

The k -independence number of graph products*

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Abstract

The concept of k -independence number is a natural generalization of classical independence number. A k -independent set is a set of vertices whose induced subgraph has maximum degree at most k . The k -independence number of G , denoted by $\alpha_k(G)$, is defined as the maximum cardinality of a k -independent set of G . In this paper, we study the k -independence number on the lexicographical, strong, Cartesian and direct product and present several upper and lower bounds for these products of graphs.

Keywords: Independence number, k -independent set, k -independence number, lexicographical product, strong product, Cartesian product, direct product.

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1 Introduction

Graphs considered in this paper are undirected, finite and simple. We refer to [1] for undefined notations and terminology. In particular, we use $\Delta(G)$ and $\delta(G)$ to denote the maximum degree and minimum degree of a graph G , respectively. If $X \subseteq V(G)$ or $X \subseteq E(G)$, then $G[X]$ is the subgraph of G induced by X . For two subsets X and Y of $V(G)$ we denote by $E_G[X, Y]$ the set of edges of G with one end in X and the other end in Y .

Independence number is one of the most basic concepts in graph theory. A subset $S \subseteq V(G)$ is said to be *independent* if $E(G[S]) = \emptyset$. The *independence number* of G denoted by $\alpha(G)$ is the size of a maximum independent set in G . In [6, 7], Fink and Jacobson generalized the concept of independent set. In this paper, k will be an integer. We say that a subset S of V is *k -independent* if $\Delta(G[S]) \leq k$, that is, the maximum degree of the subgraph induced by the vertices of S is less or equal to k . The *k -independence number*, denoted $\alpha_k(G)$, as the maximum cardinality of a k -independent set. Thus for $k = 0$, the 0-independent is the classical independent set. Every k -independent set is $(k + 1)$ -independent; so $\alpha_{k+1}(G) \geq \alpha_k(G)$ for a graph G . Moreover, the vertex set V is the only maximal Δ -independent but is not a $(\Delta - 1)$ -independent set. Thus every graph G satisfies

$$\alpha(G) = \alpha_0(G) \leq \alpha_1(G) \leq \alpha_2(G) \leq \dots \leq \alpha_{\Delta-1}(G) < \alpha_{\Delta}(G) = n.$$

For k -independent set and k -independence number, Chellali, Favaron, Hansberg, and Volkmann published a survey paper on this subject; see [3]. We must mention that the k -independence number of G is defined as the size of a largest k -colorable subgraph of G in [17].

In graph theory, Cartesian product, strong product, lexicographical product, and direct product are four of main products, each with its own set of applications and theoretical interpretations. Product networks were proposed based upon the idea of using the cross product as a tool for “combining” two known graphs with established properties to obtain a new one that inherits properties from both [5]. For more details on graph products, we refer to the book [10].

- The *Cartesian product* of two graphs G and H , written as $G \square H$, is the graph with vertex set $V(G) \times V(H)$, in which two vertices (u, v) and (u', v') are adjacent if and only if $u = u'$ and $(v, v') \in E(H)$, or $v = v'$ and $(u, u') \in E(G)$.
- The *lexicographic product* $G \circ H$ of graphs G and H has the vertex set $V(G \circ H) = V(G) \times V(H)$. Two vertices $(u, v), (u', v')$ are adjacent if $uu' \in E(G)$, or if $u = u'$ and $vv' \in E(H)$.
- The *strong product* $G \boxtimes H$ of graphs G and H has the vertex set $V(G) \times V(H)$. Two vertices (u, v) and (u', v') are adjacent whenever $uu' \in E(G)$ and $v = v'$, or $u = u'$ and $vv' \in E(H)$, or $uu' \in E(G)$ and $vv' \in E(H)$.
- The *direct product* $G \times H$ of graphs G and H has the vertex set $V(G) \times V(H)$. Two vertices (u, v) and (u', v') are adjacent if the projections on both coordinates are adjacent, i.e., $uu' \in E(G)$ and $vv' \in E(H)$.

Note that unlike the other three products, the lexicographic product is a non-commutative product since $G \circ H$ is usually not isomorphic to $H \circ G$.

For the independence number of Cartesian product graphs, Vizing [16] observed:

Theorem 1.1 ([10, 16]). *For any graphs G and H ,*

- (i) $\alpha(G \square H) \leq \min\{\alpha(G)|V(H)|, \alpha(H)|V(G)|\}$;
- (ii) $\alpha(G \square H) \geq \alpha(G)\alpha(H) + \min\{|V(G)| - \alpha(G), |V(H)| - \alpha(H)\}$.

Geller and Stahl [9] obtained the following result for the independence number of lexicographical product graphs.

Theorem 1.2 ([9]). *For any graphs G and H , $\alpha(G \circ H) = \alpha(G)\alpha(H)$.*

The following result is immediate, since $G \boxtimes H$ is a subgraph of $G \circ H$.

Corollary 1.3 ([10]). *For any graphs G and H , $\alpha(G \boxtimes H) \geq \alpha(G)\alpha(H)$.*

In 2011, Špacapan [17] proved the following theorem.

Theorem 1.4 ([17]). *For any graph G and H ,*

- (i) $\alpha(G \times H) \geq \max\{\alpha(G)|V(H)|, \alpha(H)|V(G)|\}$;
- (ii) $\alpha(G \times H) \leq \alpha(H)|V(G)| + \alpha(G)|V(H)| - \alpha(H)\alpha(G)$.

For the independence number of four graph products, Jha and Slutzki obtained the following relation in 1994.

Theorem 1.5 ([12]). *For any graphs G and H ,*

$$\alpha(G \circ H) \leq \alpha(G \boxtimes H) \leq \alpha(G \square H) \leq \alpha(G \times H).$$

In this paper, we consider four standard products: the lexicographic, the strong, the Cartesian and the direct with respect to the k -independence number. Every of these four products will be treated in one of the forthcoming subsections in Section 2. Our results can be seen as extensions of Theorems 1.1, 1.2, 1.4, 1.5 and Corollary 1.3.

2 Main results

In this section, let G and H be two connected graphs with $V(G) = \{u_1, u_2, \dots, u_n\}$ and $V(H) = \{v_1, v_2, \dots, v_m\}$, respectively. Then $V(G * H) = \{(u_i, v_j) \mid 1 \leq i \leq n, 1 \leq j \leq m\}$, where $*$ denotes lexicographic product operation, strong product operation, Cartesian product operation or direct product operation. For $v \in V(H)$, we use $G(v)$ to denote the subgraph of $G * H$ induced by the vertex set $\{(u_i, v) \mid 1 \leq i \leq n\}$. Similarly, for $u \in V(G)$, we use $H(u)$ to denote the subgraph of $G * H$ induced by the vertex set $\{(u, v_j) \mid 1 \leq j \leq m\}$.

2.1 The lexicographic product

In this subsection, we give upper and lower bounds of $\alpha_k(G \circ H)$.

Theorem 2.1. (i) *Let $k \geq 0$ be an integer. For graphs G and H ,*

$$\alpha_k(G \circ H) \leq \alpha_k(H)|V(G)|.$$

(ii) Let $k, r \geq 0$ be two integers. Let H be a graph of order m . For graphs G and H ,

$$\alpha_k(G \circ H) \geq \alpha_r(G)\alpha_{k-rm}(H)$$

where $\alpha_{k-rm}(H) = 0$ if $k \leq rm$.

Moreover, the bounds are sharp.

Proof. (i) Let I be a maximum k -independent set of $G \circ H$. We claim that $|I \cap V(H(u_i))| \leq \alpha_k(H(u_i))$ for each $u_i \in V(G)$. To see this, we observe that $H(u_i)[I \cap V(H(u_i))]$ is a subgraph of $G \circ H[I]$.

(ii) Let I be a maximum r -independent set of G , and J be a maximum $(k - rm)$ -independent set of H . Set

$$I = \{u_i \mid 1 \leq i \leq s\} \text{ and } J = \{v_j \mid 1 \leq j \leq t\}.$$

For any $(u_i, v_j) \in I \times J$, we show that the degree of (u_i, v_j) in $G \circ H[I \times J]$ is at most k . Since I is a maximum r -independent set of G , it follows that $d_{G[I]}(u_i) \leq r$, where $u_i \in V(G[I])$. Similarly, since J is a maximum $(k - mr)$ -independent set of H , it follows that $d_{H[J]}(v_j) \leq k - mr$, where $v_j \in V(H[J])$. Then

$$d_{G \circ H[I \times J]}(u_i, v_j) \leq d_{H[J]}(v_j) + md_{G[I]}(u_i) \leq k - mr + mr = k,$$

and hence $I \times J$ is a k -independent set of $G \circ H$. So $\alpha_k(G \circ H) \geq \alpha_r(G)\alpha_{k-rm}(H)$.

See Remarks 2.4 and 2.5 for the sharpness. □

2.2 The strong product

In this subsection, we derive upper and lower bounds of $\alpha_k(G \boxtimes H)$.

Theorem 2.2. (i) Let $k \geq 0$ be an integer. For graphs G and H ,

$$\alpha_k(G \boxtimes H) \leq \min\{\alpha_k(G)|V(H)|, \alpha_k(H)|V(G)|\}.$$

(ii) Let $k, r \geq 0$ be two integers. For graphs G and H ,

$$\alpha_k(G \boxtimes H) \geq \alpha_r(G)\alpha_{\lfloor \frac{k}{2r+1} \rfloor}(H)$$

and

$$\alpha_k(G \boxtimes H) \geq \alpha_r(H)\alpha_{\lfloor \frac{k}{2r+1} \rfloor}(G).$$

Moreover, the bounds are sharp.

