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The k-independence number of graph products*

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Abstract

The concept of k-independent number is a natural generalization of classical independence number. A k*-independent set* is a set of vertices whose induced subgraph has maximum degree at most k. The k*-independence number* of G, denoted by $\alpha_k(G)$, is defined as the maximum cardinality of a k-independent set of G. In this paper, we study the k-independence number on the lexicographical, strong, Cartesian and direct product and present several upper and lower bounds for these products of graphs.

Keywords: Independence number, k-independent set, k-independence number, lexicographical product, strong product, Cartesian product, direct product.

Math. Subj. Class.: 05C69, 05C76

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1 Introduction

Graphs considered in this paper are undirected, finite and simple. We refer to [1] for undefined notations and terminology. In particular, we use $\Delta(G)$ and $\delta(G)$ to denote the maximum degree and minimum degree of a graph G, respectively. If $X \subseteq V(G)$ or $X \subseteq E(G)$, then $G[X]$ is the subgraph of G induced by X. For two subsets X and Y of $V(G)$ we denote by $E_G[X,Y]$ the set of edges of G with one end in X and the other end in Y.

Independence number is one of the most basic concepts in graph theory. A subset $S \subseteq V(G)$ is said to be *independent* if $E(G[S]) = \emptyset$. The *independence number* of G denoted by $\alpha(G)$ is the size of a maximum independent set in G. In [6, 7], Fink and Jacobson generalized the concept of independent set. In this paper, k will be an integer. We say that a subset S of V is *k-independent* if $\Delta(G[S]) \leq k$, that is, the maximum degree of the subgraph induced by the vertices of S is less or equal to k. The *k-independence number*, denoted $\alpha_k(G)$, as the maximum cardinality of a k-independent set. Thus for $k = 0$, the 0-independent is the classical independent set. Every k-independent set is $(k+1)$-independent; so $\alpha_{k+1}(G) \geq \alpha_k(G)$ for a graph G. Moreover, the vertex set V is the only maximal Δ-independent but is not a $(\Delta-1)$-independent set. Thus every graph G satisfies

$$\alpha(G) = \alpha_0(G) \leq \alpha_1(G) \leq \alpha_2(G) \leq \cdots \leq \alpha_{\Delta-1}(G) < \alpha_\Delta(G) = n.$$

For k-independent set and k-independence number, Chellali, Favaron, Hansberg, and Volkmann published a survey paper on this subject; see [3]. We must mention that the k-independence number of G is defined as the size of a largest k-colorable subgraph of G in [17].

In graph theory, Cartesian product, strong product, lexicographical product, and direct product are four of main products, each with its own set of applications and theoretical interpretations. Product networks were proposed based upon the idea of using the cross product as a tool for "combining" two known graphs with established properties to obtain a new one that inherits properties from both [5]. For more details on graph products, we refer to the book [10].

- The *Cartesian product* of two graphs G and H, written as $G \,\square\, H$, is the graph with vertex set $V(G) \times V(H)$, in which two vertices (u,v) and (u',v') are adjacent if and only if $u = u'$ and $(v,v') \in E(H)$, or $v = v'$ and $(u,u') \in E(G)$.
- The *lexicographic product* $G \circ H$ of graphs G and H has the vertex set $V(G \circ H) = V(G) \times V(H)$. Two vertices $(u,v), (u',v')$ are adjacent if $uu' \in E(G)$, or if $u = u'$ and $vv' \in E(H)$.
- The *strong product* $G \boxtimes H$ of graphs G and H has the vertex set $V(G) \times V(H)$. Two vertices (u,v) and (u',v') are adjacent whenever $uu' \in E(G)$ and $v = v'$, or $u = u'$ and $vv' \in E(H)$, or $uu' \in E(G)$ and $vv' \in E(H)$.
- The *direct product* $G \times H$ of graphs G and H has the vertex set $V(G) \times V(H)$. Two vertices (u,v) and (u',v') are adjacent if the projections on both coordinates are adjacent, i.e., $uu' \in E(G)$ and $vv' \in E(H)$.

Note that unlike the other three products, the lexicographic product is a non-commutative product since $G \circ H$ is usually not isomorphic to $H \circ G$.

For the independence number of Cartesian product graphs, Vizing [16] observed:

Theorem 1.1 ([10, 16]). *For any graphs G and H,*

(i) $\alpha(G\Box H) \leq \min\{\alpha(G)|V(H)|, \alpha(H)|V(G)|\}$;

(ii) $\alpha(G\Box H) \geq \alpha(G)\alpha(H) + \min\{|V(G)| - \alpha(G), |V(H)| - \alpha(H)\}$.

Geller and Stahl [9] obtained the following result for the independence number of lexicographical product graphs.

Theorem 1.2 ([9]). *For any graphs G and H, $\alpha(G \circ H) = \alpha(G)\alpha(H)$.*

The following result is immediate, since $G \boxtimes H$ is a subgraph of $G \circ H$.

Corollary 1.3 ([10]). *For any graphs G and H, $\alpha(G \boxtimes H) \geq \alpha(G)\alpha(H)$.*

In 2011, Špacapan [17] proved the following theorem.

Theorem 1.4 ([17]). *For any graph G and H,*

(i) $\alpha(G \times H) \geq \max\{\alpha(G)|V(H)|, \alpha(H)|V(G)|\}$;

(ii) $\alpha(G \times H) \leq \alpha(H)|V(G)| + \alpha(G)|V(H)| - \alpha(H)\alpha(G)$.

For the independence number of four graph products, Jha and Slutzki obtained the following relation in 1994.

Theorem 1.5 ([12]). *For any graphs G and H,*

$$\alpha(G \circ H) \leq \alpha(G \boxtimes H) \leq \alpha(G\Box H) \leq \alpha(G \times H).$$

In this paper, we consider four standard products: the lexicographic, the strong, the Cartesian and the direct with respect to the k-independence number. Every of these four products will be treated in one of the forthcoming subsections in Section 2. Our results can be seen as extensions of Theorems 1.1, 1.2, 1.4, 1.5 and Corollary 1.3.

2 Main results

In this section, let G and H be two connected graphs with $V(G) = \{u_1, u_2, \ldots, u_n\}$ and $V(H) = \{v_1, v_2, \ldots, v_m\}$, respectively. Then $V(G*H) = \{(u_i, v_j) \,|\, 1 \leq i \leq n,\ 1 \leq j \leq m\}$, where $*$ denotes lexicographic product operation, strong product operation, Cartesian product operation or direct product operation. For $v \in V(H)$, we use $G(v)$ to denote the subgraph of $G * H$ induced by the vertex set $\{(u_i, v) \,|\, 1 \leq i \leq n\}$. Similarly, for $u \in V(G)$, we use $H(u)$ to denote the subgraph of $G * H$ induced by the vertex set $\{(u, v_j) \,|\, 1 \leq j \leq m\}$.

2.1 The lexicographic product

In this subsection, we give upper and lower bounds of $\alpha_k(G \circ H)$.

Theorem 2.1. *(i) Let $k \geq 0$ be an integer. For graphs G and H,*

$$\alpha_k(G \circ H) \leq \alpha_k(H)|V(G)|.$$

(ii) Let $k, r \geq 0$ be two integers. Let H be a graph of order m. For graphs G and H,

$$\alpha_k(G \circ H) \geq \alpha_r(G)\alpha_{k-rm}(H)$$

where $\alpha_{k-rm}(H) = 0$ if $k \leq rm$.
Moreover, the bounds are sharp.

Proof. (i) Let I be a maximum k-independent set of $G \circ H$. We claim that $|I \cap V(H(u_i))| \leq \alpha_k(H(u_i))$ for each $u_i \in V(G)$. To see this, we observe that $H(u_i)[I \cap V(H(u_i))]$ is a subgraph of $G \circ H[I]$.

(ii) Let I be a maximum r-independent set of G, and J be a maximum $(k - rm)$-independent set of H. Set

$$I = \{u_i \,|\, 1 \leq i \leq s\} \text{ and } J = \{v_j \,|\, 1 \leq j \leq t\}.$$

For any $(u_i, v_j) \in I \times J$, we show that the degree of (u_i, v_j) in $G \circ H[I \times J]$ is at most k. Since I is a maximum r-independent set of G, it follows that $d_{G[I]}(u_i) \leq r$, where $u_i \in V(G[I])$. Similarly, since J is a maximum $(k - mr)$-independent set of H, it follows that $d_{H[J]}(v_j) \leq k - mr$, where $v_j \in V(H[J])$. Then

$$d_{G \circ H[I \times J]}(u_i, v_j) \leq d_{H[J]}(v_j) + m d_{G[I]}(u_i) \leq k - mr + mr = k,$$

and hence $I \times J$ is a k-independent set of $G \circ H$. So $\alpha_k(G \circ H) \geq \alpha_r(G)\alpha_{k-rm}(H)$.

See Remarks 2.4 and 2.5 for the sharpness. □

2.2 The strong product

In this subsection, we derive upper and lower bounds of $\alpha_k(G \boxtimes H)$.

Theorem 2.2. *(i) Let $k \geq 0$ be an integer. For graphs G and H,*

$$\alpha_k(G \boxtimes H) \leq \min\{\alpha_k(G)|V(H)|, \alpha_k(H)|V(G)|\}.$$

(ii) Let $k, r \geq 0$ be two integers. For graphs G and H,

$$\alpha_k(G \boxtimes H) \geq \alpha_r(G)\alpha_{\lfloor \frac{k}{2r+1} \rfloor}(H)$$

and

$$\alpha_k(G \boxtimes H) \geq \alpha_r(H)\alpha_{\lfloor \frac{k}{2r+1} \rfloor}(G).$$

Moreover, the bounds are sharp.

