms in the permanent of the circulant $C(x_0, x_1, \ldots, x_{n-1})$ equals the number $|\mathcal{P}_n|$ of juggleable sets of size n and thus is given by (3.1). Theorem 2.1 implies that the monomial $x_0 x_1 \ldots x_{n-1}$ is a term in $\mathrm{per}(A)$ if and only if $1 \cdot 0 + 1 \cdot 1 + \cdots + 1 \cdot (n-1) \equiv 0 \pmod{n}$; since $0 + 1 + \cdots + (n-1) = n(n-1)/2$, $x_0 x_1 \ldots x_{n-1}$ is a term in $\mathrm{per}(A)$ if and only if n is odd. Now let n be even. Then a monomial of the form $x_0^{k_0} x_1^{k_1} \cdots x_{n-1}^{k_{n-1}}$ with $k_r = 2$, $k_s = 0$, and all other k_i's equal to 1, is a term in $\mathrm{per}(A)$ if and only if $|r - s| = n/2$.

In [9] it is shown that $|\mathcal{P}_n|$ equals the dimension of a certain symmetric space associated with a cyclic group of order n. See [12] for a comparison with the number of distinct terms occurring in the determinant.

The following corollary is a direct consequence of Theorem 3.2 and the definitions of a circulant matrix and the permanent.

Corollary 3.3. *Two permutations $j_1, j_2, \ldots, j_n$ and $l_1, l_2, \ldots, l_n$ of $\{1, 2, \ldots, n\}$ give the same term in the permanent of $C(x_0, x_1, \ldots, x_{n-1})$ if and only if*

$$|\{i : j_i \equiv i + r \pmod{n}\}| = |\{i : l_i \equiv i + r \pmod{n}\}| \quad \text{for each } r = 0, 1, \ldots, n-1.$$

If the common values are $k_0, k_1, \ldots, k_{n-1}$, then the term in the permanent equals $x_0^{k_0} x_1^{k_1} \cdots x_{n-1}^{k_{n-1}}$.

Example 3.4. Table 1 gives the minimal juggleable sets of size $n = 4$ and their corresponding terms in $\mathrm{per}(C(x_0, x_1, \ldots, x_{n-1}))$, along with the juggling sequences corresponding to each such pattern and their number. ◊

Table 1: Minimal juggleable sets and juggling sequences for $n = 4$.

Juggleable sets U $\{u_0, u_1, u_2, u_3\}$	Corresponding term in the permanent	Corresponding juggling sequences $\mathcal{J}_4(U)$	Cardinalities $\lvert\mathcal{J}_4(U)\rvert$ (coefficients)
$\{0,0,0,0\}$	x_0^4	$(0,0,0,0)$	1
$\{1,1,1,1\}$	x_1^4	$(1,1,1,1)$	1
$\{2,2,2,2\}$	x_2^4	$(2,2,2,2)$	1
$\{3,3,3,3\}$	x_3^4	$(3,3,3,3)$	1
$\{0,0,2,2\}$	$x_0^2 x_2^2$	$(0,2,0,2)$, $(2,0,2,0)$	2
$\{1,1,3,3\}$	$x_1^2 x_3^2$	$(1,3,1,3)$, $(3,1,3,1)$	2
$\{0,0,1,3\}$	$x_0^2 x_1 x_3$	$(0,0,1,3)$, $(0,1,3,0)$, $(1,3,0,0)$, $(3,0,0,1)$	4
$\{0,1,1,2\}$	$x_0 x_1^2 x_2$	$(0,1,1,2)$, $(1,1,2,0)$, $(1,2,0,1)$, $(2,0,1,1)$	4
$\{1,2,2,3\}$	$x_1 x_2^2 x_3$	$(1,2,2,3)$, $(2,2,3,1)$, $(2,3,1,2)$, $(3,1,2,2)$	4
$\{0,2,3,3\}$	$x_0 x_2 x_3^2$	$(0,2,3,3)$, $(2,3,3,0)$, $(3,3,0,2)$, $(3,0,2,3)$	4

As in Table 1 for $n = 4$, constant juggleable sets correspond to the n monomials terms $x_0^n, x_1^n, \ldots, x_{n-1}^n$ of the permanent of the matrix $C(x_0, x_1, \ldots, x_{n-1})$ each occuring with coefficient equal to 1.