Proof. (i) Let I be a maximum k -independent set of $G \boxtimes H$. If $|G(v_j) \cap I| > \alpha_k(G)$ for some $j \leq m$, then I is not a k -independent set in $G \boxtimes H$. It follows $\alpha_k(G \boxtimes H) \leq \alpha_k(G)|V(H)|$. From the symmetry, we have

$$\alpha_k(G \boxtimes H) \leq \min\{\alpha_k(G)|V(H)|, \alpha_k(H)|V(G)|\}.$$

(ii) Let I be a maximum r -independent set of G , and J be a maximum $(\frac{k}{2r+1})$ -independent set of H . Set

$$I = \{u_i \mid 1 \leq i \leq s\} \text{ and } J = \{v_j \mid 1 \leq j \leq t\}.$$

For any $(u_i, v_j) \in I \times J$, we show that the degree of (u_i, v_j) in $G \circ H[I \times J]$ is at most k . Since I is a maximum r -independent set of G , it follows that $d_{G[I]}(u_i) \leq r$, where $u_i \in V(G[I])$. Similarly, since J is a maximum $(\frac{k}{2r+1})$ -independent set of H , it follows that $d_{H[J]}(v_j) \leq \frac{k}{2r+1}$, where $v_j \in V(H[J])$. Then

$$d_{G \boxtimes H[I \times J]}(u_i, v_j) \leq d_{H[J]}(v_j) + \frac{2k}{2r+1} d_{G[I]}(u_i) \leq \frac{k}{2r+1} + \frac{2rk}{2r+1} = k,$$

and hence $I \times J$ is a k -independent set of $G \boxtimes H$. So $\alpha_k(G \boxtimes H) \geq \alpha_r(G) \alpha_{\lfloor \frac{k}{2r+1} \rfloor}(H)$.

See Remarks 2.4 and 2.5 for the sharpness. \square

2.3 The Cartesian product

Upper and lower bounds of $\alpha_k(G \square H)$ are derived in this subsection.

Theorem 2.3. *Let $k, r \geq 0$ be two integers. For graphs G and H ,*

- (i) $\alpha_k(G \square H) \leq \min\{\alpha_k(G)|V(H)|, \alpha_k(H)|V(G)|\}$;
- (ii) $\alpha_k(G \square H) \geq \alpha_r(G) \alpha_{k-r}(H)$

$$+ \begin{cases} st, & \text{if } k \geq s+t-2; \\ t(k-t+2), & \text{if } s \geq \lfloor \frac{k+3}{2} \rfloor, t < \lfloor \frac{k+3}{2} \rfloor, \\ & \text{and } k \leq s+t-3; \\ s(k-s+2), & \text{if } t \geq \lfloor \frac{k+3}{2} \rfloor, s < \lfloor \frac{k+3}{2} \rfloor, \\ & \text{and } k \leq s+t-3; \\ \min\{p, q\} (\lfloor \frac{k}{2} \rfloor + 1) (\lfloor \frac{k}{2} \rfloor + 1), & \text{if } s \geq \lfloor \frac{k+3}{2} \rfloor \text{ and } t \geq \lfloor \frac{k+3}{2} \rfloor, \end{cases}$$

where $0 \leq r \leq k$, $s = |V(G)| - \alpha_r(G)$, $t = |V(H)| - \alpha_{k-r}(H)$, $s = (\lfloor \frac{k}{2} \rfloor + 1)p + s'$, $t = (\lfloor \frac{k}{2} \rfloor + 1)q + t'$, $0 \leq s' < \lfloor \frac{k}{2} \rfloor + 1$ and $0 \leq t' < \lfloor \frac{k}{2} \rfloor + 1$.

Moreover, the bounds are sharp.

Proof. (i) The proof is similar to the proof of (i) of Theorem 2.1.

(ii) Suppose I is a r -independent set in G and J is a $(k-r)$ -independent set in H , respectively. We will prove that $I \times J$ is a k -independent set of $G \square H$. By commutativity, we may assume $|V(G)| - \alpha_r(G) \leq |V(H)| - \alpha_{k-r}(H)$. Say $V(H) \setminus J = \{y_1, y_2, \dots, y_t\}$, and take a subset $\{x_1, x_2, \dots, x_s\} \subseteq V(G) \setminus I$. Then $s \leq t$. Set

$$K = \{(x_i, y_j) \mid 1 \leq i \leq s, 1 \leq j \leq t\}.$$

Let $F = G \square H$. Since $F[K]$ is a spanning subgraph of $K_s \square K_t$, it follows that $\alpha_k(F[K]) \geq \alpha_k(K_s \square K_t)$, and hence there is a $\alpha_k(K_s \square K_t)$ -independent set of $F[K]$, say K' .

Claim 1: $(I \times J) \cup K'$ is a k -independent set of $G \square H$.

Proof of Claim 1. For any $(u_i, v_j) \in I \times J$ where $u_i \in V(G)$ and $v_j \in V(H)$, we have

$$d_{G \square H[I \times J]}(u_i, v_j) = d_{G[I]}(u_i) + d_{H[J]}(v_j) \leq r + (k-r) = k.$$

Therefore, $I \times J$ is a k -independent set of $G \square H$. From the structure of Cartesian product graphs, we have $E_{G \square H}[I \times J, K'] = \emptyset$. Then $(I \times J) \cup K'$ is a k -independent set of $G \square H$. \square

From Claim 1, we have $\alpha_k(G \square H) \geq |(I \times J) \cup K'| = \alpha_r(G)\alpha_{k-r}(H) + \alpha_k(K_s \square K_t)$ for graphs G and H .

If $k \geq s + t - 2$, then $(V(G) - I) \times (V(H) - J) = K_s \times K_t$ is a k -independent set of $K_s \square K_t$, and hence $\alpha_k(K_s \square K_t) \geq st$. If $s \geq \lfloor \frac{k+3}{2} \rfloor, t < \lfloor \frac{k+3}{2} \rfloor$, and $k \leq s + t - 3$, then $\alpha_k(K_s \square K_t) \geq \alpha_k(K_{k-t+2} \square K_t) \geq t(k - t + 2)$. Similarly, if $t \geq \lfloor \frac{k+3}{2} \rfloor, s < \lfloor \frac{k+3}{2} \rfloor$, and $k \leq s + t - 3$, then $\alpha_k(K_s \square K_t) \geq \alpha_k(K_s \square K_{k-s+2}) \geq s(k - s + 2)$. If $s \geq \lfloor \frac{k+3}{2} \rfloor, t \geq \lfloor \frac{k+3}{2} \rfloor$, then

$$\begin{aligned} \alpha_k(K_s \square K_t) &\geq \alpha_{\lceil \frac{k}{2} \rceil}(K_s)\alpha_{\lfloor \frac{k}{2} \rfloor}(K_t) + \alpha_k(K_{s-\lceil \frac{k}{2} \rceil-1} \square K_{t-\lfloor \frac{k}{2} \rfloor-1}) \\ &\geq \left(\left\lfloor \frac{k}{2} \right\rfloor + 1\right) \left(\left\lfloor \frac{k}{2} \right\rfloor + 1\right) + \alpha_k(K_{s-\lceil \frac{k}{2} \rceil-1} \square K_{t-\lfloor \frac{k}{2} \rfloor-1}) \\ &\geq \left(\left\lfloor \frac{k}{2} \right\rfloor + 1\right) \left(\left\lfloor \frac{k}{2} \right\rfloor + 1\right) + \alpha_{\lceil \frac{k}{2} \rceil}(K_{s-\lceil \frac{k}{2} \rceil-1})\alpha_{\lfloor \frac{k}{2} \rfloor}(K_{t-\lfloor \frac{k}{2} \rfloor-1}) \\ &\quad + \alpha_k(K_{s-2\lceil \frac{k}{2} \rceil-2} \square K_{t-2\lfloor \frac{k}{2} \rfloor-2}) \\ &= 2 \left(\left\lfloor \frac{k}{2} \right\rfloor + 1\right) \left(\left\lfloor \frac{k}{2} \right\rfloor + 1\right) + \alpha_k(K_{s-2\lceil \frac{k}{2} \rceil-2} \square K_{t-2\lfloor \frac{k}{2} \rfloor-2}) \\ &= \dots \\ &= \min\{p, q\} \left(\left\lfloor \frac{k}{2} \right\rfloor + 1\right) \left(\left\lfloor \frac{k}{2} \right\rfloor + 1\right) \\ &\quad + \alpha_k(K_{s-\min\{p, q\}(\lceil \frac{k}{2} \rceil+1)} \square K_{t-\min\{p, q\}(\lfloor \frac{k}{2} \rfloor+1)}) \\ &\geq \min\{p, q\} \left(\left\lfloor \frac{k}{2} \right\rfloor + 1\right) \left(\left\lfloor \frac{k}{2} \right\rfloor + 1\right), \end{aligned}$$

where $s = (\lceil \frac{k}{2} \rceil + 1)p + s', t = (\lfloor \frac{k}{2} \rfloor + 1)q + t', 0 \leq s' < \lceil \frac{k}{2} \rceil + 1$ and $0 \leq t' < \lfloor \frac{k}{2} \rfloor + 1$. So the result follows.