Proof. (i) Let I be a maximum k-independent set of $G \boxtimes H$. If $|G(v_j) \cap I| > \alpha_k(G)$ for some $j \leq m$, then I is not a k-independent set in $G \boxtimes H$. It follows $\alpha_k(G \boxtimes H) \leq \alpha_k(G)|V(H)|$. From the symmetry, we have

$$\alpha_k(G \boxtimes H) \leq \min\{\alpha_k(G)|V(H)|, \alpha_k(H)|V(G)|\}.$$

(ii) Let I be a maximum r-independent set of G, and J be a maximum $(\frac{k}{2r+1})$-independent set of H. Set

$$I = \{u_i \,|\, 1 \leq i \leq s\} \text{ and } J = \{v_j \,|\, 1 \leq j \leq t\}.$$

For any $(u_i, v_j) \in I \times J$, we show that the degree of (u_i, v_j) in $G \circ H[I \times J]$ is at most k. Since I is a maximum r-independent set of G, it follows that $d_{G[I]}(u_i) \leq r$, where $u_i \in V(G[I])$. Similarly, since J is a maximum $(\frac{k}{2r+1})$-independent set of H, it follows that $d_{H[J]}(v_j) \leq \frac{k}{2r+1}$, where $v_j \in V(H[J])$. Then

$$d_{G\boxtimes H[I\times J]}(u_i, v_j) \leq d_{H[J]}(v_j) + \frac{2k}{2r+1} d_{G[I]}(u_i) \leq \frac{k}{2r+1} + \frac{2rk}{2r+1} = k,$$

and hence $I \times J$ is a k-independent set of $G \boxtimes H$. So $\alpha_k(G \boxtimes H) \geq \alpha_r(G)\alpha_{\lfloor \frac{k}{2r+1} \rfloor}(H)$.

See Remarks 2.4 and 2.5 for the sharpness. □

2.3 The Cartesian product

Upper and lower bounds of $\alpha_k(G \,\Box\, H)$ are derived in this subsection.

Theorem 2.3. *Let $k, r \geq 0$ be two integers. For graphs G and H,*

(i) $\alpha_k(G \,\Box\, H) \leq \min\{\alpha_k(G)|V(H)|, \alpha_k(H)|V(G)|\}$;

(ii) $\alpha_k(G \,\Box\, H) \geq \alpha_r(G)\alpha_{k-r}(H)$

$$+\begin{cases} st, & \text{if } k \geq s+t-2; \\ t(k-t+2), & \text{if } s \geq \lfloor \frac{k+3}{2} \rfloor, t < \lfloor \frac{k+3}{2} \rfloor, \\ & \text{and } k \leq s+t-3; \\ s(k-s+2), & \text{if } t \geq \lfloor \frac{k+3}{2} \rfloor, s < \lfloor \frac{k+3}{2} \rfloor, \\ & \text{and } k \leq s+t-3; \\ \min\{p,q\}\left(\lceil \frac{k}{2} \rceil + 1\right)\left(\lfloor \frac{k}{2} \rfloor + 1\right), & \text{if } s \geq \lfloor \frac{k+3}{2} \rfloor \text{ and } t \geq \lfloor \frac{k+3}{2} \rfloor, \end{cases}$$

where $0 \leq r \leq k$, $s = |V(G)| - \alpha_r(G)$, $t = |V(H)| - \alpha_{k-r}(H)$, $s = (\lceil \frac{k}{2} \rceil + 1)p + s'$, $t = (\lfloor \frac{k}{2} \rfloor + 1)q + t'$, $0 \leq s' < \lceil \frac{k}{2} \rceil + 1$ *and* $0 \leq t' < \lfloor \frac{k}{2} \rfloor + 1$.

Moreover, the bounds are sharp.

Proof. *(i)* The proof is similar to the proof of *(i)* of Theorem 2.1.

(ii) Suppose I is a r-independent set in G and J is a $(k-r)$-independent set in H, respectively. We will prove that $I \times J$ is a k-independent set of $G \,\Box\, H$. By commutativity, we may assume $|V(G)| - \alpha_r(G) \leq |V(H)| - \alpha_{k-r}(H)$. Say $V(H) \setminus J = \{y_1, y_2, \cdots, y_t\}$, and take a subset $\{x_1, x_2, \cdots, x_s\} \subseteq V(G) \setminus I$. Then $s \leq t$. Set

$$K = \{(x_i, y_j) \,|\, 1 \leq i \leq s, \; 1 \leq j \leq t\}.$$

Let $F = G \,\Box\, H$. Since $F[K]$ is a spanning subgraph of $K_s \,\Box\, K_t$, it follows that $\alpha_k(F[K]) \geq \alpha_k(K_s \,\Box\, K_t)$, and hence there is a $\alpha_k(K_s \,\Box\, K_t)$-independent set of $F[K]$, say K'.

Claim 1: $(I \times J) \cup K'$ is a k-independent set of $G\Box H$.

Proof of Claim 1. For any $(u_i, v_j) \in I \times J$ where $u_i \in V(G)$ and $v_j \in V(H)$, we have

$$d_{G\Box H[I\times J]}(u_i, v_j) = d_{G[I]}(u_i) + d_{H[J]}(v_j) \leq r + (k-r) = k.$$

Therefore, $I \times J$ is a k-independent set of $G\Box H$. From the structure of Cartesian product graphs, we have $E_{G\Box H}[I \times J, K'] = \emptyset$. Then $(I \times J) \cup K'$ is a k-independent set of $G\Box H$. □

From Claim 1, we have $\alpha_k(G\,\square\,H) \geq |(I\times J)\cup K'| = \alpha_r(G)\alpha_{k-r}(H)+\alpha_k(K_s\,\square\,K_t)$ for graphs G and H.

If $k \geq s+t-2$, then $(V(G)-I)\times(V(H)-J) = K_s \times K_t$ is a k-independent set of $K_s\,\square\,K_t$, and hence $\alpha_k(K_s\,\square\,K_t) \geq st$. If $s \geq \lfloor\frac{k+3}{2}\rfloor$, $t < \lfloor\frac{k+3}{2}\rfloor$, and $k \leq s+t-3$, then $\alpha_k(K_s\,\square\,K_t) \geq \alpha_k(K_{k-t+2}\,\square\,K_t) \geq t(k-t+2)$. Similarly, if $t \geq \lfloor\frac{k+3}{2}\rfloor$, $s < \lfloor\frac{k+3}{2}\rfloor$, and $k \leq s+t-3$, then $\alpha_k(K_s\,\square\,K_t) \geq \alpha_k(K_s\,\square\,K_{k-s+2}) \geq s(k-s+2)$. If $s \geq \lfloor\frac{k+3}{2}\rfloor$, $t \geq \lfloor\frac{k+3}{2}\rfloor$, then

$$\begin{aligned}
\alpha_k(K_s\,\square\,K_t) &\geq \alpha_{\lceil\frac{k}{2}\rceil}(K_s)\alpha_{\lfloor\frac{k}{2}\rfloor}(K_t) + \alpha_k(K_{s-\lceil\frac{k}{2}\rceil-1}\,\square\,K_{t-\lfloor\frac{k}{2}\rfloor-1})\\
&\geq \left(\left\lceil\frac{k}{2}\right\rceil+1\right)\left(\left\lfloor\frac{k}{2}\right\rfloor+1\right) + \alpha_k(K_{s-\lceil\frac{k}{2}\rceil-1}\,\square\,K_{t-\lfloor\frac{k}{2}\rfloor-1})\\
&\geq \left(\left\lceil\frac{k}{2}\right\rceil+1\right)\left(\left\lfloor\frac{k}{2}\right\rfloor+1\right) + \alpha_{\lceil\frac{k}{2}\rceil}(K_{s-\lceil\frac{k}{2}\rceil-1})\alpha_{\lfloor\frac{k}{2}\rfloor}(K_{t-\lfloor\frac{k}{2}\rfloor-1})\\
&\quad + \alpha_k(K_{s-2\lceil\frac{k}{2}\rceil-2}\,\square\,K_{t-2\lfloor\frac{k}{2}\rfloor-2})\\
&= 2\left(\left\lceil\frac{k}{2}\right\rceil+1\right)\left(\left\lfloor\frac{k}{2}\right\rfloor+1\right) + \alpha_k(K_{s-2\lceil\frac{k}{2}\rceil-2}\,\square\,K_{t-2\lfloor\frac{k}{2}\rfloor-2})\\
&= \cdots\\
&= \min\{p,q\}\left(\left\lceil\frac{k}{2}\right\rceil+1\right)\left(\left\lfloor\frac{k}{2}\right\rfloor+1\right)\\
&\quad + \alpha_k(K_{s-\min\{p,q\}(\lceil\frac{k}{2}\rceil+1)}\,\square\,K_{t-\min\{p,q\}(\lfloor\frac{k}{2}\rfloor+1)})\\
&\geq \min\{p,q\}\left(\left\lceil\frac{k}{2}\right\rceil+1\right)\left(\left\lfloor\frac{k}{2}\right\rfloor+1\right),
\end{aligned}$$

where $s = (\lceil\frac{k}{2}\rceil+1)p+s'$, $t = (\lfloor\frac{k}{2}\rfloor+1)q+t'$, $0 \leq s' < \lceil\frac{k}{2}\rceil+1$ and $0 \leq t' < \lfloor\frac{k}{2}\rfloor+1$. So the result follows.