If $\{u_1, u_2, \ldots, u_n\}$ is a minimal juggleable set of size n, we define $c(u_1, u_2, \ldots, u_n)$ to be the number of juggling sequences of length n whose pattern is given by $\{u_1, u_2, \ldots, u_n\}$. Therefore, $c(u_1, u_2, \ldots, u_n)$ equals the number of permutations $(i_1, i_2, \ldots, i_n)$ of $\{1, 2, \ldots, n\}$ such that $i_r - r \equiv k_i \pmod{n}$ has u_i solutions for $i = 1, 2, \ldots, n$. The permanent of $C(x_0, x_1, \ldots, x_{n-1})$ is then given by the homogeneous polynomial of degree n,

$$\sum_{\{u_1,u_2,\ldots,u_n\}\in\mathcal{P}_n} c(u_1, u_2, \ldots, u_n) x_0^{u_1} x_1^{u_2} \cdots x_n^{u_{n-1}},$$

whose number of terms is given by (3.1). Thus from Table 1 we see that the permanent of $C(x_0, x_1, x_2, x_3)$ equals

$$1x_0^4x_1^0x_2^0x_3^0 + 1x_0^0x_1^4x_2^0x_3^0 + 1x_0^0x_1^0x_2^4x_3^0 + 1x_0^0x_1^0x_2^0x_3^4 + 2x_0^2x_1^0x_2^2x_3^0 + \\ 2x_0^0x_1^2x_2^0x_3^2 + 4x_0^2x_1^1x_2^0x_3^1 + 4x_0^1x_1^2x_2^1x_3^0 + 4x_0^0x_1^1x_2^2x_3^1 + 4x_0^1x_1^0x_2^1x_3^2.$$

As the referee pointed out, $c(u_1, u_2, \ldots, u_n)$ is the number of ways to arrange the multiset consisting of u_1 0's, u_2 1's, ..., u_n $(n-1)$'s into a juggling sequence. Some evaluation of these numbers can be found in sequence A006717 [11].

Theorem 3.5. *If $U = \{u_1, u_2, \ldots, u_n\}$ is a minimal juggleable set of size n, then*

$$c(u_1, u_2, \ldots, u_n) \geq 1. \tag{3.5}$$

Equality holds in (3.5) if and only if U is a constant multiset. If n is a prime p and U is not a constant multiset, then p is a divisor of $c(u_1, u_2, \ldots, u_n)$.

Proof. If U is a constant minimal juggleable set $\{k, k, \ldots, k\}$, then x_k^n occurs as a term in the permanent of $C(x_0, x_1, \ldots, x_{n-1})$ corresponding to the positions of the 1's in P_n^k, that is, the positions $(1, k+1), (2, k+2), \ldots, (n, k+n)$ taken modulo n. If $\{u_1, u_2, \ldots, u_n\}$ is a non-constant juggleable set, there is a term in the permanent of $C(x_0, x_1, \ldots, x_{n-1})$ equal to $x_0^{u_1} x_1^{u_2} \cdots x_{n-1}^{u_n}$ not arising solely from the n positions $(1, k+1), (2, k+2), \ldots, (n, k+n)$ modulo n corresponding to the 1's in the permutation matrices $I_n, P_n, P_n^2, \ldots, P_n^{n-1}$.

The $k \times k$ principal submatrix $C[i_1, i_2, \ldots, i_k \mid i_1, i_2, \ldots, i_k] = C(x_{i_1}, x_{i_2}, \ldots, x_{i_k})$ of C determined by rows and columns $i_1, i_2, \ldots, i_k$ is cyclically permutation equivalent (row and column indices are taken modulo n) to the submatrix $C[i_1+1, i_2+1, \ldots, i_k+1 \mid i_1+1, i_2+1, \ldots, i_k+1] = C(x_{i_1+1}, x_{i_2+1}, \ldots, x_{i_k+1})$ determined by rows and columns $i_1+1, i_2+1, \ldots, i_k+1$ taken modulo n. Thus if we take a monomial in the permanent corresponding to a permutation $j_1, j_2, \ldots, j_n$, we get $n-1$ other equal monomials by sequentially adding 1 modulo n to each of $j_1, j_2, \ldots, j_n$ and cyclically permuting:

$$\begin{aligned}(j_1, j_2, \ldots, j_n) &\to (j_n+1, j_1+1, \ldots, j_{n-1}+1) \\ &\to \cdots \\ &\to (j_2+(n-1), \ldots, j_n+(n-1), j_1+(n-1)).\end{aligned} \tag{3.6}$$

If U is a non-constant juggleable set, then not all these permutations can be equal. (If e.g. all of these n permutations are equal, then $(j_1, j_2, \ldots, j_n)$ is a cyclic permutation