See Remarks 2.4 and 2.5 for the sharpness. □

Remark 2.4. From Theorems 2.1, 2.2 and 2.3, we have the following upper bounds for k -independent number.

- $\alpha_k(G \circ H) \leq \alpha_k(H)|V(G)|;$
- $\alpha_k(G \boxtimes H) \leq \min\{\alpha_k(G)|V(H)|, \alpha_k(H)|V(G)|\};$
- $\alpha_k(G \square H) \leq \min\{\alpha_k(G)|V(H)|, \alpha_k(H)|V(G)|\}.$

To show the sharpness of these upper bounds, we consider the following example. Let $G = nK_1$ and $|V(H)| = m$. Then $G * H$ consists of n copies of H , where $*$ denotes the lexicographical or Cartesian or strong product operation. It is clear that $\alpha_k(G * H) = \alpha_k(H)n = \alpha_k(H)|V(G)|$. So all these upper bounds are sharp.

Remark 2.5. From Theorems 2.1, 2.2 and 2.3, we have the following lower bounds for k -independent number.

- $\alpha_k(G \circ H) \geq \alpha_r(G)\alpha_{k-rm}(H)$, where $m = |V(H)|;$
- $\alpha_k(G \boxtimes H) \geq \alpha_r(G)\alpha_{\lfloor \frac{k}{2r+1} \rfloor}(H);$

- $\alpha_k(G \square H) \geq \alpha_r(G)\alpha_{k-r}(H) + X$, where $s = |V(G)| - \alpha_r(G)$, $t = |V(H)| - \alpha_{k-r}(H)$, and

$$X = \begin{cases} st, & \text{if } k \geq s + t - 2; \\ t(k - t + 1), & \text{if } s \geq \lfloor \frac{k+3}{2} \rfloor, t < \lfloor \frac{k+3}{2} \rfloor, \\ & \text{and } k \leq s + t - 3; \\ s(k - s + 1), & \text{if } t \geq \lfloor \frac{k+3}{2} \rfloor, s < \lfloor \frac{k+3}{2} \rfloor, \\ & \text{and } k \leq s + t - 3; \\ \min\{p, q\} \left(\lfloor \frac{k}{2} \rfloor + 1 \right) \left(\lfloor \frac{k}{2} \rfloor + 1 \right), & \text{if } s \geq \lfloor \frac{k+3}{2} \rfloor \text{ and } t \geq \lfloor \frac{k+3}{2} \rfloor. \end{cases}$$

To show the sharpness of these lower bounds, we first consider the following example. Let $G = K_2$ and $H = K_2$. Then $G \circ H = G \boxtimes H = K_4$, and $\alpha_k(G \circ H) = \alpha_k(G \boxtimes H) = \alpha_k(K_4)$. For $k = 0$, $\alpha_k(G \circ H) = \alpha_k(G \boxtimes H) = \alpha_0(K_4) = 1$; for $k = 1$, $\alpha_k(G \circ H) = \alpha_k(G \boxtimes H) = \alpha_1(K_4) = 2$. From Theorems 2.1 and 2.2, $\alpha_k(G \circ H) \geq \alpha_r(G)\alpha_{k-r}(H)$ and $\alpha_k(G \boxtimes H) \geq \alpha_r(G)\alpha_{\lfloor \frac{k}{2r+1} \rfloor}(H)$. Set $r = 0$. Then $\alpha_k(G \circ H) \geq \alpha_0(K_2)\alpha_k(K_2) = \alpha_k(K_2)$ and $\alpha_k(G \boxtimes H) \geq \alpha_0(K_2)\alpha_k(K_2) = \alpha_k(K_2)$. For $k = 0$, $\alpha_0(K_2) = 1$; for $k = 1$, $\alpha_1(K_2) = 2$. For $k = 0$, $\alpha_k(G \circ H) = \alpha_k(G \boxtimes H) = \alpha_0(G)\alpha_k(H)$. This implies that the first two lower bounds are sharp.

Next, we consider the examples for Cartesian product. Let $G = K_2$ and $H = K_2$. Clearly, $G \square H = C_4$, and $\alpha_k(G \square H) = \alpha_k(C_4)$. If $k = 0$, then $r = 0$, $s = t = p = q = 1$, and $\alpha_k(G \square H) = \alpha_0(C_4) = 2 = \alpha_0(K_2)\alpha_0(K_2) + st = \alpha_0(K_2)\alpha_0(K_2) + \min\{p, q\} \left(\lfloor \frac{0}{2} \rfloor + 1 \right) \left(\lfloor \frac{0}{2} \rfloor + 1 \right)$. So the bound for the case $k \geq s + t - 2$ or $s \geq \lfloor \frac{k+3}{2} \rfloor, t \geq \lfloor \frac{k+3}{2} \rfloor$ is sharp. For the case $s \geq \lfloor \frac{k+3}{2} \rfloor, t < \lfloor \frac{k+3}{2} \rfloor$, and $k \leq s + t - 3$, we let $G = K_7$ and $H = K_4$. If $k = 3$, $r = 2$, $s = 4$, and $t = 2$, then $\alpha_3(G \square H) \geq \alpha_2(K_7)\alpha_1(K_4) + t(k - t + 2) = 12$. It suffices to show that $\alpha_3(G \square H) \leq 12$. Assume, to the contrary, that $\alpha_3(G \square H) \geq 13$. Let $V(G) = V(K_7) = \{u_i \mid 1 \leq i \leq 7\}$ and $V(H) = V(K_4) = \{v_i \mid 1 \leq i \leq 4\}$. Then $\bigcup_{i=1}^4 V(G(v_i)) = V(G \square H)$. Let I be a maximum 3-independent set in $G \square H$. Then $|I| \geq 13$. Since $k = 3$, it follows that $|I \cap V(G(v_i))| \leq 4$ for each i ($1 \leq i \leq 4$). Then there exists some $G(v_i)$ such that $|I \cap V(G(v_i))| = 4$. Without loss of generality, let $I \cap V(G(v_1)) = \{(u_j, v_1) \mid 1 \leq j \leq 4\}$. Since $k = 3$ and $|I| \geq 13$, it follows that $|I \cap V(G(v_i))| = 3$ for each i ($2 \leq i \leq 4$). Since $k = 3$, it follows that $I \cap V(G(v_i)) = \{(u_j, v_i) \mid 5 \leq j \leq 7\}$ for each i ($2 \leq i \leq 4$). Then the degree of the subgraph induced by I is at least 4, a contradiction. So $\alpha_3(G \square H) = 12$, and hence the lower bound is also sharp.

2.4 The direct product

We give upper and lower bounds for $\alpha_k(G \times H)$ in this section.

Theorem 2.6. *Let $k \geq 0$ be an integers. For graphs G and H ,*

$$(i) \alpha_k(G \times H) \geq \max \left\{ \alpha_{\lfloor \frac{k}{\Delta(H)} \rfloor}(G)|V(H)|, \alpha_{\lfloor \frac{k}{\Delta(G)} \rfloor}(H)|V(G)| \right\};$$

$$(ii) \alpha_k(G \times H)$$

$$\leq \min \left\{ \alpha_{\lfloor \frac{k}{\Delta(G)} \rfloor}(H)|V(G)| + \alpha_{\lfloor \frac{k}{\delta(H)} \rfloor}(G)|V(H)| - \alpha_{\lfloor \frac{k}{\Delta(G)} \rfloor}(H)\alpha_{\lfloor \frac{k}{\delta(H)} \rfloor}(G), \right. \\ \left. \alpha_{\lfloor \frac{k}{\Delta(G)} \rfloor}(G)|V(H)| + \alpha_{\lfloor \frac{k}{\delta(H)} \rfloor}(H)|V(G)| - \alpha_{\lfloor \frac{k}{\Delta(H)} \rfloor}(G)\alpha_{\lfloor \frac{k}{\delta(G)} \rfloor}(H) \right\}.$$

Moreover, the bounds are sharp.

Proof. (i) If I is a $\lfloor \frac{k}{\Delta(H)} \rfloor$ -independent set of G , then $I \times V(H)$ is a k -independent set of $G \times H$. Therefore, $\alpha_k(G \times H) \geq \alpha_{\lfloor \frac{k}{\Delta(H)} \rfloor}(G)|V(H)|$. By symmetry of direct product graphs, we have

$$\alpha_k(G \times H) \geq \max \left\{ \alpha_{\lfloor \frac{k}{\Delta(H)} \rfloor}(G)|V(H)|, \alpha_{\lfloor \frac{k}{\Delta(G)} \rfloor}(H)|V(G)| \right\}.$$

(ii) Let I be a k -independent set of $G \times H$. Partition I into two vertex subsets J, K such that

$$J = \left\{ (u, v) \in I \mid (u, v_j) \in I, v_j \in S_{(u,v)}, \text{ and } |S_{(u,v)}| \leq \left\lfloor \frac{k}{\Delta(G)} \right\rfloor \right\}$$

and $K = I \setminus J$, where $S_{(u,v)} = \{v_j \in N_H(v) \mid (u, v_j) \in I\}$.