See Remarks 2.4 and 2.5 for the sharpness. □

Remark 2.4. From Theorems 2.1, 2.2 and 2.3, we have the following upper bounds for k-independent number.

- $\alpha_k(G\circ H) \leq \alpha_k(H)|V(G)|$;
- $\alpha_k(G\boxtimes H) \leq \min\{\alpha_k(G)|V(H)|, \alpha_k(H)|V(G)|\}$;
- $\alpha_k(G\square H) \leq \min\{\alpha_k(G)|V(H)|, \alpha_k(H)|V(G)|\}$.

To show the sharpness of these upper bounds, we consider the following example. Let $G = nK_1$ and $|V(H)| = m$. Then $G * H$ consists of n copies of H, where $*$ denotes the lexicographical or Cartesian or strong product operation. It is clear that $\alpha_k(G*H) = \alpha_k(H)n = \alpha_k(H)|V(G)|$. So all these upper bounds are sharp.

Remark 2.5. From Theorems 2.1, 2.2 and 2.3, we have the following lower bounds for k-independent number.

- $\alpha_k(G\circ H) \geq \alpha_r(G)\alpha_{k-rm}(H)$, where $m = |V(H)|$;
- $\alpha_k(G\boxtimes H) \geq \alpha_r(G)\alpha_{\lfloor\frac{k}{2r+1}\rfloor}(H)$;

- $\alpha_k(G\Box H) \geq \alpha_r(G)\alpha_{k-r}(H) + X$, where $s = |V(G)| - \alpha_r(G)$, $t = |V(H)| - \alpha_{k-r}(H)$, and

$$X = \begin{cases} st, & \text{if } k \geq s+t-2; \\ t(k-t+1), & \text{if } s \geq \lfloor \frac{k+3}{2} \rfloor, t < \lfloor \frac{k+3}{2} \rfloor, \\ & \text{and } k \leq s+t-3; \\ s(k-s+1), & \text{if } t \geq \lfloor \frac{k+3}{2} \rfloor, s < \lfloor \frac{k+3}{2} \rfloor, \\ & \text{and } k \leq s+t-3; \\ \min\{p,q\}\left(\lceil \frac{k}{2} \rceil + 1\right)\left(\lfloor \frac{k}{2} \rfloor + 1\right), & \text{if } s \geq \lfloor \frac{k+3}{2} \rfloor \text{ and } t \geq \lfloor \frac{k+3}{2} \rfloor. \end{cases}$$

To show the sharpness of these lower bounds, we first consider the following example. Let $G = K_2$ and $H = K_2$. Then $G \circ H = G \boxtimes H = K_4$, and $\alpha_k(G \circ H) = \alpha_k(G \boxtimes H) = \alpha_k(K_4)$. For $k = 0$, $\alpha_k(G \circ H) = \alpha_k(G \boxtimes H) = \alpha_0(K_4) = 1$; for $k = 1$, $\alpha_k(G \circ H) = \alpha_k(G \boxtimes H) = \alpha_1(K_4) = 2$. From Theorems 2.1 and 2.2, $\alpha_k(G \circ H) \geq \alpha_r(G)\alpha_{k-rm}(H)$ and $\alpha_k(G \boxtimes H) \geq \alpha_r(G)\alpha_{\lfloor \frac{k}{2r+1} \rfloor}(H)$. Set $r = 0$. Then $\alpha_k(G \circ H) \geq \alpha_0(K_2)\alpha_k(K_2) = \alpha_k(K_2)$ and $\alpha_k(G \boxtimes H) \geq \alpha_0(K_2)\alpha_k(K_2) = \alpha_k(K_2)$. For $k = 0$, $\alpha_0(K_2) = 1$; for $k = 1$, $\alpha_1(K_2) = 2$. For $k = 0$, $\alpha_k(G \circ H) = \alpha_k(G \boxtimes H) = \alpha_0(G)\alpha_k(H)$. This implies that the first two lower bounds are sharp.

Next, we consider the examples for Cartesian product. Let $G = K_2$ and $H = K_2$. Clearly, $G\Box H = C_4$, and $\alpha_k(G\Box H) = \alpha_k(C_4)$. If $k = 0$, then $r = 0$, $s = t = p = q = 1$, and $\alpha_k(G\Box H) = \alpha_0(C_4) = 2 = \alpha_0(K_2)\alpha_0(K_2) + st = \alpha_0(K_2)\alpha_0(K_2) + \min\{p,q\}\left(\lceil \frac{0}{2} \rceil + 1\right)\left(\lfloor \frac{0}{2} \rfloor + 1\right)$. So the bound for the case $k \geq s+t-2$ or $s \geq \lfloor \frac{k+3}{2} \rfloor, t \geq \lfloor \frac{k+3}{2} \rfloor$ is sharp. For the case $s \geq \lfloor \frac{k+3}{2} \rfloor$, $t < \lfloor \frac{k+3}{2} \rfloor$, and $k \leq s+t-3$, we let $G = K_7$ and $H = K_4$. If $k = 3$, $r = 2$, $s = 4$, and $t = 2$, then $\alpha_3(G\Box H) \geq \alpha_2(K_7)\alpha_1(K_4) + t(k-t+2) = 12$. It suffices to show that $\alpha_3(G\Box H) \leq 12$. Assume, to the contrary, that $\alpha_3(G\Box H) \geq 13$. Let $V(G) = V(K_7) = \{u_i \,|\, 1 \leq i \leq 7\}$ and $V(H) = V(K_4) = \{v_i \,|\, 1 \leq i \leq 4\}$. Then $\bigcup_{i=1}^{4} V(G(v_i)) = V(G\Box H)$. Let I be a maximum 3-independent set in $G\Box H$. Then $|I| \geq 13$. Since $k = 3$, it follows that $|I \cap V(G(v_i))| \leq 4$ for each i $(1 \leq i \leq 4)$. Then there exists some $G(v_i)$ such that $|I \cap V(G(v_i))| = 4$. Without loss of generality, let $I \cap V(G(v_1)) = \{(u_j, v_1) \,|\, 1 \leq j \leq 4\}$. Since $k = 3$ and $|I| \geq 13$, it follows that $|I \cap V(G(v_i))| = 3$ for each i $(2 \leq i \leq 4)$. Since $k = 3$, it follows that $I \cap V(G(v_i)) = \{(u_j, v_i) \,|\, 5 \leq j \leq 7\}$ for each i $(2 \leq i \leq 4)$. Then the degree of the subgraph induced by I is at least 4, a contradiction. So $\alpha_3(G\Box H) = 12$, and hence the lower bound is also sharp.

2.4 The direct product

We give upper and lower bounds for $\alpha_k(G \times H)$ in this section.

Theorem 2.6. *Let $k \geq 0$ be an integers. For graphs G and H,*

(i) $\alpha_k(G \times H) \geq \max\left\{\alpha_{\lfloor \frac{k}{\Delta(H)} \rfloor}(G)|V(H)|, \alpha_{\lfloor \frac{k}{\Delta(G)} \rfloor}(H)|V(G)|\right\}$;

(ii) $\alpha_k(G \times H)$

$$\leq \min\Big\{\alpha_{\lfloor \frac{k}{\Delta(G)} \rfloor}(H)|V(G)| + \alpha_{\lfloor \frac{k}{\delta(H)} \rfloor}(G)|V(H)| - \alpha_{\lfloor \frac{k}{\Delta(G)} \rfloor}(H)\alpha_{\lfloor \frac{k}{\delta(H)} \rfloor}(G),$$
$$\alpha_{\lfloor \frac{k}{\Delta(G)} \rfloor}(G)|V(H)| + \alpha_{\lfloor \frac{k}{\delta(H)} \rfloor}(H)|V(G)| - \alpha_{\lfloor \frac{k}{\Delta(H)} \rfloor}(G)\alpha_{\lfloor \frac{k}{\delta(G)} \rfloor}(H)\Big\}.$$

Moreover, the bounds are sharp.

Proof. *(i)* If I is a $\lfloor \frac{k}{\Delta(H)} \rfloor$-independent set of G, then $I \times V(H)$ is a k-independent set of $G \times H$. Therefore, $\alpha_k(G \times H) \geq \alpha_{\lfloor \frac{k}{\Delta(H)} \rfloor}(G)|V(H)|$. By symmetry of direct product graphs, we have

$$\alpha_k(G \times H) \geq \max\left\{\alpha_{\lfloor \frac{k}{\Delta(H)} \rfloor}(G)|V(H)|, \alpha_{\lfloor \frac{k}{\Delta(G)} \rfloor}(H)|V(G)|\right\}.$$

(ii) Let I be a k-independent set of $G \times H$. Partition I into two vertex subsets J, K such that

$$J = \left\{(u, v) \in I \,|\, (u, v_j) \in I, v_j \in S_{(u,v)}, \text{and } |S_{(u,v)}| \leq \left\lfloor \frac{k}{\Delta(G)} \right\rfloor \right\}$$

and $K = I \setminus J$, where $S_{(u,v)} = \{v_j \in N_H(v) \,|\, (u, v_j) \in I\}$.