$a, a+1, a+2, \ldots, a+(n-1)$ modulo n giving the monomial x_i^n with coefficient equal to 1.) This amounts to simultaneously permuting rows and columns of $C(x_0, x_1, \ldots, x_{n-1})$ using the permutation matrix P_n and replacing the permutation (and its corresponding term in the permanent) with the image of $(j_1, j_2, \ldots, j_n)$ under this action. The result is a term in the permanent with the same value; basically we have that the position (i, j) moves into the position $(i+1, j+1)$ (indices taken $\bmod n$) under the action of P_n, so to position $(i+l, j+l)$ (indices taken $\bmod n$) under the action of P^l. So the set of positions in those sets corresponding to powers of P have to be invariant under a cyclic shift by l in order to get another term in the permanent with the same value. If n is a prime this cannot happen unless the term is of the form x_i^n. Since there may be other terms of equal value in the permanent of $C(x_0, x_1, \ldots, x_{n-1})$, we have that $p \mid c(u_1, u_2, \ldots, u_n)$. □

Corollary 3.6. *If n is odd, the coefficient of $c(1, 1, \ldots, 1)$ of $x_0 x_1 \cdots x_{n-1}$ in the permanent $\operatorname{per}(C(x_0, x_1, \ldots, x_{n-1}))$ is divisible by n.*

Proof. The corollary follows as in the proof of Theorem 3.5 since the term $x_0 x_1 \cdots x_{n-1}$ comes from the juggleable set $\{0, 1, \ldots, n-1\}$ and whatever order gives a juggling sequence, each of the $(n-1)$ cyclic shifts is different, resulting in a contribution of n to the coefficient. □

4 Coefficients in $\operatorname{per}(C(x_0, x_1, \ldots, x_{n-1}))$

We first consider the special case of $n = 5$.

Example 4.1. Let $n = 5$. The formula (3.1) for the number of distinct terms in the permanent of $C(x_0, x_1, x_2, x_3, x_4)$ is

$$\frac{1}{5}\left(\phi(5) + \binom{9}{5}\phi(1)\right) = \frac{1}{5}(4 + 126) = 26.$$

There are five constant terms in the permanent each with coefficient 1 and there are twenty-one terms each with coefficient divisible by 5. So either we have two terms each with coefficient 10 and nineteen terms with coefficient 5, or we have one term with coefficient 15 and twenty terms with coefficient 5.

The term $x_0 x_1 x_2 x_3 x_4$ occurs in each of the following:

$$\left[\begin{array}{c|c|c|c|c} x_0 & & & & \\ \hline & & x_1 & & \\ \hline & & & & x_2 \\ \hline & x_3 & & & \\ \hline & & & x_4 & \end{array}\right], \quad \left[\begin{array}{c|c|c|c|c} x_0 & & & & \\ \hline & & & & x_3 \\ \hline & & & x_1 & \\ \hline & & x_4 & & \\ \hline & x_2 & & & \end{array}\right], \quad \left[\begin{array}{c|c|c|c|c} x_0 & & & & \\ \hline & & & x_2 & \\ \hline & x_4 & & & \\ \hline & & & & x_1 \\ \hline & & x_3 & & \end{array}\right].$$

and thus, by cyclically simultaneously permuting rows and columns (changing the diagonal position in which x_0 occurs by shifting along the main diagonal), appears in the permanent with coefficient at least 15 and therefore exactly 15. Note the positions occupied by the x_i with $i \neq 0$ above:

$$\left[\begin{array}{c|c|c|c|c} & & & & \\ \hline & & x_1 & x_2 & x_3 \\ \hline & x_4 & & x_1 & x_2 \\ \hline & x_3 & x_4 & & x_1 \\ \hline & x_2 & x_3 & x_4 & \end{array}\right].$$

Each x_i with $i \neq 0$ occupies all the positions in the submatrix obtained by striking out row 1 and column 1 that it occupies in $C(x_0, x_1, x_2, x_3, x_4)$. Thus this simple analysis gives

$$\text{per}(C(x_0, x_1, x_2, x_3, x_4)) = \sum_{i=0}^{4} x_i^5 + 5(\text{twenty other terms}) + 15x_0x_1x_2x_3x_4. \qquad \diamondsuit$$

From calculations of $\text{per}(C(x_0, x_1, \ldots, x_{n-1}))$ using Sage, we found the following information:

