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Mixed hypergraphs and beyond*

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Dedicated to Mario Gionfriddo on the occasion of his 70th birthday.

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Abstract

Some open problems are collected on hypergraphs, graphs, and designs, presented at the HyGraDe conference celebrating Mario Gionfriddo's 70th birthday.

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The conference HyGraDe took its name from HYpergraphs, GRAphs and DEsigns, three important areas of the research activities of Mario Gionfriddo, to whom we happily dedicated all our talks. Those are also the subjects of my collaborations with colleagues in Catania. For the celebration conference I collected some open problems which are related to the coloring theory of mixed hypergraphs; here they are organized in this three-sided structure. The sources of the problems are mentioned in the text, rather than specified inside the statement of each one.

1 Hypergraph coloring

A hypergraph $\mathcal{H}$ is a pair $(X, \mathcal{E})$, where X is the underlying set called *vertex set* and $\mathcal{E}$ is a set system over X, called *edge set*. A hypergraph is *uniform* if all its edges have the same cardinality; more specifically, if $|E| = r$ for all $E \in \mathcal{E}$, then $\mathcal{H}$ is said to be r-*uniform*. (Hence, the 2-uniform hypergraphs are precisely the graphs.) In order to avoid some anomalies, we shall restrict our attention to hypergraphs in which each edge contains at least two vertices.

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As a general term, by *coloring* we mean any assignment $\varphi\colon X \to \mathbb{N}$, and call $\varphi(x)$ the *color* of vertex $x \in X$.

The classical notion of *proper coloring* means a coloring such that every edge $E \in \mathcal{E}$ contains two vertices of distinct colors; in other words, no edge is monochromatic. The *chromatic number* of $\mathcal{H}$, denoted by $\chi(\mathcal{H})$, is the smallest possible number of colors in a proper coloring of $\mathcal{H}$.

The opposite side is where each edge $E \in \mathcal{E}$ contains two vertices of the same color; i.e., no edge is multicolored.[1] Motivated by Voloshin's works, we use the term *C-coloring* for a coloring of this type, and if a hypergraph has to be colored in this way, it will be called a *C-hypergraph*. The *upper chromatic number* of a C-hypergraph $\mathcal{H}$, denoted by $\overline{\chi}(\mathcal{H})$, is the largest possible number of colors in a C-coloring of $\mathcal{H}$.

Proper hypergraph coloring is a direct generalization of the fundamental notion of proper graph coloring; research in this direction started in the mid-1960's. On the other hand, C-coloring in graphs is not really interesting as it simply means that each connected component is monochromatic. For hypergraphs, however, such problems become highly nontrivial; the first such questions arose in the first half of the 1970's. But it took two decades until Voloshin independently introduced the notion and also created a model far beyond that, as we shall discuss below.

A comparison of some basic properties of proper colorings and C-colorings is given in Table 1. It is important to emphasize that every number of colors is possible between minimum and maximum; indeed, in a proper coloring it is feasible to split any non-singleton color class into two, while in a C-coloring any two color classes may be united. This simple observation will have a relevance later.

Table 1: Some coloring properties.

	proper coloring	C-coloring
excluded edge type	monochromatic	multicolored
always colorable with	$\lvert X\rvert$ colors (= max)	1 color (= min)
interesting parameter	χ = min # of colors	$\overline{\chi}$ = max # of colors

A general overview on hypergraph colorings — not only these two types — can be found in [13]; and a comprehensive survey on C-coloring is given in [9].

Mixed hypergraphs. A new dimension in the theory of hypergraph coloring was opened in the works of Voloshin [28, 29] where he invented the following complex model. A *mixed hypergraph* has two types of edges, namely C-edges and D-edges; formally we may write $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$. The requirement for a coloring $\varphi\colon X \to \mathbb{N}$ is that every C-edge $E \in \mathcal{C}$ has to contain two vertices with **c**ommon color and every D-edge $E \in \mathcal{D}$ has to contain two vertices with **d**istinct colors. In other words, φ should be a proper coloring of $(X, \mathcal{D})$ and a C-coloring of $(X, \mathcal{C})$ at the same time.

There is no a priori assumption on the relation between $\mathcal{C}$ and $\mathcal{D}$, they may or may not be disjoint. Edges in $\mathcal{C} \cap \mathcal{D}$ are termed *bi-edges*, and if $\mathcal{C} = \mathcal{D}$ then $\mathcal{H}$ is called a *bi-*

[1] By 'multicolored' we mean that the colors of the elements are mutually distinct. Such a set is often called a *rainbow* set in the literature.

hypergraph. A coloring of a bi-hypergraph — termed *bi-coloring* — is a proper coloring and a C-coloring at the same time.