Set

$$J^{u_i} = J \cap H(u_i) \text{ and } K^{v_j} = K \cap G(v_j).$$

Let I_H be a maximum $\lfloor \frac{k}{\Delta(G)} \rfloor$ -independent set of H . Set

$$Y = (V(G) \times I_H) \cap K$$

and

$$Y^{u_i} = Y \cap H(u_i)$$

Note that $J^{u_i} \cap Y^{u_i} = \emptyset$. From the definition of $J, J^{u_i} \cup Y^{u_i}$ is a $\lfloor \frac{k}{\Delta(G)} \rfloor$ -independent set of H , and hence

$$\alpha_{\lfloor \frac{k}{\Delta(G)} \rfloor}(H) \geq |J^{u_i}| + |Y^{u_i}|. \tag{2.1}$$

Claim 1: For $v_j \in V(H)$, $\alpha_{\lfloor \frac{k}{\delta(H)} \rfloor}(G) \geq |K^{v_j}|$.

Proof of Claim 1. For $(u, v_j) \in K^{v_j}$ where $u \in V(G)$, from the definition of K^{v_j} , we have $d_H(v_j) > \lfloor \frac{k}{\Delta(G)} \rfloor$. Since $d_G(u) \cdot d_H(v_j) \leq k$, it follows that

$$d_G(u) \leq \frac{k}{d_H(v_j)} \leq \frac{k}{\delta(H)}.$$

Note that K^{v_j} is a $\lfloor \frac{k}{\delta(H)} \rfloor$ -independent set of $G(v_j)$. Therefore, $\alpha_{\lfloor \frac{k}{\delta(H)} \rfloor}(G) \geq |K^{v_j}|$. \square

Since $\sum_{u_i \in V(G)} |Y^{u_i}| = \sum_{v_j \in I(H)} |K^{v_j}|$, it follows from (2.1) and Claim 1 that

$$\begin{aligned} & \sum_{u_i \in V(G)} (\alpha_{\lfloor \frac{k}{\Delta(G)} \rfloor}(H) - |J^{u_i}|) + \sum_{v_j \in V(H)} (\alpha_{\lfloor \frac{k}{\delta(H)} \rfloor}(G) - |K^{v_j}|) \\ & \geq \sum_{u_i \in V(G)} |Y^{u_i}| + \sum_{v_j \in V(H)} (\alpha_{\lfloor \frac{k}{\delta(H)} \rfloor}(G) - |K^{v_j}|) \\ & \geq \sum_{v_j \in I(H)} |K^{v_j}| + \sum_{v_j \in I(H)} (\alpha_{\lfloor \frac{k}{\delta(H)} \rfloor}(G) - |K^{v_j}|) \\ & \geq \sum_{v_j \in I(H)} \alpha_{\lfloor \frac{k}{\delta(H)} \rfloor}(G) \end{aligned}$$

and hence

$$\begin{aligned} & \sum_{u_i \in V(G)} \alpha_{\lfloor \frac{k}{\Delta(G)} \rfloor}(H) + \sum_{v_j \in V(H)} \alpha_{\lfloor \frac{k}{\delta(H)} \rfloor}(G) - \sum_{v_j \in I(H)} \alpha_{\lfloor \frac{k}{\delta(H)} \rfloor}(G) \\ & \geq \sum_{u_i \in V(G)} |J^{u_i}| + \sum_{v_j \in V(H)} |K^{v_j}|. \end{aligned}$$

Then

$$\alpha_{\lfloor \frac{k}{\Delta(G)} \rfloor}(H)|V(G)| + \alpha_{\lfloor \frac{k}{\delta(H)} \rfloor}(G)|V(H)| - \alpha_{\lfloor \frac{k}{\Delta(G)} \rfloor}(H)\alpha_{\lfloor \frac{k}{\delta(H)} \rfloor}(G) \geq |I|.$$

From the symmetry of direct product, we have

$$\begin{aligned} & \alpha_k(G \times H) \\ & \leq \min \left\{ \alpha_{\lfloor \frac{k}{\Delta(G)} \rfloor}(H)|V(G)| + \alpha_{\lfloor \frac{k}{\delta(H)} \rfloor}(G)|V(H)| - \alpha_{\lfloor \frac{k}{\Delta(G)} \rfloor}(H)\alpha_{\lfloor \frac{k}{\delta(H)} \rfloor}(G), \right. \\ & \quad \left. \alpha_{\lfloor \frac{k}{\Delta(G)} \rfloor}(G)|V(H)| + \alpha_{\lfloor \frac{k}{\delta(H)} \rfloor}(H)|V(G)| - \alpha_{\lfloor \frac{k}{\Delta(G)} \rfloor}(G)\alpha_{\lfloor \frac{k}{\delta(H)} \rfloor}(H) \right\}. \end{aligned}$$

The proof is now complete. See Remark 2.7 for the sharpness. \square

Remark 2.7. To show the sharpness of the lower and upper bounds in Theorem 2.6, we let $G = K_2$ and $H = K_2$. Then

- $\alpha_k(G \times H) \geq \max\{\alpha_k(K_2)|V(K_2)|, \alpha_k(K_2)|V(K_2)|\} = 2\alpha_k(K_2)$;
- $\alpha_k(G \times H) \leq \min\{\alpha_k(H)|V(G)| + \alpha_k(G)|V(H)| - \alpha_k(H)\alpha_k(G),$
 $\alpha_k(G)|V(H)| + \alpha_k(H)|V(G)| - \alpha_k(G)\alpha_k(H)\}$
 $= \alpha_k(K_2)|V(K_2)| + \alpha_k(K_2)|V(K_2)| - \alpha_k(K_2)\alpha_k(K_2)$
 $= (4 - \alpha_k(K_2))\alpha_k(K_2)$.

For $k \geq 1$, we have $\alpha_k(G \times H) = 2$, which implies that the upper and lower bounds in Theorem 2.6 are sharp.

2.5 Relation of four graph products

For the k -independence number of four graph products, we have the following relation.

Proposition 2.8. For any graphs G and H ,

$$\alpha_k(G \circ H) \leq \alpha_k(G \boxtimes H) \leq \alpha_k(G \square H) \leq \min\{\alpha_{k\Delta(H)}(G \times H), \alpha_{k\Delta(G)}(G \times H)\}.$$

Proof. Since $G \boxtimes H$ is a subgraph of $G \circ H$, it follows that $\alpha_k(G \circ H) \leq \alpha_k(G \boxtimes H)$. Similarly, since $G \square H$ is a subgraph of $G \boxtimes H$, it follows that $\alpha_k(G \boxtimes H) \leq \alpha_k(G \square H)$. From Theorem 2.3, $\alpha_k(G \square H) \leq \min\{\alpha_k(G)|V(H)|, \alpha_k(H)|V(G)|\}$. From Theorem 2.6, we have

$$\begin{aligned} \alpha_{k\Delta(H)}(G \times H) & \geq \max\{\alpha_k(G)|V(H)|, \alpha_{\frac{k\Delta(H)}{\Delta(G)}}(H)|V(G)|\} \\ & \geq \min\{\alpha_k(G)|V(H)|, \alpha_k(H)|V(G)|\} \\ & \geq \alpha_k(G \square H). \end{aligned}$$

Similarly, we have $\alpha_{k\Delta(G)}(G \times H) \geq \alpha_k(G \square H)$, and hence

$$\alpha_k(G \square H) \leq \min\{\alpha_{k\Delta(H)}(G \times H), \alpha_{k\Delta(G)}(G \times H)\}.$$

The proof is now complete. \square

3 Applications

In this section, we demonstrate the usefulness of the proposed constructions by applying them to some instances of Cartesian and lexicographical product networks.

The following results are immediate.

Proposition 3.1. *Let $k \geq 0, n \geq 2$ be two integers and $\{\frac{n}{3}\}$ be the integer such that $n \equiv \{\frac{n}{3}\} \pmod{3}$.*

(i) *For a complete graph K_n ,*

$$\alpha_k(K_n) = \begin{cases} k + 1, & \text{if } 0 \leq k \leq n - 1; \\ n, & \text{if } k \geq n. \end{cases}$$

(ii) *For a path P_n ,*

$$\alpha_k(P_n) = \begin{cases} \lceil \frac{n}{2} \rceil, & \text{if } k = 0; \\ 2\lfloor \frac{n}{3} \rfloor + \{\frac{n}{3}\}, & \text{if } k = 1; \\ n, & \text{if } k \geq 2. \end{cases}$$

(iii) *For a cycle C_n ,*

$$\alpha_k(C_n) = \begin{cases} \lfloor \frac{n}{2} \rfloor, & \text{if } k = 0; \\ 2\lfloor \frac{n}{3} \rfloor, & \text{if } k = 1 \text{ and } n \equiv 0, 1 \pmod{3}; \\ 2\lfloor \frac{n}{3} \rfloor + 1, & \text{if } k = 1 \text{ and } n \equiv 2 \pmod{3}; \\ n, & \text{if } k \geq 2. \end{cases}$$

3.1 n -dimensional generalized hypercube

Let K_m be a clique of m vertices, $m \geq 2$. An n -dimensional generalized hypercube [5, 8] is the product of n cliques.

We first focus our attention on 2-dimensional generalized hypercube.