- $(n = 5)$: largest coefficient is 15 occuring uniquely for

  $$x_0x_1x_2x_3x_4.$$

  Coefficients are $1, 5, 15$. This confirms the calculations in Example 4.1.
- $(n = 6)$: largest coefficient is 24 occuring for the six terms of the form

  $$x_0^2x_1x_2x_3^0x_4x_5.$$

  Coefficients are $1, 2, 3, 6, 9, 12, 18, 24$.
- $(n = 7)$: largest coefficient is 133 occuring uniquely for

  $$x_0x_1x_2x_3x_4x_5x_6.$$

  Coefficients are $1, 7, 14, 21, 35, 42, 49, 133$.
- $(n = 8)$: largest coefficient is 256 occuring for the 8 terms

  $$\begin{array}{ll} x_0^2x_1x_2x_3x_4^0x_5x_6x_7, & x_0x_1^2x_2x_3x_4x_5^0x_6x_7, \\ x_0x_1x_2^2x_3x_4x_5x_6^0x_7, & x_0x_1x_2x_3^2x_4x_5x_6x_7^0, \\ x_0^0x_1x_2x_3x_4^2x_5x_6x_7, & x_0x_1^0x_2x_3x_4x_5^2x_6x_7, \\ x_0x_1x_2^0x_3x_4x_5x_6^2x_7, & x_0x_1x_2x_3^0x_4x_5x_6x_7^2. \end{array}$$

  For instance, $x_0^2x_1x_2x_3x_4^0x_5x_6x_7$ occurs in the term

  $$\left[\begin{array}{c|c|c|c|c|c|c|c} x_0 & & & & & & & \\ \hline & & & x_2 & & & & \\ \hline & & & & & x_3 & & \\ \hline & x_6 & & & & & & \\ \hline & & & & x_0 & & & \\ \hline & & x_5 & & & & & \\ \hline & & & & & & & x_1 \\ \hline & & & & & & x_7 & \end{array}\right].$$

  There are 810 different terms that occur in $\text{per}(C(x_0, x_1, \ldots, x_7))$. The full set of coefficients in the permanent are

  $$\{1, 2, 4, 6, 8, 12, 16, 20, 24, 32, 40, 48, 56, 64, 72, 80, 96, 128, 160, 256\}.$$

  Note that the differences of consecutive coefficients in this list are:

  $$1, 2, 2, 2, 4, 4, 4, 4, 8, 8, 8, 8, 8, 8, 8, 16, 32, 32, 96.$$

  Only the last is not a power of 2.

Conjecture 4.2. *Our calculations have shown that for $n = 4$, the largest coefficient in* $\mathrm{per}(C(x_0, x_1, \ldots, x_{n-1}))$ *is $4 = 2^2$ occuring 4 times, and for $n = 8$, the largest coefficient is $256 = 2^8$ occuring 8 times. We conjecture that if n is a power of 2, then the largest coefficient is also a power of 2 occuring for the terms of the form $x_0^2 x_1 x_2 \cdots \widehat{x_{n/2}} \cdots x_{n-1}$, and cyclical translates of terms of this form (total number of different terms is n). Unfortunately, the occurrence of these terms does not seem to have a pattern. For instance, with $n = 8$, we have*

$$\begin{bmatrix} \boxed{x_0} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ x_7 & \boxed{x_0} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ x_6 & x_7 & x_0 & x_1 & x_2 & x_3 & x_4 & \boxed{x_5} \\ x_5 & x_6 & x_7 & x_0 & \boxed{x_1} & x_2 & x_3 & x_4 \\ x_4 & x_5 & x_6 & x_7 & x_0 & x_1 & \boxed{x_2} & x_3 \\ x_3 & x_4 & x_5 & \boxed{x_6} & x_7 & x_0 & x_1 & x_2 \\ x_2 & x_3 & x_4 & x_5 & x_6 & \boxed{x_7} & x_0 & x_1 \\ x_1 & x_2 & \boxed{x_3} & x_4 & x_5 & x_6 & x_7 & x_0 \end{bmatrix} \qquad (x_0^2 x_1 x_2 x_3 x_5 x_6 x_7).$$

This corresponds to the permutation σ of $\{1, 2, 3, 4, 5, 6, 7, 8\}$ *such that*

$$\begin{array}{llll} \sigma(1) = 1, & \sigma(2) = 2, & \sigma(3) = 8, & \sigma(4) = 5, \\ \sigma(5) = 7, & \sigma(6) = 4, & \sigma(7) = 6, & \sigma(8) = 3. \end{array}$$

The coefficient of $x_0^2 x_1 x_2 x_3 x_5 x_6 x_7$ is the number of permutations π of $\{1, 2, 3, 4, 5, 6, 7, 8\}$*, such that*

$$\pi(i) - i \equiv j \pmod 8$$

has two solutions for $j = 0$, no solutions for $j = 4$, and one solution for $j = 1, 2, 3, 5, 6, 7$. A similar statement holds for all even n, and we seek the number of such solutions.