Set

$$J^{u_i} = J \cap H(u_i) \text{ and } K^{v_j} = K \cap G(v_j).$$

Let I_H be a maximum $\lfloor \frac{k}{\Delta(G)} \rfloor$-independent set of H. Set

$$Y = (V(G) \times I_H) \cap K$$

and

$$Y^{u_i} = Y \cap H(u_i)$$

Note that $J^{u_i} \cap Y^{u_i} = \varnothing$. From the definition of J, $J^{u_i} \cup Y^{u_i}$ is a $\lfloor \frac{k}{\Delta(G)} \rfloor$-independent set of H, and hence

$$\alpha_{\lfloor \frac{k}{\Delta(G)} \rfloor}(H) \geq |J^{u_i}| + |Y^{u_i}|. \tag{2.1}$$

Claim 1: For $v_j \in V(H)$, $\alpha_{\lfloor \frac{k}{\delta(H)} \rfloor}(G) \geq |K^{v_j}|$.

Proof of Claim 1. For $(u, v_j) \in K^{v_j}$ where $u \in V(G)$, from the definition of K^{v_j}, we have $d_H(v_j) > \lfloor \frac{k}{\Delta(G)} \rfloor$. Since $d_G(u) \cdot d_H(v_j) \leq k$, it follows that

$$d_G(u) \leq \frac{k}{d_H(v_j)} \leq \frac{k}{\delta(H)}.$$

Note that K^{v_j} is a $\lfloor \frac{k}{\delta(H)} \rfloor$-independent set of $G(v_j)$. Therefore, $\alpha_{\frac{k}{\delta(H)}}(G) \geq |K^{v_j}|$. □

Since $\sum_{u_i \in V(G)} |Y^{u_i}| = \sum_{v_j \in I(H)} |K^{v_j}|$, it follows from (2.1) and Claim 1 that

$$\begin{aligned}
&\sum_{u_i \in V(G)} (\alpha_{\lfloor \frac{k}{\Delta(G)} \rfloor}(H) - |J^{u_i}|) + \sum_{v_j \in V(H)} (\alpha_{\lfloor \frac{k}{\delta(H)} \rfloor}(G) - |K^{v_j}|) \\
&\geq \sum_{u_i \in V(G)} |Y^{u_i}| + \sum_{v_j \in V(H)} (\alpha_{\lfloor \frac{k}{\delta(H)} \rfloor}(G) - |K^{v_j}|) \\
&\geq \sum_{v_j \in I(H)} |K^{v_j}| + \sum_{v_j \in I(H)} (\alpha_{\lfloor \frac{k}{\delta(H)} \rfloor}(G) - |K^{v_j}|) \\
&\geq \sum_{v_j \in I(H)} \alpha_{\lfloor \frac{k}{\delta(H)} \rfloor}(G)
\end{aligned}$$

and hence

$$\sum_{u_i \in V(G)} \alpha_{\lfloor \frac{k}{\Delta(G)} \rfloor}(H) + \sum_{v_j \in V(H)} \alpha_{\lfloor \frac{k}{\delta(H)} \rfloor}(G) - \sum_{v_j \in I(H)} \alpha_{\lfloor \frac{k}{\delta(H)} \rfloor}(G) \geq \sum_{u_i \in V(G)} |J^{u_i}| + \sum_{v_j \in V(H)} |K^{v_j}|.$$

Then

$$\alpha_{\lfloor \frac{k}{\Delta(G)} \rfloor}(H)|V(G)| + \alpha_{\lfloor \frac{k}{\delta(H)} \rfloor}(G)|V(H)| - \alpha_{\lfloor \frac{k}{\Delta(G)} \rfloor}(H)\alpha_{\lfloor \frac{k}{\delta(H)} \rfloor}(G) \geq |I|.$$

From the symmetry of direct product, we have

$$\begin{aligned} &\alpha_k(G \times H) \\ &\leq \min\Big\{\alpha_{\lfloor \frac{k}{\Delta(G)} \rfloor}(H)|V(G)| + \alpha_{\lfloor \frac{k}{\delta(H)} \rfloor}(G)|V(H)| - \alpha_{\lfloor \frac{k}{\Delta(G)} \rfloor}(H)\alpha_{\lfloor \frac{k}{\delta(H)} \rfloor}(G), \\ &\quad \alpha_{\lfloor \frac{k}{\Delta(G)} \rfloor}(G)|V(H)| + \alpha_{\lfloor \frac{k}{\delta(H)} \rfloor}(H)|V(G)| - \alpha_{\lfloor \frac{k}{\Delta(H)} \rfloor}(G)\alpha_{\lfloor \frac{k}{\delta(G)} \rfloor}(H)\Big\}. \end{aligned}$$

The proof is now complete. See Remark 2.7 for the sharpness. □

Remark 2.7. To show the sharpness of the lower and upper bounds in Theorem 2.6, we let $G = K_2$ and $H = K_2$. Then

- $\alpha_k(G \times H) \geq \max\{\alpha_k(K_2)|V(K_2)|, \alpha_k(K_2)|V(K_2)|\} = 2\alpha_k(K_2)$;
- $$\begin{aligned} \alpha_k(G \times H) &\leq \min\{\alpha_k(H)|V(G)| + \alpha_k(G)|V(H)| - \alpha_k(H)\alpha_k(G), \\ &\qquad \alpha_k(G)|V(H)| + \alpha_k(H)|V(G)| - \alpha_k(G)\alpha_k(H)\} \\ &= \alpha_k(K_2)|V(K_2)| + \alpha_k(K_2)|V(K_2)| - \alpha_k(K_2)\alpha_k(K_2) \\ &= (4 - \alpha_k(K_2))\alpha_k(K_2). \end{aligned}$$

For $k \geq 1$, we have $\alpha_k(G \times H) = 2$, which implies that the upper and lower bounds in Theorem 2.6 are sharp.

2.5 Relation of four graph products

For the k-independence number of four graph products, we have the following relation.

Proposition 2.8. *For any graphs G and H,*

$$\alpha_k(G \circ H) \leq \alpha_k(G \boxtimes H) \leq \alpha_k(G \,\square\, H) \leq \min\{\alpha_{k\Delta(H)}(G \times H), \alpha_{k\Delta(G)}(G \times H)\}.$$

Proof. Since $G \boxtimes H$ is a subgraph of $G \circ H$, it follows that $\alpha_k(G \circ H) \leq \alpha_k(G \boxtimes H)$. Similarly, since $G \,\square\, H$ is a subgraph of $G \boxtimes H$, it follows that $\alpha_k(G \boxtimes H) \leq \alpha_k(G \,\square\, H)$. From Theorem 2.3, $\alpha_k(G \,\square\, H) \leq \min\{\alpha_k(G)|V(H)|, \alpha_k(H)|V(G)|\}$. From Theorem 2.6, we have

$$\begin{aligned} \alpha_{k\Delta(H)}(G \times H) &\geq \max\{\alpha_k(G)|V(H)|, \alpha_{\frac{k\Delta(H)}{\Delta(G)}}(H)|V(G)|\} \\ &\geq \min\{\alpha_k(G)|V(H)|, \alpha_k(H)|V(G)|\} \\ &\geq \alpha_k(G \,\square\, H). \end{aligned}$$

Similarly, we have $\alpha_{k\Delta(G)}(G \times H) \geq \alpha_k(G \,\square\, H)$, and hence

$$\alpha_k(G \,\square\, H) \leq \min\{\alpha_{k\Delta(H)}(G \times H), \alpha_{k\Delta(G)}(G \times H)\}.$$

The proof is now complete. □

3 Applications

In this section, we demonstrate the usefulness of the proposed constructions by applying them to some instances of Cartesian and lexicographical product networks.

The following results are immediate.

Proposition 3.1. *Let $k \geq 0, n \geq 2$ be two integers and $\{\frac{n}{3}\}$ be the integer such that $n \equiv \{\frac{n}{3}\}(mod\, 3)$.*

(i) For a complete graph K_n,

$$\alpha_k(K_n) = \begin{cases} k+1, & \text{if } 0 \leq k \leq n-1; \\ n, & \text{if } k \geq n. \end{cases}$$

(ii) For a path P_n,

$$\alpha_k(P_n) = \begin{cases} \lceil \frac{n}{2} \rceil, & \text{if } k = 0; \\ 2\lfloor \frac{n}{3} \rfloor + \{\frac{n}{3}\}, & \text{if } k = 1; \\ n, & \text{if } k \geq 2. \end{cases}$$

(iii) For a cycle C_n,

$$\alpha_k(C_n) = \begin{cases} \lfloor \frac{n}{2} \rfloor, & \text{if } k = 0; \\ 2\lfloor \frac{n}{3} \rfloor, & \text{if } k = 1 \text{ and } n \equiv 0, 1\ (mod\ 3); \\ 2\lfloor \frac{n}{3} \rfloor + 1, & \text{if } k = 1 \text{ and } n \equiv 2\ (mod\ 3); \\ n, & \text{if } k \geq 2. \end{cases}$$

3.1 n-dimensional generalized hypercube

Let K_m be a clique of m vertices, $m \geq 2$. An *n-dimensional generalized hypercube* [5, 8] is the product of n cliques.

We first focus our attention on 2-dimensional generalized hypercube.