Contrary to proper colorings and C-colorings, which always exist for every hypergraph, a mixed hypergraph may not admit any coloring; in this case it is called *uncolorable*. For instance, the bi-hypergraph whose bi-edges are the ten 3-element subsets of a 5-element vertex set, is uncolorable because either at least three colors occur (violating the condition of C-coloring) or some color occurs on at least three vertices (violating proper coloring).

If a mixed hypergraph $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ is colorable, its *lower chromatic number* denoted by $\chi(\mathcal{H})$ is the smallest possible number of colors, and its *upper chromatic number* $\overline{\chi}(\mathcal{H})$ is the largest possible number of colors. The feasible set $\Phi(\mathcal{H})$ of $\mathcal{H}$ is the set of those integers k for which $\mathcal{H}$ admits a coloring with precisely k colors.

A comprehensive account on the first decade of results and methods concerning mixed hypergraphs is the monograph [30].

Stably bounded hypergraphs. A structure more general than mixed hypergraphs was introduced in two steps, in the papers [5, 7] and [6], and studied further in a series of papers. A *stably bounded hypergraph* is a hypergraph $\mathcal{H} = (X, \mathcal{E})$ for which also four functions $s, t, a, b \colon \mathcal{E} \to \mathbb{N}$ are given. The first two of them prescribe lower and upper bounds on the number of colors occurring inside the edges, and the other two prescribe bounds for each edge on the multiplicity of the color occurring most frequently in it. We assume

$$1 \le s(E) \le t(E) \le |E|$$

and

$$1 \le a(E) \le b(E) \le |E|$$

for every $E \in \mathcal{E}$. A coloring φ is feasible if, for each $E \in \mathcal{E}$, we have:

- φ uses at least $s(E)$ colors inside E,
- φ uses at most $t(E)$ colors inside E,
- there exists a color which is assigned to least $a(E)$ vertices of E,
- no color is assigned to more than $b(E)$ vertices of E.

Hence, if E is a C-edge of a mixed hypergraph then its requirements are $t(E) = |E| - 1$ and $a(E) = 2$; and if it is a D-edge, then $s(E) = 2$ and $b(E) = |E| - 1$. In fact one of a and t suffices to describe a C-edge, and one of s and b suffices to describe a D-edge.

Stating the conditions in other words, the functions s and t restrict the sizes of largest multicolored subsets inside the edges, while a and b restrict the sizes of their largest monochromatic subsets.

The lower chromatic number $\chi(\mathcal{H})$, the upper chromatic number $\overline{\chi}(\mathcal{H})$, and the feasible set $\Phi(\mathcal{H})$ are naturally defined in the same way as for mixed hypergraphs.

The conditions $s(E) = 1$, $t(E) = |E|$, $a(E) = 1$, $b(E) = |E|$ put no restriction on the coloring of edge E. We obtain functional subclasses of stably bounded hypergraphs if we prescribe the set of functions which are allowed to be restrictive. For instance, (S,T,A)-hypergraph means that the functions s, t, and a can put restrictions on (some of) the edges, but b must be non-restrictive for all edges. An interesting subclass is that of (S,T)-hypergraphs, termed *color-bounded hypergraphs*. Earlier examples in the literature may be interpreted as B-hypergaphs [1, 24] and S-hypergaphs [15].

Table 2: Coloring restrictions determined by the functions s, t, a, b.

function	meaning
s	at least $s(E)$ colors inside E
t	at most $t(E)$ colors inside E
a	some color at least $a(E)$ times inside E
b	each color at most $b(E)$ times inside E

Interval hypergraphs. A hypergraph $\mathcal{H} = (X, \mathcal{E})$ is called an *interval hypergraph* if its vertex set X admits an ordering $x_1, x_2, \ldots, x_n$ such that every edge $E \in \mathcal{E}$ is a set of consecutive vertices in this order. Interval hypergraphs have many nice properties and admit efficient algorithms for various problems which are intractable on general structures.

Problem 1.1. Determine the time complexity of the following problems over the given functional subclasses of stably bounded *interval* hypergraphs:

1. Colorability of (S,T)-hypergraphs.
2. Lower chromatic number of (S,A)-hypergraphs.
3. Lower chromatic number of (S,T,A)-hypergraphs.
4. Upper chromatic number of (S,T)-hypergraphs.
5. Upper chromatic number of A-hypergraphs.
6. Upper chromatic number of (T,A)-hypergraphs.
7. Upper chromatic number of (T,B)-hypergraphs.
8. Upper chromatic number of (S,T,B)-hypergraphs.