Conjecture 4.3. *Three* conjectures/problems *for n even (or perhaps just n a power of 2):*

(a) There exists a term $x_0^2 x_1 x_2 \cdots \widehat{x_{n/2}} \cdots x_{n-1}$ in $\mathrm{per}(C(x_0, x_1, \ldots, x_{n-1}))$ *arising from every choice of the two x_0's on the main diagonal.*

(b) In reference to (a), the largest number of terms occurs when the x_0's are chosen to be $n/2$ apart (cyclically, the same number of elements on the main diagonal between them). In the case of $n = 8$, there are 16 terms for a choice of x_0's which are 4 apart (5th x_0 on the main diagonal minus 1st x_0 on main diagonal) and 8 terms for all other choices of x_0's.

(c) If n is a power of 2, the coefficient of $x_0^2 x_1 x_2 \cdots \widehat{x_{n/2}} \cdots x_{n-1}$ is a power of 2.

Problem 4.4. The matrix $C(x_0, x_1, \ldots, x_{n-1})$ can be regarded as a special latin square. The coefficient of $x_0 x_1 \cdots x_{n-1}$ in $\mathrm{per}(C(x_0, x_1, \ldots, x_{n-1}))$ equals the number of transversals of this latin square. In [5] it is shown that if n is odd and sufficiently large, the coefficient of $x_0 x_1 \cdots x_{n-1}$ in $\mathrm{per}(C(x_0, x_1, \ldots, x_{n-1}))$ is greater than $(3.246)^n$. In the cases of $n = 5$ and $n = 7$, the number of latin square transversals of $C(x_0, x_1, \ldots, x_{n-1})$ equals $15 = 5 \times 3$ and $133 = 7 \times 19$, respectively. Since a latin square transversal is mapped into a latin square transversal by multiplying $C(x_0, x_1, \ldots, x_{n-1})$ by the full cycle permutation matrix P_n, it follows that for odd n, the number of latin square transversals, that is, $c(1, 1, \ldots, 1)$ is divisible by n. See also Theorem 3.5 and Corollary 3.6.

If n is odd, the term $x_0x_1\cdots x_{n-1}$ occurs in $\mathrm{per}(C(x_0, x_1, \ldots, x_{n-1})$ with a nonzero coefficient. A *conjecture* would be that this term has the largest coefficient. Thinking of the x_i as n different colors giving $n!$ multicolored transversals, the conjecture is saying that the number of multicolored transversals with all colors different is greater than the number of multicolored transversals of any other prescribed color type (so at least two colors the same). This coefficient is equal to the number of transversals of $C(x_0, x_1, \ldots, x_{n-1})$ considered as a latin square, so finding this exactly is probably not attainable (see [5]).

Remark 4.5. Concerning Problem 4.4 and the juggleable set $\{1, 2, \ldots, n\}$ with n odd, corresponding to the term $x_0x_1\cdots x_{n-1}$ in $\mathrm{per}(C(x_0, x_1, \ldots, x_{n-1}))$. A permutation $(i_1, i_2, \ldots, i_n)$ of this juggleable set is a juggling sequence giving the term $x_0x_1\cdots x_{n-1}$ in $\mathrm{per}(C(x_0, x_1, \ldots, x_{n-1}))$ provided $1+i_1, 2+i_2, \ldots, n+i_n$ are distinct modulo n. If this is the case, then any cyclic permutation of $(i_1, i_2, \ldots, i_n)$ is also a juggling sequence (since subtracting 1 modulo n from distinct integers modulo n gives distinct integers modulo n, thereby giving n terms in $\mathrm{per}(C(x_0, x_1, \ldots, x_{n-1}))$ equal to $x_0x_1\cdots x_{n-1}$. The difficulty in calculating the coefficient of $x_0x_1\cdots x_{n-1}$ in $\mathrm{per}(C(x_0, x_1, \ldots, x_{n-1}))$ is knowing how many permutations $i_1, i_2, \ldots, i_n$ of the set $\{1, 2, \ldots, n\}$ have the property that $1+i_1, 2+i_2, \ldots, n+i_n$ are distinct modulo n. So one might consider the additive group $\mathbb{Z}_n^{(n)} = \mathbb{Z}_n \times \mathbb{Z}_n \times \cdots \times \mathbb{Z}_n$ (n copies of $\mathbb{Z}_n$) and the mapping

$$T\colon \mathbb{Z}_n^{(n)} \to \mathbb{Z}_n^{(n)}$$

given by

$$\begin{aligned} T(i_1, i_2, \ldots, i_n) &= (1+i_1, 2+i_2, \ldots, n+i_n) \\ &= (1, 2, \ldots, n) + (i_1, i_2, \ldots, i_n) \bmod n. \end{aligned}$$

Unfortunately, this mapping is not a homomorphism and so does not seem useful. But it does seem that for a juggleable set $\{u_1, u_2, \ldots, u_n\}$ with at least one repeat, that is, the number of permutations $(u_1, u_2, \ldots, u_n)$ of this pattern such that $1+u_1, 2+u_2, \ldots, n+u_n$ are distinct modulo n is smaller than when there is no repeat in $\{u_1, u_2, \ldots, u_n\}$. But it seems difficult to make a comparison.