Proposition 3.2. *For network $K_{m_1} \Box K_{m_2}$,*

$$\begin{aligned} &\min\{m_1, \lceil k/2 \rceil + 1\} \min\{m_2, \lfloor k/2 \rfloor + 1\} \\ &\quad \leq \alpha_k(K_{m_1} \Box K_{m_2}) \\ &\quad \leq \begin{cases} \min\{m_2, m_1\}(k+1), & \text{if } k \leq m_i - 1\ (i = 1, 2); \\ (k+1)m_1, & \text{if } k \leq m_2 - 1,\ k \geq m_1; \\ (k+1)m_2, & \text{if } k \leq m_1 - 1,\ k \geq m_2; \\ m_1 m_2, & \text{if } k \geq m_1,\ k \geq m_2. \end{cases} \end{aligned}$$

Proof. We first investigate the upper bound of $\alpha_k(K_{m_1} \Box K_{m_2})$. If $k \geq m_i$ $(i = 1, 2)$, then $\alpha_k(K_{m_i}) = m_i$ and $\alpha_k(K_{m_1} \Box K_{m_2}) \leq \min\{\alpha_k(K_{m_1})|V(K_{m_2})|, \alpha_k(K_{m_2})|V(K_{m_1})|\} = m_1 m_2$ by Theorem 2.3. If $k \leq m_2 - 1$ and $k \geq m_1$, then $\alpha_k(K_{m_1}) = m_1$ and $\alpha_k(K_{m_2}) = k+1$ and $\alpha_k(K_{m_1} \Box K_{m_2}) \leq \min\{\alpha_k(K_{m_1})|V(K_{m_2})|, \alpha_k(K_{m_2})|V(K_{m_1})|\} = \min\{m_1 m_2, (k+1)m_1\} = (k+1)m_1$. Similarly, if $k \leq m_1 - 1$ and $k \geq m_2$, then $\alpha_k(K_{m_1} \Box K_{m_2}) \leq (k+1)m_2$. If $k \leq m_i - 1$ $(i = 1, 2)$, then $\alpha_k(K_{m_i}) = k+1$, and hence $\alpha_k(K_{m_1} \Box K_{m_2}) \leq \min\{(k+1)m_2, (k+1)m_1\} = \min\{m_2, m_1\}(k+1)$.

Next, we consider the lower bound of $\alpha_k(K_{m_1}\Box K_{m_2})$. From Theorem 2.3, we have $\alpha_k(K_{m_1}\Box K_{m_2}) \geq \alpha_r(K_{m_1})\alpha_{k-r}(K_{m_2})$, where $0 \leq r \leq k$. If $r = \lceil k/2\rceil$, then $k - r = \lfloor k/2\rfloor$, $\alpha_r(K_{m_1}) = \min\{m_1, \lceil k/2\rceil + 1\}$, and $\alpha_{k-r}(K_{m_2}) = \min\{m_2, \lfloor k/2\rfloor + 1\}$. Furthermore, we have $\alpha_k(K_{m_1}\Box K_{m_2}) \geq \alpha_r(K_{m_1})\alpha_{k-r}(K_{m_2}) = \min\{m_1, \lceil k/2\rceil + 1\}\min\{m_2, \lfloor k/2\rfloor + 1\}$, as desired. $\Box$

Next, we consider n-dimensional generalized hypercube.

Proposition 3.3. *For network $K_{m_1}\Box K_{m_2}\Box\cdots\Box K_{m_n}$, we have the following.*

(i) If $m_i \leq k$ $(1 \leq i \leq n)$, then

$$m_1 \leq \alpha_k(K_{m_1}\Box K_{m_2}\Box\cdots\Box K_{m_n}) \leq \prod_{i=1}^{n} m_i.$$

(ii) If $k \leq m_j - 1$ $(1 \leq j \leq n)$, then

$$k+1 \leq \alpha_k(K_{m_1}\Box K_{m_2}\Box\cdots\Box K_{m_n}) \leq (k+1)\prod_{i=2}^{n} m_i.$$

Proof. *(i)* Since $m_i \leq k$ $(1 \leq i \leq n)$, it follows that $\alpha_k(K_{m_i}) = m_i$, where $1 \leq i \leq n$. From Theorem 2.3, we have $\alpha_k(G\Box H) \leq \min\{\alpha_k(G)|V(H)|, \alpha_k(H)|V(G)|\} \leq \alpha_k(G)|V(H)|$ for any two graphs G and H, and hence

$$\begin{aligned}
\alpha_k(K_{m_1}\Box K_{m_2}\Box\cdots\Box K_{m_n}) &= \alpha_k((K_{m_1}\Box K_{m_2}\Box\cdots\Box K_{m_{n-1}})\Box K_{m_n}) \\
&\leq \alpha_k((K_{m_1}\Box K_{m_2}\Box\cdots\Box K_{m_{n-1}})m_n \\
&\leq \alpha_k((K_{m_1}\Box K_{m_2}\Box\cdots\Box K_{m_{n-2}})m_{n-1}m_n \\
&\leq \cdots \\
&\leq \alpha_k(K_{m_1})m_2\cdots m_{n-1}m_n \\
&= \prod_{i=1}^{n} m_i.
\end{aligned}$$

From Theorem 2.3, we have $\alpha_k(G\Box H) \geq \alpha_r(G)\alpha_{k-r}(H)$ for any two graphs G and H. Set $r = k$. Then $\alpha_k(G\Box H) \geq \alpha_k(G)\alpha_0(H)$ for any two graphs G and H, and hence

$$\begin{aligned}
\alpha_k(K_{m_1}\Box K_{m_2}\Box\cdots\Box K_{m_n}) &= \alpha_k((K_{m_1}\Box K_{m_2}\Box\cdots\Box K_{m_{n-1}})\Box K_{m_n}) \\
&\geq \alpha_k(K_{m_1}\Box K_{m_2}\Box\cdots\Box K_{m_{n-1}})\alpha_0(K_{m_n}) \\
&\geq \alpha_k(K_{m_1}\Box K_{m_2}\Box\cdots\Box K_{m_{n-1}}) \\
&\geq \alpha_k((K_{m_1}\Box K_{m_2}\Box\cdots\Box K_{m_{n-2}}) \\
&\geq \cdots \\
&\geq \alpha_k(K_{m_1}) \\
&= m_1.
\end{aligned}$$

(ii) Since $k \leq m_j - 1$ $(1 \leq j \leq n)$, it follows that $\alpha_k(K_{m_j}) = k+1$, where $1 \leq j \leq n$. From Theorem 2.3, we have $\alpha_k(G\Box H) \leq \min\{\alpha_k(G)|V(H)|, \alpha_k(H)|V(G)|\} \leq$

$\alpha_k(G)|V(H)|$ for any two graphs G and H, and hence

$$\begin{aligned}\alpha_k(K_{m_1}\Box K_{m_2}\Box\cdots\Box K_{m_n}) &= \alpha_k((K_{m_1}\Box K_{m_2}\Box\cdots\Box K_{m_{n-1}})\Box K_{m_n})\\ &\le \alpha_k((K_{m_1}\Box K_{m_2}\Box\cdots\Box K_{m_{n-1}})m_n\\ &\le \alpha_k((K_{m_1}\Box K_{m_2}\Box\cdots\Box K_{m_{n-2}})m_{n-1}m_n\\ &\le \cdots\\ &\le \alpha_k((K_{m_1})m_2\cdots m_{n-1}m_n\\ &= (k+1)\prod_{i=2}^{n} m_i.\end{aligned}$$

From Theorem 2.3, we have $\alpha_k(G\Box H)\ge \alpha_r(G)\alpha_{k-r}(H)$ for any two graphs G and H. Set $r=k$. Then $\alpha_k(G\Box H)\ge\alpha_k(G)\alpha_0(H)$ for any two graphs G and H, and hence

$$\begin{aligned}\alpha_k(K_{m_1}\Box K_{m_2}\Box\cdots\Box K_{m_n}) &= \alpha_k((K_{m_1}\Box K_{m_2}\Box\cdots\Box K_{m_{n-1}})\Box K_{m_n})\\ &\ge \alpha_k(K_{m_1}\Box K_{m_2}\Box\cdots\Box K_{m_{n-1}})\alpha_0(K_{m_n})\\ &\ge \alpha_k(K_{m_1}\Box K_{m_2}\Box\cdots\Box K_{m_{n-1}})\\ &\ge \alpha_k((K_{m_1}\Box K_{m_2}\Box\cdots\Box K_{m_{n-2}})\\ &\ge \cdots\\ &\ge \alpha_k(K_{m_1}) = k+1,\end{aligned}$$

as desired. □

Proposition 3.4. *For network $K_{m_1}\circ K_{m_2}\circ\cdots\circ K_{m_n}$,*

$$\alpha_k(K_{m_1}\circ K_{m_2}\circ\cdots\circ K_{m_n}) = \begin{cases} k+1, & \text{if } 0\le k\le \prod_{i=1}^n m_i - 1;\\ \prod_{i=1}^n m_i, & \text{if } k\ge \sum_{i=1}^n m_i.\end{cases}$$

Proof. From the definition of lexicographical product, $K_{m_1}\circ K_{m_2}\circ\cdots\circ K_{m_n}$ is a complete graph. From Proposition 3.1, if $0\le k\le\prod_{i=1}^n m_i-1$, then $\alpha_k(K_{m_1}\circ K_{m_2}\circ\cdots\circ K_{m_n}) = k+1$; if $k+1k\ge\sum_{i=1}^n m_i$, then $\alpha_k(K_{m_1}\circ K_{m_2}\circ\cdots\circ K_{m_n}) = \prod_{i=1}^n m_i$. □

3.2 Two-dimensional grid graph

A *two-dimensional grid graph* is the Cartesian product $P_n\Box P_m$ of path graphs on m and n vertices. For more details on grid graph, we refer to [2, 11]. The network $P_n\circ P_m$ is the lexicographical product $P_n\circ P_m$ of path graphs on m and n vertices; see [15]. Let $\{m/3\}$ be the integer such that $m\equiv\{m/3\}(mod\ 3)$.