Proposition 3.2. *For network $K_{m_1} \square K_{m_2}$,*

$$\begin{aligned} & \min\{m_1, \lceil k/2 \rceil + 1\} \min\{m_2, \lfloor k/2 \rfloor + 1\} \\ & \leq \alpha_k(K_{m_1} \square K_{m_2}) \\ & \leq \begin{cases} \min\{m_2, m_1\}(k + 1), & \text{if } k \leq m_i - 1 \ (i = 1, 2); \\ (k + 1)m_1, & \text{if } k \leq m_2 - 1, k \geq m_1; \\ (k + 1)m_2, & \text{if } k \leq m_1 - 1, k \geq m_2; \\ m_1m_2, & \text{if } k \geq m_1, k \geq m_2. \end{cases} \end{aligned}$$

Proof. We first investigate the upper bound of $\alpha_k(K_{m_1} \square K_{m_2})$. If $k \geq m_i$ ($i = 1, 2$), then $\alpha_k(K_{m_i}) = m_i$ and $\alpha_k(K_{m_1} \square K_{m_2}) \leq \min\{\alpha_k(K_{m_1})|V(K_{m_2})|, \alpha_k(K_{m_2})|V(K_{m_1})|\} = m_1m_2$ by Theorem 2.3. If $k \leq m_2 - 1$ and $k \geq m_1$, then $\alpha_k(K_{m_1}) = m_1$ and $\alpha_k(K_{m_2}) = k + 1$ and $\alpha_k(K_{m_1} \square K_{m_2}) \leq \min\{\alpha_k(K_{m_1})|V(K_{m_2})|, \alpha_k(K_{m_2})|V(K_{m_1})|\} = \min\{m_1m_2, (k + 1)m_1\} = (k + 1)m_1$. Similarly, if $k \leq m_1 - 1$ and $k \geq m_2$, then $\alpha_k(K_{m_1} \square K_{m_2}) \leq (k + 1)m_2$. If $k \leq m_i - 1$ ($i = 1, 2$), then $\alpha_k(K_{m_i}) = k + 1$, and hence $\alpha_k(K_{m_1} \square K_{m_2}) \leq \min\{(k + 1)m_2, (k + 1)m_1\} = \min\{m_2, m_1\}(k + 1)$.

Next, we consider the lower bound of $\alpha_k(K_{m_1} \square K_{m_2})$. From Theorem 2.3, we have $\alpha_k(K_{m_1} \square K_{m_2}) \geq \alpha_r(K_{m_1})\alpha_{k-r}(K_{m_2})$, where $0 \leq r \leq k$. If $r = \lceil k/2 \rceil$, then $k - r = \lfloor k/2 \rfloor$, $\alpha_r(K_{m_1}) = \min\{m_1, \lceil k/2 \rceil + 1\}$, and $\alpha_{k-r}(K_{m_2}) = \min\{m_2, \lfloor k/2 \rfloor + 1\}$. Furthermore, we have $\alpha_k(K_{m_1} \square K_{m_2}) \geq \alpha_r(K_{m_1})\alpha_{k-r}(K_{m_2}) = \min\{m_1, \lceil k/2 \rceil + 1\} \min\{m_2, \lfloor k/2 \rfloor + 1\}$, as desired. \square

Next, we consider n -dimensional generalized hypercube.

Proposition 3.3. *For network $K_{m_1} \square K_{m_2} \square \cdots \square K_{m_n}$, we have the following.*

(i) *If $m_i \leq k$ ($1 \leq i \leq n$), then*

$$m_1 \leq \alpha_k(K_{m_1} \square K_{m_2} \square \cdots \square K_{m_n}) \leq \prod_{i=1}^n m_i.$$

(ii) *If $k \leq m_j - 1$ ($1 \leq j \leq n$), then*

$$k + 1 \leq \alpha_k(K_{m_1} \square K_{m_2} \square \cdots \square K_{m_n}) \leq (k + 1) \prod_{i=2}^n m_i.$$

Proof. (i) Since $m_i \leq k$ ($1 \leq i \leq n$), it follows that $\alpha_k(K_{m_i}) = m_i$, where $1 \leq i \leq n$. From Theorem 2.3, we have $\alpha_k(G \square H) \leq \min\{\alpha_k(G)|V(H)|, \alpha_k(H)|V(G)|\} \leq \alpha_k(G)|V(H)|$ for any two graphs G and H , and hence

$$\begin{aligned} \alpha_k(K_{m_1} \square K_{m_2} \square \cdots \square K_{m_n}) &= \alpha_k((K_{m_1} \square K_{m_2} \square \cdots \square K_{m_{n-1}}) \square K_{m_n}) \\ &\leq \alpha_k((K_{m_1} \square K_{m_2} \square \cdots \square K_{m_{n-1}}) m_n) \\ &\leq \alpha_k((K_{m_1} \square K_{m_2} \square \cdots \square K_{m_{n-2}}) m_{n-1} m_n) \\ &\leq \cdots \\ &\leq \alpha_k(K_{m_1}) m_2 \cdots m_{n-1} m_n \\ &= \prod_{i=1}^n m_i. \end{aligned}$$

From Theorem 2.3, we have $\alpha_k(G \square H) \geq \alpha_r(G)\alpha_{k-r}(H)$ for any two graphs G and H . Set $r = k$. Then $\alpha_k(G \square H) \geq \alpha_k(G)\alpha_0(H)$ for any two graphs G and H , and hence

$$\begin{aligned} \alpha_k(K_{m_1} \square K_{m_2} \square \cdots \square K_{m_n}) &= \alpha_k((K_{m_1} \square K_{m_2} \square \cdots \square K_{m_{n-1}}) \square K_{m_n}) \\ &\geq \alpha_k(K_{m_1} \square K_{m_2} \square \cdots \square K_{m_{n-1}})\alpha_0(K_{m_n}) \\ &\geq \alpha_k(K_{m_1} \square K_{m_2} \square \cdots \square K_{m_{n-1}}) \\ &\geq \alpha_k((K_{m_1} \square K_{m_2} \square \cdots \square K_{m_{n-2}}) \\ &\geq \cdots \\ &\geq \alpha_k(K_{m_1}) \\ &= m_1. \end{aligned}$$

(ii) Since $k \leq m_j - 1$ ($1 \leq j \leq n$), it follows that $\alpha_k(K_{m_j}) = k + 1$, where $1 \leq j \leq n$. From Theorem 2.3, we have $\alpha_k(G \square H) \leq \min\{\alpha_k(G)|V(H)|, \alpha_k(H)|V(G)|\} \leq$

$\alpha_k(G)|V(H)|$ for any two graphs G and H , and hence

$$\begin{aligned} \alpha_k(K_{m_1} \square K_{m_2} \square \dots \square K_{m_n}) &= \alpha_k((K_{m_1} \square K_{m_2} \square \dots \square K_{m_{n-1}}) \square K_{m_n}) \\ &\leq \alpha_k((K_{m_1} \square K_{m_2} \square \dots \square K_{m_{n-1}})m_n) \\ &\leq \alpha_k((K_{m_1} \square K_{m_2} \square \dots \square K_{m_{n-2}})m_{n-1}m_n) \\ &\leq \dots \\ &\leq \alpha_k((K_{m_1})m_2 \dots m_{n-1}m_n) \\ &= (k + 1) \prod_{i=2}^n m_i. \end{aligned}$$

From Theorem 2.3, we have $\alpha_k(G \square H) \geq \alpha_r(G)\alpha_{k-r}(H)$ for any two graphs G and H . Set $r = k$. Then $\alpha_k(G \square H) \geq \alpha_k(G)\alpha_0(H)$ for any two graphs G and H , and hence

$$\begin{aligned} \alpha_k(K_{m_1} \square K_{m_2} \square \dots \square K_{m_n}) &= \alpha_k((K_{m_1} \square K_{m_2} \square \dots \square K_{m_{n-1}}) \square K_{m_n}) \\ &\geq \alpha_k(K_{m_1} \square K_{m_2} \square \dots \square K_{m_{n-1}})\alpha_0(K_{m_n}) \\ &\geq \alpha_k(K_{m_1} \square K_{m_2} \square \dots \square K_{m_{n-1}}) \\ &\geq \alpha_k((K_{m_1} \square K_{m_2} \square \dots \square K_{m_{n-2}})) \\ &\geq \dots \\ &\geq \alpha_k(K_{m_1}) = k + 1, \end{aligned}$$

as desired. □

Proposition 3.4. For network $K_{m_1} \circ K_{m_2} \circ \dots \circ K_{m_n}$,

$$\alpha_k(K_{m_1} \circ K_{m_2} \circ \dots \circ K_{m_n}) = \begin{cases} k + 1, & \text{if } 0 \leq k \leq \prod_{i=1}^n m_i - 1; \\ \prod_{i=1}^n m_i, & \text{if } k \geq \sum_{i=1}^n m_i. \end{cases}$$

Proof. From the definition of lexicographical product, $K_{m_1} \circ K_{m_2} \circ \dots \circ K_{m_n}$ is a complete graph. From Proposition 3.1, if $0 \leq k \leq \prod_{i=1}^n m_i - 1$, then $\alpha_k(K_{m_1} \circ K_{m_2} \circ \dots \circ K_{m_n}) = k + 1$; if $k + 1 \geq \sum_{i=1}^n m_i$, then $\alpha_k(K_{m_1} \circ K_{m_2} \circ \dots \circ K_{m_n}) = \prod_{i=1}^n m_i$. □

3.2 Two-dimensional grid graph

A two-dimensional grid graph is the Cartesian product $P_n \square P_m$ of path graphs on m and n vertices. For more details on grid graph, we refer to [2, 11]. The network $P_n \circ P_m$ is the lexicographical product $P_n \circ P_m$ of path graphs on m and n vertices; see [15]. Let $\{m/3\}$ be the integer such that $m \equiv \{m/3\} \pmod{3}$.

Proposition 3.5. For network $P_n \square P_m$ ($n \geq 3, m \geq 3$), we have the following.