Table 3: Solved and unsolved cases — time complexity of basic coloring problems on seven functional subclasses of stably bounded interval hypergraphs; ??? = open, o = obvious, lin = solvable in linear time, NP-c = NP-complete, NP-h = NP-hard.

	S,T	A / T,A	S,A / S,T,A	T,B / S,T,B
exists?	???	o	NP-c	NP-c
min	lin	o	???	NP-h
max	???	???	NP-h	???

On interval hypergraphs, complexity is known for all the other combinations of the four functions s, t, a, b. Subsets of those results are proved in different papers, the last pieces appearing in [10], where also a detailed summary for several further classes of hypergraphs is given. Note that each of T, A, and (T,A) admits a monochromatic X, whereas each of S, B, and (S,B) admits a multicolored X.

Table 4: The other functional subclasses, time complexity solved completely on interval hypergraphs.

	T	S / B / S,B	A,B / any larger
exists?	o	o	NP-c
min	o	lin	NP-h
max	lin	o	NP-h

Gaps. The feasible set $\Phi(\mathcal{H})$ of a colorable hypergraph $\mathcal{H}$ is called *gap-free* if it is an interval of integers. If this property does not hold, then we say that $\mathcal{H}$ has a *gap* at k (also called 'gap in the chromatic spectrum') if k is an integer such that $\chi(\mathcal{H}) < k < \overline{\chi}(\mathcal{H})$ and $k \notin \Phi(\mathcal{H})$.

If $1 \in \Phi(\mathcal{H})$, then the feasible set is gap-free. On the other hand, for every finite set W of positive integers with $1 \notin W$, in [20] a mixed hypergraph $\mathcal{H}$ is constructed such that $\Phi(\mathcal{H}) = W$. Since $|X| \in \Phi(\mathcal{H})$ also guarantees that $\Phi(\mathcal{H})$ is gap-free, a hypergraph with gaps in $\Phi(\mathcal{H})$ necessarily has both C-edges and D-edges.

It is interesting to investigate which classes of hypergraphs have members with gaps in the chromatic spectrum, and which are completely gap-free. For instance, a gap-free class is that of interval hypergraphs [21], and the property remains valid also for interval (S,T)-hypergraphs. Also, mixed 'hypertrees' — hypergraphs $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ which can be represented over a tree graph such that each hyperedge $E \in \mathcal{C} \cup \mathcal{D}$ induces a subtree — have a gap-free $\Phi(\mathcal{H})$ [23], but this property does not extend for (S,T)-hypertrees [8].

Another famous example is the class of planar mixed hypergraphs, which admit constructions with $\Phi(\mathcal{H}) = \{2, 4\}$, hence a gap at 3 [22].

Problem 1.2.

1. Can interval (S,A)-hypergraphs have gaps?
2. Can interval (T,B)-hypergraphs have gaps?
3. Can interval (A,B)-hypergraphs have gaps?
4. What gaps can occur in stably bounded planar hypergraphs and in their functional subclasses?

The planar case is open also for color-bounded hypergraphs.

Chromatic polynomials. Let $\mathcal{H} = (X, \mathcal{E})$ be a hypergraph in any of the models above (mixed, stably bounded, etc.), and assume that $\mathcal{H}$ is colorable. For λ running over the natural numbers, it is known that the number of allowed colorings

$$\varphi \colon X \to \{1, \dots, \lambda\}$$

is a polynomial in λ, more precisely a polynomial of degree $\overline{\chi}(\mathcal{H})$. It is called the *chromatic polynomial* of $\mathcal{H}$, denoted by $P(\mathcal{H}, \lambda)$.

For any class $\mathfrak{H}$ of hypergraphs, one can consider the class

$$\{P(\mathcal{H}, \lambda) \mid \mathcal{H} \in \mathfrak{H}\}$$

of chromatic polynomials. From this point of view, the partial order for functional subclasses of mixed and stably bounded hypergraphs is determined in [6], as illustrated in Figure 1. Also, the chromatic polynomials of non-1-colorable hypergraphs (i.e., of those containing at least one D-edge) is characterized [7], in terms of Stirling numbers of the second kind.

$$
\begin{tikzcd}[column sep=small]
& \text{any } Y \subseteq \{\text{S,T,A,B}\} \text{ such that } \{\text{S,A}\} \subseteq Y \text{ or } \{\text{A,B}\} \subseteq Y \arrow[dl] \arrow[dr] & & \\
\text{A / A,T} \arrow[dr] & & \text{S,T,B / S,T / T,B / M} \arrow[dl] \arrow[dr] & \\
& \text{T / C} & & \text{S / S,B} \arrow[d] \\
& & & \text{B / D}
\end{tikzcd}
$$

Figure 1: Hierarchy of classes of chromatic polynomials; M = mixed hypergraphs, C = only C-edges, D = only D-edges.