Proposition 3.5. *For network $P_n\Box P_m$ $(n\ge 3, m\ge 3)$, we have the following.*

(i) If $k\ge 4$, then $\alpha_k(P_n\Box P_m)=mn$.

(ii) If $k=2,3$, then $\min\{m\lceil n/2\rceil, n\lceil m/2\rceil\}\le\alpha_k(P_n\Box P_m)\le mn$.

(iii) If $k=1$, then

$$\begin{aligned}\lceil n/2\rceil(2\lfloor m/3\rfloor+\{m/3\}) &\le \alpha_k(P_n\Box P_m)\\ &\le \min\{(2\lfloor n/3\rfloor+\{n/3\})m, (2\lfloor m/3\rfloor+\{m/3\})n\}.\end{aligned}$$

(iv) If $k = 0$*, then* $\lceil n/2\rceil\lceil m/2\rceil \le \alpha_k(P_n \Box P_m) \le \min\{\lceil n/2\rceil m, \lceil m/2\rceil n\}$.

Proof. *(i)* Choose all vertices in $P_n \Box P_m$. Since the degree of each vertex in the induced subgraph induced by these vertices is at most 4, it follows that $\alpha_k(P_n \Box P_m) = mn$.

(ii) From Theorem 2.3, $\alpha_2(P_n \Box P_m) \le \min\{\alpha_2(P_n)|V(P_m)|, \alpha_2(P_m)|V(P_n)|\} = \min\{nm, mn\} = mn$ and $\alpha_2(P_n \Box P_m) \ge \alpha_r(P_n)\alpha_{2-r}(P_m)$. If $r = 0$, then we have $\alpha_2(P_n \Box P_m) \ge \alpha_0(P_n)\alpha_2(P_m) = \lceil n/2\rceil m$. If $r = 2$, then $\alpha_2(P_n \Box P_m) \ge \alpha_2(P_n)\alpha_0(P_m) = \lceil m/2\rceil n$. So, we have $\alpha_2(P_n \Box P_m) \ge \min\{m\lceil n/2\rceil, n\lceil m/2\rceil\}$. Similarly, if $k = 3$, then $\min\{m\lceil n/2\rceil, n\lceil m/2\rceil\} \le \alpha_3(P_n \Box P_m) \le mn$.

(iii) From Theorem 2.3, $\alpha_1(P_n \Box P_m) \le \min\{\alpha_1(P_n)|V(P_m)|, \alpha_1(P_m)|V(P_n)|\} = \min\{(2\lfloor n/3\rfloor + \{n/3\})m, (2\lfloor m/3\rfloor + \{m/3\})n\}$. From Theorem 2.3, $\alpha_1(P_n \Box P_m) \ge \alpha_r(P_n)\alpha_{1-r}(P_m)$. If $r = 0$, then $\alpha_1(P_n \Box P_m) \ge \alpha_0(P_n)\alpha_1(P_m) = \lceil n/2\rceil(2\lfloor m/3\rfloor + \{m/3\})$.

(iv) From Theorem 2.3, $\alpha_0(P_n \Box P_m) \le \min\{\alpha_0(P_n)|V(P_m)|, \alpha_0(P_m)|V(P_n)|\} = \min\{\lceil n/2\rceil m, \lceil m/2\rceil n\}$, and $\alpha_0(P_n \Box P_m) \ge \alpha_0(P_n)\alpha_0(P_m) = \lceil n/2\rceil\lceil m/2\rceil$. □

Proposition 3.6. *For network* $P_n \circ P_m$ $(n \ge 4, m \ge 3)$*, we have the following.*

(i) If $k \ge 2m + 2$*, then* $\alpha_k(P_n \circ P_m) = mn$.

(ii) If $2 \le k < 2m + 2$*, then* $\lceil n/2\rceil m \le \alpha_k(P_n \circ P_m) \le mn$.

(iii) If $k = 1$*, then*

$$\lceil n/2\rceil(2\lfloor m/3\rfloor + \{m/3\}) \le \alpha_1(P_n \circ P_m) \le n(2\lfloor m/3\rfloor + \{m/3\}).$$

(iv) If $k = 0$*, then*

$$\lceil n/2\rceil\lceil m/2\rceil \le \alpha_k(P_n \circ P_m) \le n\lceil m/2\rceil.$$

Proof. From Theorem 2.1, we have $\alpha_k(P_n \circ P_m) \le \alpha_k(P_m)|V(P_n)| = n\alpha_k(P_m)$ and $\alpha_k(P_n \circ P_m) \ge \alpha_r(P_n)\alpha_{k-rm}(P_m)$. Let $r = 0$. Then $\alpha_k(P_n \circ P_m) \ge \alpha_0(P_n)\alpha_k(P_m)$, and hence

$$\lceil n/2\rceil\alpha_k(P_m) \le \alpha_k(P_n \circ P_m) \le n\alpha_k(P_m). \tag{3.1}$$

(i) For $k \ge 2m + 2$, we choose all vertices in $P_n \circ P_m$. Since the degree of each vertex in the induced subgraph induced by these vertices is at most $2m + 2$, it follows that $\alpha_k(P_n \circ P_m) = mn$.

(ii) Since $2 \le k < 2m + 2$, it follows that $\alpha_k(P_m) = m$. From (3.1), $\lceil n/2\rceil m \le \alpha_k(P_n \circ P_m) \le mn$.

(iii) For $k = 1$, $\alpha_k(P_m) = 2\lfloor m/3\rfloor + \{m/3\}$. From (3.1), $\lceil n/2\rceil(2\lfloor m/3\rfloor + \{m/3\}) \le \alpha_1(P_n \circ P_m) \le n(2\lfloor m/3\rfloor + \{m/3\})$.

(iv) For $k = 0$, $\alpha_k(P_m) = \lfloor m/2\rfloor$. From (3.1), we have $\lceil n/2\rceil\lceil m/2\rceil \le \alpha_k(P_n \circ P_m) \le n\lceil m/2\rceil$. □

3.3 n-dimensional mesh

An *n-dimensional mesh* is the Cartesian product of n paths. By this definition, two-dimensional grid graph is a 2-dimensional mesh. An n-dimensional hypercube is a special case of an n-dimensional mesh, in which the n linear arrays are all of size 2; see [13].

Proposition 3.7. *For n-dimensional mesh $P_{m_1}\Box P_{m_2}\Box\cdots\Box P_{m_n}$,*

$$\alpha_k(P_{m_1}\Box P_{m_2}\Box\cdots\Box P_{m_n}) \leq \begin{cases} \lceil \frac{m_1}{2}\rceil \prod_{i=2}^n m_i, & \text{if } k=0;\\ (2\lfloor \frac{m_1}{3}\rfloor + \{\frac{m_1}{3}\}) \prod_{i=2}^n m_i, & \text{if } k=1;\\ \prod_{i=1}^n m_i, & \text{if } k\geq 2, \end{cases}$$

and

$$\alpha_k(P_{m_1}\Box P_{m_2}\Box\cdots\Box P_{m_n}) \geq \begin{cases} \lceil \frac{m_1}{2}\rceil \prod_{i=2}^n \lceil m_i/2\rceil, & \text{if } k=0;\\ (2\lfloor \frac{m_1}{3}\rfloor + \{\frac{m_1}{3}\}) \prod_{i=2}^n \lceil m_i/2\rceil, & \text{if } k=1;\\ m_1 \prod_{i=2}^n \lceil m_i/2\rceil, & \text{if } k\geq 2. \end{cases}$$

Proof. From Theorem 2.3, we have $\alpha_k(G\Box H) \leq \min\{\alpha_k(G)|V(H)|, \alpha_k(H)|V(G)|\} \leq \alpha_k(G)|V(H)|$ for any two graphs G and H, and hence

$$\begin{aligned} \alpha_k(P_{m_1}\Box P_{m_2}\Box\cdots\Box P_{m_n}) &= \alpha_k((P_{m_1}\Box P_{m_2}\Box\cdots\Box P_{m_{n-1}})\Box P_{m_n})\\ &\leq \alpha_k((P_{m_1}\Box P_{m_2}\Box\cdots\Box P_{m_{n-1}})m_n\\ &\leq \alpha_k((P_{m_1}\Box P_{m_2}\Box\cdots\Box P_{m_{n-2}})m_{n-1}m_n\\ &\leq \cdots\\ &\leq \alpha_k(P_{m_1})m_2\cdots m_{n-1}m_n. \end{aligned}$$

So, the result follows.

From Theorem 2.3, we have $\alpha_k(G\Box H) \geq \alpha_r(G)\alpha_{k-r}(H)$ for any two graphs G and H. Set $r=k$. Then $\alpha_k(G\Box H)\geq \alpha_k(G)\alpha_0(H)$ for any two graphs G and H, and hence

$$\begin{aligned} \alpha_k(P_{m_1}\Box P_{m_2}\Box\cdots\Box P_{m_n}) &= \alpha_k((P_{m_1}\Box P_{m_2}\Box\cdots\Box P_{m_{n-1}})\Box P_{m_n})\\ &\geq \alpha_k(P_{m_1}\Box P_{m_2}\Box\cdots\Box P_{m_{n-1}})\alpha_0(P_{m_n})\\ &\geq \alpha_k(P_{m_1}\Box P_{m_2}\Box\cdots\Box P_{m_{n-1}})\lceil m_n/2\rceil\\ &\geq \alpha_k((P_{m_1}\Box P_{m_2}\Box\cdots\Box P_{m_{n-2}})\lceil m_{n-1}/2\rceil\lceil m_n/2\rceil\\ &\geq \cdots\\ &\geq \alpha_k(P_{m_1})\prod_{i=2}^n \lceil m_i/2\rceil, \end{aligned}$$

and hence the result holds. □

Similarly to the proof of Proposition 3.7, we can obtain the following result.