Remark 4.6. Assume n is odd. Then $x_0^1x_1^1x_2^1\cdots x_{n-1}^1$ occurs in $\mathrm{per}(C(x_0, x_1, \ldots, x_{n-1}))$ with a nonzero coefficient. We can think of this term as *generating* other terms that occur in $\mathrm{per}(C(x_0, x_1, \ldots, x_{n-1}))$ as follows:

We increase or decrease (by 1) some of the exponents of this term to get

$$x_0^{1+a_0}x_1^{1+a_1}x_2^{1+a_2}\cdots x_{n-1}^{1+a_{n-1}}$$

where each $a_i \in \{1, 0, -1\}$, and

$$\sum_{i=0}^{n-1} a_i = 0 \tag{4.1}$$

and, in order that the result is a term in $\mathrm{per}(C(x_0, x_1, \ldots, x_{n-1}))$, we must have

$$\sum_{i=0}^{n-1} i a_i \equiv 0 \pmod{n}. \tag{4.2}$$

(By (4.2), $\sum_{i-0}^{n-1}(1+a_i)=0$ and $\sum_{i=0}^{n-1} i(a_i+1)\equiv 0 \pmod n$ and thus gives a term in this permanent.) We can do a similar operation on the resulting term but then we need to be sure that the resulting exponents are always between 0 and n. Continuing like this we can generate all terms that occur in this permanent.

So in this operation we increase $s\geq 1$ exponents by 1 and decrease s exponents by -1, so adding $(a_0,a_1,a_2,\dots,a_{n-1})$, subject to the condition (4.2), to the vector of exponents in a term in our permanent. One line of investigation is to try to determine when this operation increases/decreases the coefficient of the corresponding terms in our permanent. In particular, when with one application starting with the term $x_0^1x_1^1x_2^1\cdots x_{n-1}^1$, does the coefficient decrease? Note that in one application, we must reduce two exponents to 0 in order that we satisfy (4.2); in general there must be at least four changes in exponents. See the following example.

Example 4.7. Let $n=9$. We start with the term $x_0x_1x_2x_3x_4x_5x_6x_7x_8$. We can change exponents by using the vector $(0,0,1,-1,0,-1,1,0,0)$. Since $1\cdot 2-1\cdot 3+(-1)\cdot 5+1\cdot 6=0\equiv 0 \pmod 9$,

$$x_0x_1x_2^2x_4x_6^2x_7x_8$$

is a term in our permanent. ♢

Problem 4.8. If n is even, then we can also ask for the term(s) with the largest coefficient. If $n=4$, there are four terms that appear with the largest coefficient of 4, namely

$$x_0^2x_1x_3,\, x_0x_1^2x_2,\, x_1x_2^2x_3,\, x_0x_2x_3^2.$$

A *conjecture* might be:

If n is even then the terms in $\operatorname{per}(C(x_0,x_1,\dots,x_{n-1}))$ that occur with the largest coefficient are the terms with the property that x_i occurs with exponent 2, $x_{i+n/2}$ (subscript $\bmod n$) occurs with exponent 0, and all other x_i appear with exponent 1.

Remark 4.9. We have that there is a nonzero term in $\operatorname{per}(C(x_0,x_1,\dots,x_{n-1}))$ with exactly two nonzero exponents (so binomials) if and only if n is not a prime. The reason is as follows: Suppose $x_i^ax_j^b$ occurs with a nonzero coefficient where $0\leq j<i\leq n-1$ and $i\neq j$, and $a,b\geq 1$, and $a+b=n$ (and so $a,b\leq n-1$). Then by Hall's theorem

$$ai+bj=ai+(n-a)j\equiv 0 \pmod n, \quad \text{that is,} \quad a(i-j)\equiv 0 \pmod n.$$

If n is a prime p, this is a contradiction since $p\nmid a$ and $p\nmid (i-j)$. If n is not a prime, say $n=uv$ where $1<u,v<n-1$. Then we may choose $a=u$, and i and j so that $i-j=v$, and get a term $x_i^ax_j^{(n-a)}$ with a nonzero coefficient.