Problem 1.3.

1. Characterize those polynomials which are chromatic polynomials of a given type of 1-*colorable* hypergraphs.
2. Determine the hierarchy analogous to the one exhibited in Figure 1 when the hypergraphs have a structural property (e.g., interval hypergraphs). How does the hierarchy depend on the structure?

The requirement of 1-colorability in Problem 1.3.1 means the restriction to C-hypergraphs for mixed, T-hypergraphs for color-bounded, and A-hypergraphs or (T,A)-hypergraphs for those subclasses of stably bounded hypergraphs which are not color-bounded.

2 Graphs

There are many problems in graph theory which can be interpreted in terms of colorings of mixed hypergraphs. Here we discuss only one of them.

***F*-WORM colorings.** Let F be a fixed graph with at least three vertices. For a graph $G = (V, E)$, a vertex coloring φ is an F-*WORM coloring* if the vertex set of every subgraph isomorphic to F in G is neither monochromatic nor multicolored. ('WORM' abbreviates '**wit**hout **r**ainbow or **m**onochromatic'.)

The notion was introduced not much time ago, in [19], which actually appeared later than the second paper [18]. Further early works on the subject are [11] and [12].

Three basic coloring problems, also for F-WORM colorings, are whether a given G is colorable, and if it is, then what is the minimum and maximum number of colors in an F-WORM coloring of G. This similarity to the previous section is no surprise because one can observe that F-WORM coloring of G precisely means a feasible coloring of the bi-hypergraph whose bi-edges are the subsets $B \subset V$ such that $|B| = |V(F)|$ and the induced subgraph $G[B]$ contains a subgraph isomorphic to F.

Many aspects of mixed hypergraphs can be raised for F-WORM colorings as well, and also further questions arise. Here we mention only some of the interesting problems.

Problem 2.1. Let F be a connected graph with at least three vertices.

1. Is it NP-complete to decide whether a generic graph G admits an F-WORM coloring?
2. What is the necessary and sufficient condition for F to ensure that the minimum number of colors in an F-WORM coloring is bounded above by a universal constant for all F-WORM colorable graphs?
3. What is the time complexity of computing the minimum number of colors?
4. What is the complexity of deciding whether the feasible set (set of those numbers k of colors for which a generic input graph G admits an F-free coloring with precisely k colors) is gap-free?
5. Can the F-WORM feasible set contain any large gaps?
6. Study similar problems assuming that G belongs to a particular class of graphs.

Partial results are known to these questions, but the case of general F seems to be open.

3 Designs

A *Steiner system* $\mathrm{S}(t, k, v)$ is a k-uniform hypergraph with v vertices, such that each t-tuple of vertices is contained in precisely one edge (also called block). Viewing such systems from the direction of mixed hypergraphs, several interesting approaches arise. For instance, if each block is considered as a C-edge, we obtain a C-S(t, k, v) system. Another possibility[2] is to assume that each block is a bi-edge; then we have a B-S(t, k, v) system. Particular types are the systems B-STS(v), C-STS(v), B-SQS(v), C-SQS(v), derived from *Steiner triple* and *quadrulpe systems* (where $(t, k) = (2, 3)$ or $(t, k) = (3, 4)$, respectively), cf. also [27]. Besides, we consider here finite geometries, too.

Finite projective planes. It is proved in [3] that if the points of a projective plane of order q are colored in such a way that no line is multicolored, then the number of colors cannot exceed $q^2 - q - \Theta(q^{1/2})$ as $q \to \infty$; i.e., this function is an upper bound on the upper chromatic number. The bound is tight for an infinite sequence of planes, and it is even proved in [2] that an optimal C-coloring is obtained by making a 'double blocking set' (a set that meets every line in at least two points) monochromatic and assigning a distinct color to every point outside this set, provided that the plane is a Desarguesian plane $\mathrm{PG}(2, q)$ of sufficiently large order.

Problem 3.1.

1. Find a tight general *lower* bound on the upper chromatic number for every finite projective plane of order q.
2. Find estimates on the upper chromatic number of other types of finite geometries.
3. Study further types of colorings of finite geometries.

[2]In fact many more possibilities arise when larger block sizes are considered.