Proposition 3.8. *For n-dimensional mesh $P_{m_1}\circ P_{m_2}\circ\cdots\circ P_{m_n}$,*

$$\begin{cases} \lceil \frac{m_1}{2}\rceil \leq \alpha_k(P_{m_1}\circ\cdots\circ P_{m_n}) \leq \lceil \frac{m_1}{2}\rceil \prod_{i=2}^n m_i, & \text{if } k=0;\\ 2\lfloor \frac{m_1}{3}\rfloor + \{\frac{m_1}{3}\} \leq \alpha_k(P_{m_1}\circ\cdots\circ P_{m_n}) \leq (2\lfloor \frac{m_1}{3}\rfloor + \{\frac{m_1}{3}\}) \prod_{i=2}^n m_i, & \text{if } k=1;\\ m_1 \leq \alpha_k(P_{m_1}\circ\cdots\circ P_{m_n}) \leq \prod_{i=1}^n m_i, & \text{if } k\geq 2. \end{cases}$$

3.4 n-dimensional torus

An *n-dimensional torus* is the Cartesian product of n cycles $C_{m_1}, C_{m_2}, \cdots, C_{m_n}$ of size at least three. The cycles C_{m_i} are not necessary to have the same size. Ku et al. [14] showed that there are n edge-disjoint spanning trees in an n-dimensional torus. The network $C_{m_1} \circ C_{m_2} \circ \cdots \circ C_{m_n}$ is investigated in [15]. Here, we consider the networks constructed by $C_{m_1} \Box C_{m_2} \Box \cdots \Box C_{m_n}$ and $C_{m_1} \circ C_{m_2} \circ \cdots \circ C_{m_n}$, respectively.

Proposition 3.9. *For network $C_n \Box C_m$ $(n \geq 3, m \geq 3)$, we have the following.*

(i) *If $k \geq 4$, then $\alpha_k(C_n \Box C_m) = mn$.*

(ii) *If $k = 3$ or $k = 2$, then $\min\{m\lfloor n/2\rfloor, n\lfloor m/2\rfloor\} \leq \alpha_k(C_n \Box C_m) \leq mn$.*

(iii) *If $k = 1$, then $2\lfloor n/2\rfloor\lfloor \frac{m}{3}\rfloor \leq \alpha_k(C_n \Box C_m) \leq \min\{m(2\lfloor \frac{n}{3}\rfloor + 1), n(2\lfloor \frac{m}{3}\rfloor + 1)\}$.*

(iv) *If $k = 0$, then $\lfloor n/2\rfloor\lfloor m/2\rfloor \leq \alpha_k(C_n \Box C_m) \leq \min\{\lfloor n/2\rfloor m, \lfloor m/2\rfloor n\}$.*

Proof. *(i)* Choose all vertices in $C_n \Box C_m$. Since the degree of each vertex in the induced subgraph induced by these vertices is at most 4, it follows that $\alpha_k(C_n \Box C_m) = mn$.

(ii) From Theorem 2.3, $\alpha_3(C_n \Box C_m) \leq \min\{\alpha_3(C_n)|V(C_m)|, \alpha_3(C_m)|V(C_n)|\} = \min\{nm, mn\} = mn$, and $\alpha_3(P_n \Box P_m) \geq \alpha_r(C_n)\alpha_{3-r}(C_m)$. If $r = 0$, then we have $\alpha_3(C_n \Box C_m) \geq \alpha_0(C_n)\alpha_3(C_m) = \lfloor n/2\rfloor m$. If $r = 3$, then $\alpha_3(C_n \Box C_m) \geq \alpha_3(C_n)\alpha_0(C_m) = \lfloor m/2\rfloor n = \lfloor m/2\rfloor n$. So, $\alpha_3(C_n \Box C_m) \geq \min\{m\lfloor n/2\rfloor, n\lfloor m/2\rfloor\}$. The case $k = 2$ can be similarly proved.

(iii) From Theorem 2.3, $\alpha_1(C_n \Box C_m) \geq \alpha_r(C_n)\alpha_{1-r}(C_m)$. If $r = 0$, then we have $\alpha_1(C_n \Box C_m) \geq \alpha_0(C_n)\alpha_1(C_m) = \lfloor n/2\rfloor(2\lfloor \frac{m}{3}\rfloor)$, and $\alpha_1(C_n \Box C_m) \leq \min\{\alpha_1(C_n)|V(C_m)|, \alpha_1(C_m)|V(C_n)|\} = \min\{m(2\lfloor \frac{n}{3}\rfloor + 1), n(2\lfloor \frac{m}{3}\rfloor + 1)\}$.

(iv) From Theorem 2.3, $\alpha_0(C_n \Box C_m) \leq \min\{\lfloor n/2\rfloor m, \lfloor m/2\rfloor n\}$, and $\alpha_0(C_n \Box C_m) \geq \alpha_0(C_n)\alpha_0(C_m) = \lfloor n/2\rfloor\lfloor m/2\rfloor$. □

For network $C_n \circ C_m$, we have the following result.

Proposition 3.10. *For network $C_n \circ C_m$ $(n \geq 4, m \geq 3)$, we have the following.*

(i) *If $k \geq 2m + 2$, then $\alpha_k(C_n \circ C_m) = mn$.*

(ii) *If $2 \leq k < 2m + 2$, then $\lfloor n/2\rfloor m \leq \alpha_k(C_n \circ C_m) \leq mn$.*

(iii) *If $k = 1$ and $n \equiv 0, 1 \pmod 3$, then $2\lfloor n/2\rfloor\lfloor n/3\rfloor \leq \alpha_k(C_n \circ C_m) \leq 2n\lfloor n/3\rfloor$.*

(iv) *If $k = 1$ and $n \equiv 2 \pmod 3$, then*

$$\lfloor n/2\rfloor(2\lfloor n/3\rfloor + 1) \leq \alpha_k(C_n \circ C_m) \leq n(2\lfloor n/3\rfloor + 1).$$

(v) *If $k = 0$, then $\lfloor m/2\rfloor\lfloor n/2\rfloor \leq \alpha_0(C_n \circ C_m) \leq n\lfloor m/2\rfloor$.*

Proof. From Theorem 2.1, we have $\alpha_k(C_n \circ C_m) \leq \alpha_k(C_m)|V(C_n)| = n\alpha_k(C_m)$ and $\alpha_k(C_n \circ C_m) \geq \alpha_r(C_n)\alpha_{k-rm}(C_m)$. Let $r = 0$. Then $\alpha_k(C_n \circ C_m) \geq \alpha_0(C_n)\alpha_k(C_m)$, and hence

$$\lfloor n/2\rfloor\alpha_k(C_m) \leq \alpha_k(C_n \circ C_m) \leq n\alpha_k(C_m). \tag{3.2}$$

(i) For $k \geq 2m + 2$, we choose all vertices in $C_n \circ C_m$. Since the degree of each vertex in the induced subgraph induced by these vertices is at most $2m + 2$, it follows that $\alpha_k(C_n \circ C_m) = mn$.

(ii) Since $2 \leq k < 2m+2$, it follows that $\alpha_k(C_m) = m$, and hence $\lfloor n/2 \rfloor m \leq \alpha_k(C_n \circ C_m) \leq mn$ by (3.2).

(iii) Since $k = 1$ and $n \equiv 0, 1$ $(mod\ 3)$, we have $\alpha_k(C_m) = 2\lfloor \frac{n}{3} \rfloor$. From (3.2), $2\lfloor n/2 \rfloor \lfloor n/3 \rfloor \leq \alpha_k(C_n \circ C_m) \leq 2n\lfloor n/3 \rfloor$.

(iv) For $k = 1$ and $n \equiv 2$ $(mod\ 3)$, $\alpha_k(C_m) = 2\lfloor \frac{n}{3} \rfloor + 1$. From (3.2), $\lfloor n/2 \rfloor (2\lfloor n/3 \rfloor + 1) \leq \alpha_k(C_n \circ C_m) \leq n(2\lfloor n/3 \rfloor + 1)$.

(v) For $k = 0$, $\alpha_k(C_m) = \lfloor m/2 \rfloor$. From (3.2), $\lfloor m/2 \rfloor \lfloor n/2 \rfloor \leq \alpha_k(C_n \circ C_m) \leq n\lfloor m/2 \rfloor$. □

For general case, we have the following two results.

Proposition 3.11. *For network $C_{m_1} \square C_{m_2} \square \cdots \square C_{m_n}$,*

$$\alpha_k(C_{m_1} \square C_{m_2} \square \cdots \square C_{m_n}) \leq \begin{cases} \lfloor \frac{m_1}{2} \rfloor \prod_{i=2}^{n} m_i, & \text{if } k = 0; \\ 2\lfloor \frac{m_1}{3} \rfloor \prod_{i=2}^{n} m_i, & \text{if } k = 1, m_1 \equiv 0, 1\ (mod\ 3); \\ (2\lfloor \frac{m_1}{3} \rfloor + 1) \prod_{i=2}^{n} m_i, & \text{if } k = 1, m_1 \equiv 2\ (mod\ 3); \\ \prod_{i=1}^{n} m_i, & \text{if } k \geq 2, \end{cases}$$

and

$$\alpha_k(C_{m_1} \square \cdots \square C_{m_n}) \geq \begin{cases} \lfloor \frac{m_1}{2} \rfloor \prod_{i=2}^{n} \lfloor m_i/2 \rfloor, & \text{if } k = 0; \\ 2\lfloor \frac{m_1}{3} \rfloor \prod_{i=2}^{n} \lfloor m_i/2 \rfloor, & \text{if } k = 1, m_1 \equiv 0, 1\ (mod\ 3); \\ (2\lfloor \frac{m_1}{3} \rfloor + 1) \prod_{i=2}^{n} \lfloor m_i/2 \rfloor, & \text{if } k = 1, m_1 \equiv 2\ (mod\ 3); \\ m_1 \prod_{i=2}^{n} \lfloor m_i/2 \rfloor, & \text{if } k \geq 2, \end{cases}$$

where m_i is the order of C_{m_i} and $1 \leq i \leq n$.