In investigating binomials in $\operatorname{per}(C(x_0,x_1,\dots,x_{n-1}))$ it is sufficient to consider binomials of the form $x_0^ax_k^b$ where $1\leq k\leq n-1$. Thus we consider the terms of $\operatorname{per}(x_0I_n+x_kP_n^k)$ different from x_0^n and x_k^n. This permanent is easily computed:

$$\operatorname{per}(x_0I_n+x_kP_n^k)=\sum_{t=0}^{d}\binom{d}{t}x_0^{t\frac{n}{d}}x_k^{(d-t)\frac{n}{d}} \text{ where } d=\gcd(n,k).$$

Thus the largest coefficient of a binomial is $\binom{d}{\frac{d}{2}}$.

More generally, let $H\subseteq\{0,1,\dots,n-1\}$. If we set $x_j=0$ if $j\notin H$, then the permanent of the resulting matrix $C_H(x_0,x_1,\dots,x_{n-1})$ gives all the terms that occur in

$\text{per}(C(x_0, x_1, \ldots, x_{n-1}))$ and their coefficients in which the only x_i that can occur are those with $i \in H$. By also setting $x_i = 1$ for $i \in H$, the permanent equals the number of terms in $\text{per}(C_H(x_0, x_1, \ldots, x_{n-1}))$.

Remark 4.10. Now consider terms in $\text{per}(C(x_0, x_1, \ldots, x_{n-1}))$ where there are exactly three nonzero exponents (so in the juggling context, three different heights in throwing the balls). These terms are then *trinomials*. *Which trinomial has the largest coefficient among all trinomials that occur in* $\text{per}(C(x_0, x_1, \ldots, x_{n-1}))$*?* The *conjecture* is that the maximum coefficient occurs when the exponents are as equal as possible; in particular if $n = 3k$, then the trinomial will largest coefficient is conjectured to be $x_0^k x_k^k x_{2k}^k$ and its cyclic permutations. In investigating trinomials in $\text{per}(C(x_0, x_1, \ldots, x_{n-1}))$ it suffices to consider terms of the form $x_0^a x_r^b x_s^c$ where $0 < r < s < n$ and $a + b + c = n$, that is it suffices to consider the trinomials in

$$\text{per}(x_0 I_n + x_r P_n^r + x_s P_n^s).$$

The *conjecture* is that the largest coefficient of a trinomial in this permanent occurs when the exponents are as equal as possible and the powers of P_n, i.e. the subscripts of the x's are as equally spaced as possible (in the cyclic sense). If $n = 3k$, then after permutations $x_0 I_n + x_k P_n^k + x_{2k} P_n^{2k}$ becomes a direct sum of k 3×3 matrices of the form

$$x_0 I_3 + x_k P_3 + x_{2k} P_3^2.$$

5 Juggling sequences with additional properties

Let $U = \{u_1, u_2, \ldots, u_n\}$ be a minimal juggleable set, and let $u_{\tau(1)}, u_{\tau(2)}, \ldots, u_{\tau(n)}$ be a juggling sequence corresponding to U. Thus τ is a permutation of $\{1, 2, \ldots, n\}$ and it is natural to ask about the existence of such permutations τ with additional properties, equivalently, extensions of Theorem 2.1 by imposing additional restrictions on the permutation τ. Juggling sequences correspond to transversals in the circulant $C(x_0, x_1, \ldots, x_{n-1})$ and thus we seeks transversals of $C(x_0, x_1, \ldots, x_{n-1})$ whose pattern has additional properties.

Two natural permutations to consider are *involutions* and *centrosymmetric permutations* of σ of $\{1, 2, \ldots, n\}$. Involutions are permutations σ of $\{1, 2, \ldots, n\}$ where for all i and j, $\sigma(i) = j$ implies $\sigma(j) = i$, and these correspond to transversals of $C(x_0, x_1, \ldots, x_{n-1})$ whose positions have a symmetric matrix pattern, that is, transversal patterns invariant under a reflection about the main diagonal. A permutation σ is centrosymmetric provided that for all i, $\sigma(i) + \sigma(n + 1 - i) = n + 1$ and these correspond to transversals of $C(x_0, x_1, \ldots, x_{n-1})$ whose positions have a centrosymmetric matrix pattern, that is, transversal patterns invariant under a 180 degree rotation. There are permutations that are both symmetric and centrosymmetric.