- (i) If $k \geq 4$, then $\alpha_k(P_n \square P_m) = mn$.
- (ii) If $k = 2, 3$, then $\min\{m \lceil n/2 \rceil, n \lceil m/2 \rceil\} \leq \alpha_k(P_n \square P_m) \leq mn$.
- (iii) If $k = 1$, then

$$\begin{aligned} \lceil n/2 \rceil (2 \lfloor m/3 \rfloor + \{m/3\}) &\leq \alpha_k(P_n \square P_m) \\ &\leq \min\{(2 \lfloor n/3 \rfloor + \{n/3\})m, (2 \lfloor m/3 \rfloor + \{m/3\})n\}. \end{aligned}$$

(iv) If $k = 0$, then $\lceil n/2 \rceil \lceil m/2 \rceil \leq \alpha_k(P_n \square P_m) \leq \min\{\lceil n/2 \rceil m, \lceil m/2 \rceil n\}$.

Proof. (i) Choose all vertices in $P_n \square P_m$. Since the degree of each vertex in the induced subgraph induced by these vertices is at most 4, it follows that $\alpha_k(P_n \square P_m) = mn$.

(ii) From Theorem 2.3, $\alpha_2(P_n \square P_m) \leq \min\{\alpha_2(P_n)|V(P_m)|, \alpha_2(P_m)|V(P_n)|\} = \min\{nm, mn\} = mn$ and $\alpha_2(P_n \square P_m) \geq \alpha_r(P_n)\alpha_{2-r}(P_m)$. If $r = 0$, then we have $\alpha_2(P_n \square P_m) \geq \alpha_0(P_n)\alpha_2(P_m) = \lceil n/2 \rceil m$. If $r = 2$, then $\alpha_2(P_n \square P_m) \geq \alpha_2(P_n)\alpha_0(P_m) = \lceil m/2 \rceil n$. So, we have $\alpha_2(P_n \square P_m) \geq \min\{\lceil m/2 \rceil n, \lceil n/2 \rceil m\}$. Similarly, if $k = 3$, then $\min\{\lceil m/2 \rceil n, \lceil n/2 \rceil m\} \leq \alpha_3(P_n \square P_m) \leq mn$.

(iii) From Theorem 2.3, $\alpha_1(P_n \square P_m) \leq \min\{\alpha_1(P_n)|V(P_m)|, \alpha_1(P_m)|V(P_n)|\} = \min\{(2\lfloor n/3 \rfloor + \{n/3\})m, (2\lfloor m/3 \rfloor + \{m/3\})n\}$. From Theorem 2.3, $\alpha_1(P_n \square P_m) \geq \alpha_r(P_n)\alpha_{1-r}(P_m)$. If $r = 0$, then $\alpha_1(P_n \square P_m) \geq \alpha_0(P_n)\alpha_1(P_m) = \lceil n/2 \rceil (2\lfloor m/3 \rfloor + \{m/3\})$.

(iv) From Theorem 2.3, $\alpha_0(P_n \square P_m) \leq \min\{\alpha_0(P_n)|V(P_m)|, \alpha_0(P_m)|V(P_n)|\} = \min\{\lceil n/2 \rceil m, \lceil m/2 \rceil n\}$, and $\alpha_0(P_n \square P_m) \geq \alpha_0(P_n)\alpha_0(P_m) = \lceil n/2 \rceil \lceil m/2 \rceil$. \square

Proposition 3.6. For network $P_n \circ P_m$ ($n \geq 4, m \geq 3$), we have the following.

(i) If $k \geq 2m + 2$, then $\alpha_k(P_n \circ P_m) = mn$.

(ii) If $2 \leq k < 2m + 2$, then $\lceil n/2 \rceil m \leq \alpha_k(P_n \circ P_m) \leq mn$.

(iii) If $k = 1$, then

$$\lceil n/2 \rceil (2\lfloor m/3 \rfloor + \{m/3\}) \leq \alpha_1(P_n \circ P_m) \leq n(2\lfloor m/3 \rfloor + \{m/3\}).$$

(iv) If $k = 0$, then

$$\lceil n/2 \rceil \lceil m/2 \rceil \leq \alpha_k(P_n \circ P_m) \leq n\lceil m/2 \rceil.$$

Proof. From Theorem 2.1, we have $\alpha_k(P_n \circ P_m) \leq \alpha_k(P_m)|V(P_n)| = n\alpha_k(P_m)$ and $\alpha_k(P_n \circ P_m) \geq \alpha_r(P_n)\alpha_{k-rm}(P_m)$. Let $r = 0$. Then $\alpha_k(P_n \circ P_m) \geq \alpha_0(P_n)\alpha_k(P_m)$, and hence

$$\lceil n/2 \rceil \alpha_k(P_m) \leq \alpha_k(P_n \circ P_m) \leq n\alpha_k(P_m). \quad (3.1)$$

(i) For $k \geq 2m + 2$, we choose all vertices in $P_n \circ P_m$. Since the degree of each vertex in the induced subgraph induced by these vertices is at most $2m + 2$, it follows that $\alpha_k(P_n \circ P_m) = mn$.

(ii) Since $2 \leq k < 2m + 2$, it follows that $\alpha_k(P_m) = m$. From (3.1), $\lceil n/2 \rceil m \leq \alpha_k(P_n \circ P_m) \leq mn$.

(iii) For $k = 1$, $\alpha_k(P_m) = 2\lfloor m/3 \rfloor + \{m/3\}$. From (3.1), $\lceil n/2 \rceil (2\lfloor m/3 \rfloor + \{m/3\}) \leq \alpha_1(P_n \circ P_m) \leq n(2\lfloor m/3 \rfloor + \{m/3\})$.

(iv) For $k = 0$, $\alpha_k(P_m) = \lceil m/2 \rceil$. From (3.1), we have $\lceil n/2 \rceil \lceil m/2 \rceil \leq \alpha_k(P_n \circ P_m) \leq n\lceil m/2 \rceil$. \square

3.3 n -dimensional mesh

An n -dimensional mesh is the Cartesian product of n paths. By this definition, two-dimensional grid graph is a 2-dimensional mesh. An n -dimensional hypercube is a special case of an n -dimensional mesh, in which the n linear arrays are all of size 2; see [13].

Proposition 3.7. For n -dimensional mesh $P_{m_1} \square P_{m_2} \square \dots \square P_{m_n}$,

$$\alpha_k(P_{m_1} \square P_{m_2} \square \dots \square P_{m_n}) \leq \begin{cases} \lceil \frac{m_1}{2} \rceil \prod_{i=2}^n m_i, & \text{if } k = 0; \\ (2\lfloor \frac{m_1}{3} \rfloor + \{\frac{m_1}{3}\}) \prod_{i=2}^n m_i, & \text{if } k = 1; \\ \prod_{i=1}^n m_i, & \text{if } k \geq 2, \end{cases}$$

and

$$\alpha_k(P_{m_1} \square P_{m_2} \square \dots \square P_{m_n}) \geq \begin{cases} \lceil \frac{m_1}{2} \rceil \prod_{i=2}^n \lceil m_i/2 \rceil, & \text{if } k = 0; \\ (2\lfloor \frac{m_1}{3} \rfloor + \{\frac{m_1}{3}\}) \prod_{i=2}^n \lceil m_i/2 \rceil, & \text{if } k = 1; \\ m_1 \prod_{i=2}^n \lceil m_i/2 \rceil, & \text{if } k \geq 2. \end{cases}$$

Proof. From Theorem 2.3, we have $\alpha_k(G \square H) \leq \min\{\alpha_k(G)|V(H)|, \alpha_k(H)|V(G)|\} \leq \alpha_k(G)|V(H)|$ for any two graphs G and H , and hence

$$\begin{aligned} \alpha_k(P_{m_1} \square P_{m_2} \square \dots \square P_{m_n}) &= \alpha_k((P_{m_1} \square P_{m_2} \square \dots \square P_{m_{n-1}}) \square P_{m_n}) \\ &\leq \alpha_k((P_{m_1} \square P_{m_2} \square \dots \square P_{m_{n-1}})m_n) \\ &\leq \alpha_k((P_{m_1} \square P_{m_2} \square \dots \square P_{m_{n-2}})m_{n-1}m_n) \\ &\leq \dots \\ &\leq \alpha_k(P_{m_1})m_2 \dots m_{n-1}m_n. \end{aligned}$$

So, the result follows.

From Theorem 2.3, we have $\alpha_k(G \square H) \geq \alpha_r(G)\alpha_{k-r}(H)$ for any two graphs G and H . Set $r = k$. Then $\alpha_k(G \square H) \geq \alpha_k(G)\alpha_0(H)$ for any two graphs G and H , and hence

$$\begin{aligned} \alpha_k(P_{m_1} \square P_{m_2} \square \dots \square P_{m_n}) &= \alpha_k((P_{m_1} \square P_{m_2} \square \dots \square P_{m_{n-1}}) \square P_{m_n}) \\ &\geq \alpha_k(P_{m_1} \square P_{m_2} \square \dots \square P_{m_{n-1}})\alpha_0(P_{m_n}) \\ &\geq \alpha_k(P_{m_1} \square P_{m_2} \square \dots \square P_{m_{n-1}})\lceil m_n/2 \rceil \\ &\geq \alpha_k((P_{m_1} \square P_{m_2} \square \dots \square P_{m_{n-2}})\lceil m_{n-1}/2 \rceil)\lceil m_n/2 \rceil \\ &\geq \dots \\ &\geq \alpha_k(P_{m_1}) \prod_{i=2}^n \lceil m_i/2 \rceil, \end{aligned}$$

and hence the result holds. □

Similarly to the proof of Proposition 3.7, we can obtain the following result.