Proof. From Theorem 2.3, we have $\alpha_k(G \square H) \leq \min\{\alpha_k(G)|V(H)|, \alpha_k(H)|V(G)|\} \leq \alpha_k(G)|V(H)|$ for any two graphs G and H, and hence

$$\begin{aligned} \alpha_k(C_{m_1} \square C_{m_2} \square \cdots \square C_{m_n}) &= \alpha_k((C_{m_1} \square C_{m_2} \square \cdots \square C_{m_{n-1}}) \square C_{m_n}) \\ &\leq \alpha_k((C_{m_1} \square C_{m_2} \square \cdots \square C_{m_{n-1}})m_n \\ &\leq \alpha_k((C_{m_1} \square C_{m_2} \square \cdots \square C_{m_{n-2}})m_{n-1}m_n \\ &\leq \cdots \\ &\leq \alpha_k(C_{m_1})m_2 \cdots m_{n-1}m_n. \end{aligned}$$

From (iii) of Proposition 3.1, the result follows.

From Theorem 2.3, we have $\alpha_k(G \square H) \geq \alpha_r(G)\alpha_{k-r}(H)$ for any two graphs G and H. Set $r = k$. Then $\alpha_k(G \square H) \geq \alpha_k(G)\alpha_0(H)$ for any two graphs G and H, and hence

$$\begin{aligned} \alpha_k(C_{m_1} \square C_{m_2} \square \cdots \square C_{m_n}) &= \alpha_k((C_{m_1} \square C_{m_2} \square \cdots \square C_{m_{n-1}}) \square C_{m_n}) \\ &\geq \alpha_k(C_{m_1} \square C_{m_2} \square \cdots \square C_{m_{n-1}})\alpha_0(C_{m_n}) \\ &\geq \alpha_k(C_{m_1} \square C_{m_2} \square \cdots \square C_{m_{n-1}})\lfloor m_n/2 \rfloor \\ &\geq \alpha_k((C_{m_1} \square C_{m_2} \square \cdots \square C_{m_{n-2}})\lfloor m_{n-1}/2 \rfloor \lfloor m_n/2 \rfloor \\ &\geq \cdots \\ &\geq \alpha_k(C_{m_1}) \prod_{i=2}^{n} \lfloor m_i/2 \rfloor. \end{aligned}$$

From (3.2) of Proposition 3.1, the result holds. □

Similarly to the proof of Proposition 3.11, we can prove the following result.

Proposition 3.12. *For network* $C_{m_1} \circ C_{m_2} \circ \cdots \circ C_{m_n}$,

$$\begin{cases} \lfloor \frac{m_1}{2} \rfloor \leq \alpha_k(C_{m_1} \circ \cdots \circ C_{m_n}) \leq \lfloor \frac{m_1}{2} \rfloor \prod_{i=2}^{n} m_i, & \text{if } k=0; \\ 2\lfloor \frac{m_1}{3} \rfloor \leq \alpha_k(C_{m_1} \circ \cdots \circ C_{m_n}) \leq 2\lfloor \frac{m_1}{3} \rfloor \prod_{i=2}^{n} m_i, & \text{if } k=1 \\ & \quad \text{and } m_1 \equiv 0,1 \,(mod\, 3); \\ 2\lfloor \frac{m_1}{3} \rfloor + 1 \leq \alpha_k(C_{m_1} \circ \cdots \circ C_{m_n}) \leq (2\lfloor \frac{m_1}{3} \rfloor + 1) \prod_{i=2}^{n} m_i, & \text{if } k=1 \\ & \quad \text{and } m_1 \equiv 2 \,(mod\, 3); \\ m_1 \leq \alpha_k(C_{m_1} \circ \cdots \circ C_{m_n}) \leq \prod_{i=1}^{n} m_i, & \text{if } k \geq 2, \end{cases}$$

where m_i *is the order of* C_{m_i} *and* $1 \leq i \leq n$.

3.5 n-dimensional hyper Petersen network

An *n-dimensional hyper Petersen network* HP_n is the product of the well-known Petersen graph and Q_{n-3} [4], where $n \geq 3$ and Q_{n-3} denotes an $(n-3)$-dimensional hypercube. Note that HP_3 is just the Petersen graph.

The network HL_n is the lexicographical product of the Petersen graph and Q_{n-3}, where $n \geq 3$ and Q_{n-3} denotes an $(n-3)$-dimensional hypercube; see [15]. Note that HL_3 is just the Petersen graph, and HL_4 is a graph obtained from two copies of the Petersen graph by adding the edges between all the vertices from different copies of the Petersen graph.

Proposition 3.13. *(i) For network* HP_3 *and* HL_3,

$$\alpha_k(HP_3) = \alpha_k(HL_3) = \begin{cases} 4, & \text{if } k=0; \\ 5, & \text{if } k=1; \\ 5, & \text{if } k=2; \\ 10, & \text{if } k \geq 3. \end{cases}$$

(ii) For network HP_4,

$$\begin{cases} 5 \leq \alpha_k(HP_4) \leq 8, & \text{if } k=0; \\ 6 \leq \alpha_k(HP_4) \leq 10, & \text{if } k=1; \\ 6 \leq \alpha_k(HP_4) \leq 10, & \text{if } k=2; \\ 11 \leq \alpha_k(HP_4) \leq 20, & \text{if } k=3; \\ \alpha_k(HP_4) = 30, & \text{if } k \geq 4. \end{cases}$$

(iii) For network HL_4,

$$\begin{cases} 4 \leq \alpha_k(HP_4) \leq 15, & \text{if } k=0; \\ 8 \leq \alpha_k(HP_4) \leq 30, & \text{if } k \geq 1. \end{cases}$$

Proof. (i) Note that HL_3 or HP_3 is just the Petersen graph, and its maximum degree is 3. Since $|V(HP_3)| = 10$, it follows that $\alpha_k(HP_3) = 10$ for $k \geq 3$. One can also check that

$$\alpha_k(HP_3) = \alpha_k(HL_3) = \begin{cases} 4, & \text{if } k=0; \\ 5, & \text{if } k=1; \\ 5, & \text{if } k=2. \end{cases}$$

(ii) For network HP_4, $HP_4 = HP_3 \Box K_2$. From Theorem 2.3, we have $\alpha_k(HP_4) = \alpha_k(HP_3 \Box K_2) \leq \min\{2\alpha_k(HP_3), 10\alpha_k(K_2)\}$. Note that $\alpha_k(K_2) = 1$ for $k = 0$; $\alpha_k(K_2) = 2$ for $k \geq 1$. Combining this with (i) of this proposition, we have

$$\alpha_k(HP_4) \leq \begin{cases} 8, & \text{if } k = 0; \\ 10, & \text{if } k = 1; \\ 10, & \text{if } k = 2; \\ 20, & \text{if } k \geq 3. \end{cases}$$

From Theorem 2.3, $\alpha_k(HP_4) \geq \alpha_r(HP_3)\alpha_{k-r}(K_2) + \alpha_k(K_s \Box K_t)$, where $s = |V(HP_3)| - \alpha_r(HP_3)$ and $t = |V(K_2)| - \alpha_{k-r}(K_2)$. Set $r = k$. Then $t = 1$ and $\alpha_k(HP_4) \geq \alpha_k(HP_3)\alpha_0(K_2) + \alpha_k(K_s \Box K_1) \geq \alpha_k(HP_3) + 1$, and hence

$$\alpha_k(HP_4) \geq \begin{cases} 5, & \text{if } k = 0; \\ 6, & \text{if } k = 1; \\ 6, & \text{if } k = 2; \\ 11, & \text{if } k \geq 3. \end{cases}$$

(iii) For network HL_4, $HL_4 = K_2 \circ HL_3$. From Theorem 2.3, we have $\alpha_k(HL_4) = \alpha_k(K_2 \circ HL_3) \leq |V(HL_3)|\alpha_k(K_2) = 10\alpha_k(K_2)$. Note that $\alpha_k(K_2) = 1$ for $k = 0$; $\alpha_k(K_2) = 2$ for $k \geq 1$. Combining this with (i) of this proposition, we have

$$\alpha_k(HL_4) \leq \begin{cases} 15, & \text{if } k = 0; \\ 20, & \text{if } k \geq 1. \end{cases}$$

From Theorem 2.3, $\alpha_k(HL_4) \geq \alpha_r(HL_3)\alpha_{k-2r}(K_2)$. Set $r = 0$. Then $\alpha_k(HL_4) \geq \alpha_0(HL_3)\alpha_k(K_2) = 4\alpha_k(K_2)$, and hence

$$\alpha_k(HL_4) \geq \begin{cases} 4, & \text{if } k = 0; \\ 8, & \text{if } k \geq 1. \end{cases}$$

$\square$

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