Example 5.1. Let $n = 4$ and let $\sigma = (2, 1, 4, 3)$. As a permutation matrix, σ equals

$$\begin{bmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{bmatrix}$$

which is invariant under a reflection about the diagonal and a rotation of 180 degrees. Thus σ is both an involution (invariant under a reflection about the main diagonal) and a centrosymmetric permutation (invariant under a 180 degree rotation). Notice that σ is also invariant under reflection about the antidiagonal running from the lower left to the upper right, and this holds in general for permutations that are both symmetric and centrosymmetric. ♢

Let $U = \{u_1, u_2, \ldots, u_n\}$ be a multiset where $u_i \in \{0, 1, \ldots, n-1\}$ for $0 \leq i \leq n-1$. We say that U is *balanced* $\bmod n$ provided that its nonzero elements can be paired as $\{a, b\}$ so that $a + b \equiv 0 \pmod{n}$. Thus if n is even, 0 and $n/2$ each occur an even, possibly zero, number of times, and if n is odd, 0 occurs an odd number of times. If U is balanced $\bmod n$, then it is an immediate consequence of Theorem 2.1 that U is a juggleable set with each x_i with $i \neq 0$ occurring with an even, possibly zero, exponent in $\mathrm{per}(C(x_0, x_1, \ldots, x_{n-1}))$.

Example 5.2. Let $n = 8$ and let $U = \{0, 0, 1, 7, 1, 7, 4, 4\}$. Then U is balanced $\bmod 8$ and hence is a juggleable set. In $C(x_0, x_1, \ldots, x_7)$ below we have realizations

	1	2	3	4	5	6	7	8
1	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7
2	x_7	x_0	x_1	x_2	x_3	x_4	x_5	x_6
3	x_6	x_7	x_0	x_1	x_2	x_3	x_4	x_5
4	x_5	x_6	x_7	x_0	x_1	x_2	x_3	x_4
5	x_4	x_5	x_6	x_7	x_0	x_1	x_2	x_3
6	x_3	x_4	x_5	x_6	x_7	x_0	x_1	x_2
7	x_2	x_3	x_4	x_5	x_6	x_7	x_0	x_1
8	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_0

corresponding to the term $x_0^2 x_1^2 x_4^2 x_7^2$ in $\mathrm{per}(C(x_0, x_1, \ldots, x_7))$, achieved in the permanent $\mathrm{per}(C(x_0, x_1, \ldots, x_7))$ by an involution (dark gray) and by a centrosymmetric permutation (light gray). ♢

Example 5.3. Let $n = 6$ and consider the multiset $U = \{2, 2, 2, 4, 4, 4\}$ balanced $\bmod 6$ with the pairing $\{2, 4\}, \{2, 4\}, \{2, 4\}$. In both case we seek a corresponding transversal in

$$\left[\begin{array}{c|c|c|c|c|c} x_0 & x_1 & x_2 & x_3 & x_4 & x_5 \\ \hline x_5 & x_0 & x_1 & x_2 & x_3 & x_4 \\ \hline x_4 & x_5 & x_0 & x_1 & x_2 & x_3 \\ \hline x_3 & x_4 & x_5 & x_0 & x_1 & x_2 \\ \hline x_2 & x_3 & x_4 & x_5 & x_0 & x_1 \\ \hline x_1 & x_2 & x_3 & x_4 & x_5 & x_0 \end{array}\right],$$

consisting of three x_2's and three x_4's. We have indicated such a realization in the centrosymmetric case, but it is straightforward to check that it cannot be attained by an involution. ♢

We have done a substantial amount of calculation with the following consequences:

(i) For $n \leq 19$ a prime, all balanced $\bmod n$ multisets can be achieved by a transversal with a symmetric pattern. When $n = 15$, there are 16 balanced $\bmod 15$ multisets that cannot be achieved by a transversal with a symmetric pattern, e.g. the multiset $\{0, 6, 6, 6, 6, 6, 6, 6, 9, 9, 9, 9, 9, 9, 9\}$ cannot be so achieved. On the other hand, for $n = 18$, there are 48 620 balanced $\bmod 18$ multisets satisfying (2.1) and only 36 195 can be achieved with a symmetric pattern.

(ii) For odd $n \leq 21$, all balanced $\bmod n$ multisets can be achieved by a transversal with a centrosymmetric pattern.

As a consequence of the data obtained we make two conjectures:

Conjecture 5.4. *If n is a prime, then every balanced $\bmod n$ multiset can be achieved by a transversal with a symmetric pattern.*

Conjecture 5.5. *If n is odd, then a balanced $\bmod n$ multiset can be achieved by a transversal with a centrosymmetric pattern. If n is even, then the unachievable balanced $\bmod n$ multisets only have terms with the same parity.*

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