Proposition 3.8. For n -dimensional mesh $P_{m_1} \circ P_{m_2} \circ \dots \circ P_{m_n}$,

$$\begin{cases} \lceil \frac{m_1}{2} \rceil \leq \alpha_k(P_{m_1} \circ \dots \circ P_{m_n}) \leq \lceil \frac{m_1}{2} \rceil \prod_{i=2}^n m_i, & \text{if } k = 0; \\ 2\lfloor \frac{m_1}{3} \rfloor + \{\frac{m_1}{3}\} \leq \alpha_k(P_{m_1} \circ \dots \circ P_{m_n}) \leq (2\lfloor \frac{m_1}{3} \rfloor + \{\frac{m_1}{3}\}) \prod_{i=2}^n m_i, & \text{if } k = 1; \\ m_1 \leq \alpha_k(P_{m_1} \circ \dots \circ P_{m_n}) \leq \prod_{i=1}^n m_i, & \text{if } k \geq 2. \end{cases}$$

3.4 n -dimensional torus

An n -dimensional torus is the Cartesian product of n cycles $C_{m_1}, C_{m_2}, \dots, C_{m_n}$ of size at least three. The cycles C_{m_i} are not necessary to have the same size. Ku et al. [14] showed that there are n edge-disjoint spanning trees in an n -dimensional torus. The network $C_{m_1} \circ C_{m_2} \circ \dots \circ C_{m_n}$ is investigated in [15]. Here, we consider the networks constructed by $C_{m_1} \square C_{m_2} \square \dots \square C_{m_n}$ and $C_{m_1} \circ C_{m_2} \circ \dots \circ C_{m_n}$, respectively.

Proposition 3.9. For network $C_n \square C_m$ ($n \geq 3, m \geq 3$), we have the following.

- (i) If $k \geq 4$, then $\alpha_k(C_n \square C_m) = mn$.
- (ii) If $k = 3$ or $k = 2$, then $\min\{m \lfloor n/2 \rfloor, n \lfloor m/2 \rfloor\} \leq \alpha_k(C_n \square C_m) \leq mn$.
- (iii) If $k = 1$, then $2 \lfloor n/2 \rfloor \lfloor \frac{m}{3} \rfloor \leq \alpha_k(C_n \square C_m) \leq \min\{m(2 \lfloor \frac{n}{3} \rfloor + 1), n(2 \lfloor \frac{m}{3} \rfloor + 1)\}$.
- (iv) If $k = 0$, then $\lfloor n/2 \rfloor \lfloor m/2 \rfloor \leq \alpha_k(C_n \square C_m) \leq \min\{\lfloor n/2 \rfloor m, \lfloor m/2 \rfloor n\}$.

Proof. (i) Choose all vertices in $C_n \square C_m$. Since the degree of each vertex in the induced subgraph induced by these vertices is at most 4, it follows that $\alpha_k(C_n \square C_m) = mn$.

(ii) From Theorem 2.3, $\alpha_3(C_n \square C_m) \leq \min\{\alpha_3(C_n)|V(C_m)|, \alpha_3(C_m)|V(C_n)|\} = \min\{nm, mn\} = mn$, and $\alpha_3(P_n \square P_m) \geq \alpha_r(C_n)\alpha_{3-r}(C_m)$. If $r = 0$, then we have $\alpha_3(C_n \square C_m) \geq \alpha_0(C_n)\alpha_3(C_m) = \lfloor n/2 \rfloor m$. If $r = 3$, then $\alpha_3(C_n \square C_m) \geq \alpha_3(C_n)\alpha_0(C_m) = \lfloor m/2 \rfloor n = \lfloor m/2 \rfloor n$. So, $\alpha_3(C_n \square C_m) \geq \min\{m \lfloor n/2 \rfloor, n \lfloor m/2 \rfloor\}$. The case $k = 2$ can be similarly proved.

(iii) From Theorem 2.3, $\alpha_1(C_n \square C_m) \geq \alpha_r(C_n)\alpha_{1-r}(C_m)$. If $r = 0$, then we have $\alpha_1(C_n \square C_m) \geq \alpha_0(C_n)\alpha_1(C_m) = \lfloor n/2 \rfloor (2 \lfloor \frac{m}{3} \rfloor)$, and $\alpha_1(C_n \square C_m) \leq \min\{\alpha_1(C_n)|V(C_m)|, \alpha_1(C_m)|V(C_n)|\} = \min\{m(2 \lfloor \frac{n}{3} \rfloor + 1), n(2 \lfloor \frac{m}{3} \rfloor + 1)\}$.

(iv) From Theorem 2.3, $\alpha_0(C_n \square C_m) \leq \min\{\lfloor n/2 \rfloor m, \lfloor m/2 \rfloor n\}$, and $\alpha_0(C_n \square C_m) \geq \alpha_0(C_n)\alpha_0(C_m) = \lfloor n/2 \rfloor \lfloor m/2 \rfloor$. \square

For network $C_n \circ C_m$, we have the following result.

Proposition 3.10. For network $C_n \circ C_m$ ($n \geq 4, m \geq 3$), we have the following.

- (i) If $k \geq 2m + 2$, then $\alpha_k(C_n \circ C_m) = mn$.
- (ii) If $2 \leq k < 2m + 2$, then $\lfloor n/2 \rfloor m \leq \alpha_k(C_n \circ C_m) \leq mn$.
- (iii) If $k = 1$ and $n \equiv 0, 1 \pmod{3}$, then $2 \lfloor n/2 \rfloor \lfloor n/3 \rfloor \leq \alpha_k(C_n \circ C_m) \leq 2n \lfloor n/3 \rfloor$.
- (iv) If $k = 1$ and $n \equiv 2 \pmod{3}$, then

$$\lfloor n/2 \rfloor (2 \lfloor n/3 \rfloor + 1) \leq \alpha_k(C_n \circ C_m) \leq n(2 \lfloor n/3 \rfloor + 1).$$

- (v) If $k = 0$, then $\lfloor m/2 \rfloor \lfloor n/2 \rfloor \leq \alpha_0(C_n \circ C_m) \leq n \lfloor m/2 \rfloor$.

Proof. From Theorem 2.1, we have $\alpha_k(C_n \circ C_m) \leq \alpha_k(C_m)|V(C_n)| = n\alpha_k(C_m)$ and $\alpha_k(C_n \circ C_m) \geq \alpha_r(C_n)\alpha_{k-r}(C_m)$. Let $r = 0$. Then $\alpha_k(C_n \circ C_m) \geq \alpha_0(C_n)\alpha_k(C_m)$, and hence

$$\lfloor n/2 \rfloor \alpha_k(C_m) \leq \alpha_k(C_n \circ C_m) \leq n\alpha_k(C_m). \quad (3.2)$$

(i) For $k \geq 2m + 2$, we choose all vertices in $C_n \circ C_m$. Since the degree of each vertex in the induced subgraph induced by these vertices is at most $2m + 2$, it follows that $\alpha_k(C_n \circ C_m) = mn$.

(ii) Since $2 \leq k < 2m + 2$, it follows that $\alpha_k(C_m) = m$, and hence $\lfloor n/2 \rfloor m \leq \alpha_k(C_n \circ C_m) \leq mn$ by (3.2).

(iii) Since $k = 1$ and $n \equiv 0, 1 \pmod{3}$, we have $\alpha_k(C_m) = 2\lfloor \frac{n}{3} \rfloor$. From (3.2), $2\lfloor n/2 \rfloor \lfloor n/3 \rfloor \leq \alpha_k(C_n \circ C_m) \leq 2n\lfloor n/3 \rfloor$.

(iv) For $k = 1$ and $n \equiv 2 \pmod{3}$, $\alpha_k(C_m) = 2\lfloor \frac{n}{3} \rfloor + 1$. From (3.2), $\lfloor n/2 \rfloor (2\lfloor n/3 \rfloor + 1) \leq \alpha_k(C_n \circ C_m) \leq n(2\lfloor n/3 \rfloor + 1)$.

(v) For $k = 0$, $\alpha_k(C_m) = \lfloor m/2 \rfloor$. From (3.2), $\lfloor m/2 \rfloor \lfloor n/2 \rfloor \leq \alpha_k(C_n \circ C_m) \leq n\lfloor m/2 \rfloor$. □

For general case, we have the following two results.

Proposition 3.11. For network $C_{m_1} \square C_{m_2} \square \dots \square C_{m_n}$,

$$\alpha_k(C_{m_1} \square C_{m_2} \square \dots \square C_{m_n}) \leq \begin{cases} \lfloor \frac{m_1}{2} \rfloor \prod_{i=2}^n m_i, & \text{if } k = 0; \\ 2\lfloor \frac{m_1}{3} \rfloor \prod_{i=2}^n m_i, & \text{if } k = 1, m_1 \equiv 0, 1 \pmod{3}; \\ (2\lfloor \frac{m_1}{3} \rfloor + 1) \prod_{i=2}^n m_i, & \text{if } k = 1, m_1 \equiv 2 \pmod{3}; \\ \prod_{i=1}^n m_i, & \text{if } k \geq 2, \end{cases}$$

and

$$\alpha_k(C_{m_1} \square \dots \square C_{m_n}) \geq \begin{cases} \lfloor \frac{m_1}{2} \rfloor \prod_{i=2}^n \lfloor m_i/2 \rfloor, & \text{if } k = 0; \\ 2\lfloor \frac{m_1}{3} \rfloor \prod_{i=2}^n \lfloor m_i/2 \rfloor, & \text{if } k = 1, m_1 \equiv 0, 1 \pmod{3}; \\ (2\lfloor \frac{m_1}{3} \rfloor + 1) \prod_{i=2}^n \lfloor m_i/2 \rfloor, & \text{if } k = 1, m_1 \equiv 2 \pmod{3}; \\ m_1 \prod_{i=2}^n \lfloor m_i/2 \rfloor, & \text{if } k \geq 2, \end{cases}$$

where m_i is the order of C_{m_i} and $1 \leq i \leq n$.