Steiner quadruple systems. It is known that for every fixed $t \geq 2$ the upper chromatic number of a C-S$(t, t+1, v)$ system is at most $c_t \log v$ for some constant c_t [26]. However, a tight estimate is available only for triple systems, as we shall mention below. For quadruple systems of order $v = 2^m$ a repeated application of the 'doubling construction' shows that the upper chromatic number can be at least $m+1$ in general. The method is: start with two vertex-disjoint systems $\mathcal{H}_1 = (X_1, \mathcal{E}_1)$ and $\mathcal{H}_2 = (X_2, \mathcal{E}_2)$ of order v, take 1-factorizations $(F_1^i, \ldots, F_{v-1}^i)$ of the complete graphs whose vertex set is X_i for $i = 1, 2$; and then the blocks in the system of order $2v$ are those in $\mathcal{H}_1 \cup \mathcal{H}_2$ moreover the 4-tuples of the form $e_j^1 \cup e_k^2$ where $e_j^1 \in F_j^1$ and $e_k^2 \in F_k^2$, for all combinations (j, k) with $1 \leq j, k \leq v-1$.

Problem 3.2.

1. Do there exist uncolorable B-SQS(v) systems?
2. Does every $\mathcal{H} =$ C-SQS(2^m) have $\overline{\chi}(\mathcal{H}) \leq m+1$?
3. Does there exist an infinite sequence of B-SQS(v) systems with unbounded upper chromatic number?

A complete answer to parts 2 and 3 seems to be unknown even for quadruple systems obtained by the repeated application of the doubling construction, starting from a single 4-element block on four vertices. (Such systems always admit a bi-coloring — their feasible set is $\{2, 3\}$ when viewed as bi-hypergraphs — hence they are not relevant concerning part 1.)

Steiner triple systems. The 'doubling plus one' construction builds an STS$(2v+1)$ from an STS(v). The method is: start with a triple system $\mathcal{H} = (X, \mathcal{E})$ of order v, where $X = \{x_1, \ldots, x_v\}$; let X' be a set of $v+1$ further vertices, disjoint from X; take a 1-factorization[3] $(F_1, \ldots, F_v)$ of the complete graphs whose vertex set is X'; and create the triples of the form $x_i \cup e$ where $e \in F_i$. Together with the edges of $\mathcal{H}$, this yields a Steiner triple system over $X \cup X'$.

The coloring requirement on a B-STS system means that each block (triple) has to contain precisely two colors. In a C-STS system, monochromatic blocks may also occur (and the lower chromatic number is 1).

In [25], the first paper dealing with C-STS and B-STS (and also with SQS) systems, it is proved that if $v < 2^m$, then the upper chromatic number of every STS(v) is at most m; moreover, $\overline{\chi} = m$ is attained for exactly those systems which are obtained by a sequence of doubling plus one constructions starting from the trivial system of order 3 with one triple.

For such B-STS systems, Mario Gionfriddo raised the following attractive problem in [16].

Conjecture 3.3. *If a B-STS$(2^m - 1)$ system $\mathcal{H}$ is obtained from B-STS(3) by a sequence of doubling plus one constructions, then it has $\chi(\mathcal{H}) = \overline{\chi}(\mathcal{H}) = m$.*

In other words, no bi-coloring of B-STS$(2^{m-1} - 1)$ can be extended to the B-STS$(2^m - 1)$ without increasing the number of colors, i.e., the latter system does not admit any 'extended bi-coloring'.

[3]Since v is odd — more precisely $v \equiv 1 \vee 3 \pmod 6$ — we have $v+1$ even, therefore the edge set of the complete graph K_{v+1} can be decomposed into 1-factors.

One approach to the conjecture is to assume that a B-STS$(2^m - 1)$, obtained from a B-STS$(2^{m-1} - 1)$ by doubling plus one, admits a bi-coloring with $m - 1$ colors, and to investigate what types of size distributions of the color classes might occur. The first necessary conditions of this kind are given in [14]. The recent paper [4] makes further steps in this direction, and also describes a doubling-plus-one sequence constructed explicitly over $\mathrm{GF}(2)$, which is proved to not admit any extended bi-coloring.

It is important that the construction be started from B-STS(3), because other systems may admit extended bi-colorings [17]. The smallest known example is $v = 13 \to 2v + 1 = 27$, which has an extended bi-coloring with 3 colors.

4 Conclusion

Mixed hypergraph is a great invention. By the combination of two antipodal concepts a new dimension has been opened for coloring theory. The above collection of problems is just an appetizer, lots of interesting further ones remain unsolved, for instance to characterize nice classes of colorable hypergraphs. Moreover, mixed hypergraphs and their generalizations can describe several issues in graph theory as well. WORM coloring considered above is just one example; one can mention areas in Ramsey theory, and more.

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