Proof. From Theorem 2.3, we have $\alpha_k(G \square H) \leq \min\{\alpha_k(G)|V(H)|, \alpha_k(H)|V(G)|\} \leq \alpha_k(G)|V(H)|$ for any two graphs G and H , and hence

$$\begin{aligned} \alpha_k(C_{m_1} \square C_{m_2} \square \dots \square C_{m_n}) &= \alpha_k((C_{m_1} \square C_{m_2} \square \dots \square C_{m_{n-1}}) \square C_{m_n}) \\ &\leq \alpha_k((C_{m_1} \square C_{m_2} \square \dots \square C_{m_{n-1}})m_n) \\ &\leq \alpha_k((C_{m_1} \square C_{m_2} \square \dots \square C_{m_{n-2}})m_{n-1}m_n) \\ &\leq \dots \\ &\leq \alpha_k(C_{m_1})m_2 \dots m_{n-1}m_n. \end{aligned}$$

From (iii) of Proposition 3.1, the result follows.

From Theorem 2.3, we have $\alpha_k(G \square H) \geq \alpha_r(G)\alpha_{k-r}(H)$ for any two graphs G and H . Set $r = k$. Then $\alpha_k(G \square H) \geq \alpha_k(G)\alpha_0(H)$ for any two graphs G and H , and hence

$$\begin{aligned} \alpha_k(C_{m_1} \square C_{m_2} \square \dots \square C_{m_n}) &= \alpha_k((C_{m_1} \square C_{m_2} \square \dots \square C_{m_{n-1}}) \square C_{m_n}) \\ &\geq \alpha_k(C_{m_1} \square C_{m_2} \square \dots \square C_{m_{n-1}})\alpha_0(C_{m_n}) \\ &\geq \alpha_k(C_{m_1} \square C_{m_2} \square \dots \square C_{m_{n-1}})\lfloor m_n/2 \rfloor \\ &\geq \alpha_k((C_{m_1} \square C_{m_2} \square \dots \square C_{m_{n-2}})\lfloor m_{n-1}/2 \rfloor \lfloor m_n/2 \rfloor) \\ &\geq \dots \\ &\geq \alpha_k(C_{m_1}) \prod_{i=2}^n \lfloor m_i/2 \rfloor. \end{aligned}$$

From (3.2) of Proposition 3.1, the result holds. □

Similarly to the proof of Proposition 3.11, we can prove the following result.

Proposition 3.12. For network $C_{m_1} \circ C_{m_2} \circ \cdots \circ C_{m_n}$,

$$\left\{ \begin{array}{ll} \lfloor \frac{m_1}{2} \rfloor \leq \alpha_k(C_{m_1} \circ \cdots \circ C_{m_n}) \leq \lfloor \frac{m_1}{2} \rfloor \prod_{i=2}^n m_i, & \text{if } k = 0; \\ 2\lfloor \frac{m_1}{3} \rfloor \leq \alpha_k(C_{m_1} \circ \cdots \circ C_{m_n}) \leq 2\lfloor \frac{m_1}{3} \rfloor \prod_{i=2}^n m_i, & \text{if } k = 1 \\ & \text{and } m_1 \equiv 0, 1 \pmod{3}; \\ 2\lfloor \frac{m_1}{3} \rfloor + 1 \leq \alpha_k(C_{m_1} \circ \cdots \circ C_{m_n}) \leq (2\lfloor \frac{m_1}{3} \rfloor + 1) \prod_{i=2}^n m_i, & \text{if } k = 1 \\ & \text{and } m_1 \equiv 2 \pmod{3}; \\ m_1 \leq \alpha_k(C_{m_1} \circ \cdots \circ C_{m_n}) \leq \prod_{i=1}^n m_i, & \text{if } k \geq 2, \end{array} \right.$$

where m_i is the order of C_{m_i} and $1 \leq i \leq n$.

3.5 n -dimensional hyper Petersen network

An n -dimensional hyper Petersen network HP_n is the product of the well-known Petersen graph and Q_{n-3} [4], where $n \geq 3$ and Q_{n-3} denotes an $(n-3)$ -dimensional hypercube. Note that HP_3 is just the Petersen graph.

The network HL_n is the lexicographical product of the Petersen graph and Q_{n-3} , where $n \geq 3$ and Q_{n-3} denotes an $(n-3)$ -dimensional hypercube; see [15]. Note that HL_3 is just the Petersen graph, and HL_4 is a graph obtained from two copies of the Petersen graph by adding the edges between all the vertices from different copies of the Petersen graph.

Proposition 3.13. (i) For network HP_3 and HL_3 ,

$$\alpha_k(HP_3) = \alpha_k(HL_3) = \begin{cases} 4, & \text{if } k = 0; \\ 5, & \text{if } k = 1; \\ 5, & \text{if } k = 2; \\ 10, & \text{if } k \geq 3. \end{cases}$$

(ii) For network HP_4 ,

$$\begin{cases} 5 \leq \alpha_k(HP_4) \leq 8, & \text{if } k = 0; \\ 6 \leq \alpha_k(HP_4) \leq 10, & \text{if } k = 1; \\ 6 \leq \alpha_k(HP_4) \leq 10, & \text{if } k = 2; \\ 11 \leq \alpha_k(HP_4) \leq 20, & \text{if } k = 3; \\ \alpha_k(HP_4) = 30, & \text{if } k \geq 4. \end{cases}$$

(iii) For network HL_4 ,

$$\begin{cases} 4 \leq \alpha_k(HP_4) \leq 15, & \text{if } k = 0; \\ 8 \leq \alpha_k(HP_4) \leq 30, & \text{if } k \geq 1. \end{cases}$$

Proof. (i) Note that HL_3 or HP_3 is just the Petersen graph, and its maximum degree is 3. Since $|V(HP_3)| = 10$, it follows that $\alpha_k(HP_3) = 10$ for $k \geq 3$. One can also check that

$$\alpha_k(HP_3) = \alpha_k(HL_3) = \begin{cases} 4, & \text{if } k = 0; \\ 5, & \text{if } k = 1; \\ 5, & \text{if } k = 2. \end{cases}$$

(ii) For network HP_4 , $HP_4 = HP_3 \square K_2$. From Theorem 2.3, we have $\alpha_k(HP_4) = \alpha_k(HP_3 \square K_2) \leq \min\{2\alpha_k(HP_3), 10\alpha_k(K_2)\}$. Note that $\alpha_k(K_2) = 1$ for $k = 0$; $\alpha_k(K_2) = 2$ for $k \geq 1$. Combining this with (i) of this proposition, we have

$$\alpha_k(HP_4) \leq \begin{cases} 8, & \text{if } k = 0; \\ 10, & \text{if } k = 1; \\ 10, & \text{if } k = 2; \\ 20, & \text{if } k \geq 3. \end{cases}$$

From Theorem 2.3, $\alpha_k(HP_4) \geq \alpha_r(HP_3)\alpha_{k-r}(K_2) + \alpha_k(K_s \square K_t)$, where $s = |V(HP_3)| - \alpha_r(HP_3)$ and $t = |V(K_2)| - \alpha_{k-r}(K_2)$. Set $r = k$. Then $t = 1$ and $\alpha_k(HP_4) \geq \alpha_k(HP_3)\alpha_0(K_2) + \alpha_k(K_s \square K_1) \geq \alpha_k(HP_3) + 1$, and hence

$$\alpha_k(HP_4) \geq \begin{cases} 5, & \text{if } k = 0; \\ 6, & \text{if } k = 1; \\ 6, & \text{if } k = 2; \\ 11, & \text{if } k \geq 3. \end{cases}$$

(iii) For network HL_4 , $HL_4 = K_2 \circ HL_3$. From Theorem 2.3, we have $\alpha_k(HL_4) = \alpha_k(K_2 \circ HL_3) \leq |V(HL_3)|\alpha_k(K_2) = 10\alpha_k(K_2)$. Note that $\alpha_k(K_2) = 1$ for $k = 0$; $\alpha_k(K_2) = 2$ for $k \geq 1$. Combining this with (i) of this proposition, we have

$$\alpha_k(HL_4) \leq \begin{cases} 15, & \text{if } k = 0; \\ 20, & \text{if } k \geq 1. \end{cases}$$

From Theorem 2.3, $\alpha_k(HL_4) \geq \alpha_r(HL_3)\alpha_{k-2r}(K_2)$. Set $r = 0$. Then $\alpha_k(HL_4) \geq \alpha_0(HL_3)\alpha_k(K_2) = 4\alpha_k(K_2)$, and hence

$$\alpha_k(HL_4) \geq \begin{cases} 4, & \text{if } k = 0; \\ 8, & \text{if } k \geq 1. \end{cases}$$

□

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